The Gambler's Ruin

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The Game Played

A gambler plays a game in which, with each round, the gambler wins one unit or loses one unit, with the probabilities p and q respectively. The gambler starts with some initial stake S and keeps betting, one unit at a time, until the gambler's total fortune has arrived at a targeted maximum M, "winning"; or the gambler has lost the entire fortune, "ruin." The gambler's ruin problem is to determine the probability of ruin and to estimate how long the play will last.

The results below are drawn from William Feller, *An Introduction to Probability Theory and Its Applications*. Vol. 1. 3rd ed. New York: Wiley, 1968. The proofs differ from Feller's text and are my own.

Main Results

Condition for a fair game

(1) $p = q = 1/2$

since then the expectation per bet is

$$p(+1) + q(-1) = 1/2 - 1/2 = 0.$$

Condition for an unfair game that disadvantages the gambler

(2) $q > p$ and $r = q/p > 1$

and the expectation per bet is

$$p(+1) + q(-1) = p - q = -(q - p).$$

Probability of ruin

In a fair game:

(3) $P_{\text{ruin}}(S) = P_{\text{ruin}}(S) = (M\text{-}S)/M$

If the gambler just seeks to double the stake so that $M = 2S$, then

$$P_{\text{ruin}} = S/2S = 1/2$$

If the gambler is more ambitious and seeks a fortune M=10S, then

$$P_{\text{ruin}} = 9S/10S = 9/10$$

In an unfair game, in which the gambler is disadvantaged:

(4) $$P_{ruin}(S) = \frac{r^M - r^S}{r^M - 1}$$

A small amount of unfairness in the game is enough to make the chances of ruin very great. In American roulette, for odd/even or red/black bets, with a 0 and 00 on the wheel

p = 18/38, q = 20/38, so that r = 20/18 = 10/9

q-p = 20/38-18/38=1/19

If a gambler starts with S=50 and seeks to double this fortune to M= 100, then the prospects are remote from the chance of 1/2 in the fair game:

$r^M = (10/9)^{100} = 37{,}649$ $\qquad r^S = (10/9)^{50} = 194$

$$P_{ruin} = \frac{r^M - r^S}{r^M - 1} = \frac{37{,}649 - 194}{37{,}649 - 1} = 0.9949$$

Thus, if the gambler bets one unit at a time, the chance of ruin is very high. This should be compared to the situation if the gambler were to bet the entire stake of S=50 at once. Then

$$P_{\text{ruin}} = 20/38 = 0.5263$$

which is close to the probability of ruin in a fair game. The practical moral is that, when gambling in an unfair game, if you must, make the largest bets possible.

Duration of play

The number of bets made by the gambler before winning or ruin has many values with various probabilities. The probabilistic expectation D is as follows:

In a fair game:

(5) $$D(S) = S(M - S)$$

If the gambler merely seeks to double the stake, so that $M = 2S$, then

$$D(S) = S^2$$

When $S = 50$, we have

$$D = 2500$$

In an unfair game that disadvantages the gambler:

(6) $$D(S) = \frac{S}{q-p} - \frac{M}{q-p} \cdot \frac{r^S-1}{r^M-1}$$

For the same game with S = 50 and M = 100, but with American roulette odds so that q-p = 20/38-18/38=2/17, we have

$$D(S) = \frac{50}{1/19} - \frac{100}{\frac{1}{19}} \cdot \frac{194-1}{37{,}649-1} = \frac{50}{1/19} - \frac{100}{\frac{1}{19}} \cdot 0.005126$$

$$= 950 - 1900 \text{ x } 0.005126 = 950 - 9.74 = 940.3$$

That is, ruin is expected to come much faster than the completion of the fair game, whose expected duration is 2,500.

This last calculation indicates that in many real cases, the first term, in (6), $S/(q-p)$ comprises the bulk of the result. More generally, in the common case in which $r^M >> r^S$. Then the second term in (6) is close to zero and we have

(6a) $$D(S) \approx \frac{S}{q-p}$$

This result is easy to interpret informally. In an unfair game with $q > p$, on each unit bet the gambler expects to lose on average $q - p$. Thus, the approximate duration of play in (16) is just:

Duration is roughly (initial stake S) / (average loss per bet (q-p))

for then the gambler loses q-p, over and over, $D(S)$ times, after which all the initial stake is exhausted. Notably, this expected duration of play is independent of M. This reflects that fact that in these cases, ruin is all but assured.

This approximation (6a) is an upper bound since it will always be greater than the value in (6), for finite M. This suggests that smaller values of M lead to shorter durations of play through the happy circumstance that the gambler may actually win and raise the fortune to M.

