Bad behavior of length in taking the infinite limit.

How does the bad behavior of these limits appear if we set them in the context of the methods used to determine lengths in the calculus? The short answer is that we can identify a specific place in which a limit fails. However, I do not see that this failure gives a better diagnosis of the paradox than is given in the main text. Here are the details.

In the failed "proofs" of the text, we consider the limiting behavior of a family of curves C_1, C_2, C_3, …. Represent these curves in a space with Cartesian coordinates (x,y) as the functions $y_1(x)$, $y_2(x)$, $y_3(x)$, … The length $L_n(a,b)$ of the curve $y_n(x)$ between $x = a$ and $x = b$ is given by

$$L_n(a,b) = \int_{x=a}^{b} \sqrt{1 + \left(\frac{dy_n(x)}{dx}\right)^2}\, dx$$

As long as we consider the curves $y_n(x)$ directly, nothing goes amiss. The limit of the curves as n → ∞ is the constant function. That is

$$\lim_{n\to\infty} y_n(x) = y_\infty(x) = \text{ constant.}$$

However, if we consider the length $L_n(a,b)$ as defined by the integral above, then we have a failure. It arises because the integral is not well defined as we take the infinite limit of n. This follows since the length $L_n(a,b)$ as defined by the integral is not a function of $y_n(x)$ directly. Rather the length $L_n(a,b)$ is defined through the integral in terms of the derivative $\frac{dy_n(x)}{dx}$. As we proceed along the sequence of curves $y_1(x)$, $y_2(x)$, $y_3(x)$, … , the values of $y_n(x)$ oscillate more and more widely so that

$$\lim_{n\to\infty} \left(\frac{dy_n}{dx}\right) \text{ is not defined}$$

It then follows that the integral

$$\int_{x=a}^{b} \sqrt{1 + \left(\frac{dy_n(x)}{dx}\right)^2}\, dx$$

has no definite value as n → ∞. If, however, we substitute directly $y_\infty(x)$ = constant for $y_n(x)$, then we get the result expected. For then

$$\frac{dy_{\infty}(x)}{dx} = 0$$

so that

$$L_{\infty}(a, b) = \int_{x=a}^{b} \sqrt{1 + \left(\frac{dy_{\infty}}{dx}\right)^2}\, dx = \int_{x=a}^{b} \sqrt{1+0}\, dx = b - a$$

While these are all secure results in the calculus, I do not see that they explain the flaw in the "proofs" any better than merely observing that this: The numerical sequence of lengths, understood as numbers only,

$$L_1(a,b),\ L_2(a,b),\ L_3(a,b),\ \ldots$$

has a different limit from the value of $L_{\infty}(a,b)$ computed directly from $y_{\infty}(x) =$ constant.

The error of reasoning arises from expecting that the two would match. While in similar cases the two do often match, there is in general no compulsion for the two always to match. In this case, they do not.

$$L_{\infty}(a, b) = \int_{x=a}^{b} \sqrt{1 + \left(\frac{dy_{\infty}}{dx}\right)^2}\, dx = \int_{x=a}^{b} \sqrt{1+0}\, dx = b - a$$