Proofs

Probability of Ruin in Fair Game

With each gamble, the expected return is zero, so the total expectation is zero. Play ends with either winning M-S, with probability $P_{\text{win}}(S)$ or ruin, a loss of -S, with $P_{\text{ruin}}(S)$. We have:

$$P_{\text{win}}(S) + P_{\text{ruin}}(S) = 1$$

and from the zero expectation

$(M\text{-}S).P_{\text{win}}(S) - S.P_{\text{ruin}}(S) = 0$

That is

$(M\text{-}S).(1\text{-}P_{\text{ruin}}(S)) = S.P_{\text{ruin}}(S)$

$(M\text{-}S) = P_{\text{ruin}}(S)\ (M\text{-}S) + S.P_{\text{ruin}}(S) = M.P_{\text{ruin}}(S)$

So that $P_{\text{ruin}}(S) = (M\text{-}S)/M$

Probability of Ruin in an Unfair Game that Disadvantages the Gambler

In the course of play, the gambler's fortune will move up and down over many values lying between M and zero. Using z to represent this fluctuating fortune, we connect the probability of ruin, $P_{\text{ruin}}(z)$, for different values of z by considering the two possible outcomes of the first bet taken by the gambler who starts with a stake of z:

$$P_{\text{ruin}}(z) = P(\text{win, so that z} \rightarrow \text{z+1}).\ P_{\text{ruin}}(z+1)$$
$$+ P(\text{lose, so that z} \rightarrow \text{z-1}).\ P_{\text{ruin}}(z-1)$$

(7) $$P_{\text{ruin}}(z) = p.\ P_{\text{ruin}}(z+1) + q.\ P_{\text{ruin}}(z-1)$$

where $z > 0$, since otherwise if z=0, z-1 would be an impossible negative value for $P_{\text{ruin}}(z\text{-}1)$. Using $1 = p + q$, we can re-express $P_{\text{ruin}}(z)$ as:

$$P_{\text{ruin}}(z) = p.\ P_{\text{ruin}}(z) + q.\ P_{\text{ruin}}(z)$$

Substituting into (7), the difference equation (7) can be rewritten as

$$P_{\text{ruin}}(z+1) - P_{\text{ruin}}(z) = (q/p)\ [P_{\text{ruin}}(z) - P_{\text{ruin}}(z-1)]$$
$$= r.\ [P_{\text{ruin}}(z) - P_{\text{ruin}}(z-1)]$$

for $z > 0$. This set of difference equations is to be solved for $0 \le z \le M$. We have the boundary conditions:

$P_{\text{ruin}}(0) = 1$, since at z=0, the gambler has lost the stake

$P_{\text{ruin}}(M) = 0$, since at z=M, the gamblers halts playing with a win.

We arrive at a set of equations for $0 \le z \le S$:

(8) $P_{\text{ruin}}(2) - P_{\text{ruin}}(1) = r.\ [P_{\text{ruin}}(1) - P_{\text{ruin}}(0)]$

$P_{\text{ruin}}(3) - P_{\text{ruin}}(2) = r.\ [P_{\text{ruin}}(2) - P_{\text{ruin}}(1)]$

…

$P_{ruin}(S)$ - $P_{ruin}(S$-1) = r. [$P_{ruin}(S$-1) - $P_{ruin}(S$-2)]

Substituting for sequentially for [P_{ruin}(2) - P_{ruin}(1)], [P_{ruin}(3) - P_{ruin}(2)], … and using P_{ruin}(0)=1, this set becomes[1]

(9) P_{ruin}(1) - P_{ruin}(0) = [P_{ruin}(1) - 1]

P_{ruin}(2) - P_{ruin}(1) = r. [P_{ruin}(1) - 1]

P_{ruin}(3) - P_{ruin}(2) = r^2. [P_{ruin}(1) - 1]

…

$P_{ruin}(S)$ - $P_{ruin}(S$-1) = r^{S-1}. [P_{ruin}(1) - 1]

Summing these S equations, we have:

$P_{ruin}(S)$ - P_{ruin}(0) = (1 + r + r^2 + … + r^{S-1}). [P_{ruin}(1) - 1]

or

(9) $P_{ruin}(S)$ - 1 = (1 + r + r^2 + … + r^{S-1}). [P_{ruin}(1) - 1]

As long as the game is unfair and r does not equal 1, the geometric series can be summed to give

(10) $$P_{ruin}(S) - 1 = \frac{r^S - 1}{r-1}[P_{ruin}(1) - 1]$$

To complete the computation of $P_{ruin}(S)$, the value of the remaining variable $P_{ruin}(1)$ needs to be found. Its value is introduced into the calculation with the remaining boundary condition, $P_{ruin}(M) = 0$. That is, setting S=M in (10) we have:

$$P_{ruin}(M) - 1 = \frac{r^M - 1}{r-1}[P_{ruin}(1) - 1] = -1$$

Solving for $P_{ruin}(1)$ we have

(11) $$P_{ruin}(1) - 1 = -\frac{r-1}{r^M - 1}$$

Substituing (11) into (10), we have

$$P_{ruin}(S) - 1 = \frac{r^S - 1}{r-1}\left(-\frac{r-1}{r^M - 1}\right)$$

so that

$$P_{ruin}(S) = 1 + \frac{r^S - 1}{r-1}\left(-\frac{r-1}{r^M - 1}\right) = 1 - \frac{r^S - 1}{r^M - 1} = \frac{r^M - r^S}{r^M - 1}$$

which completed the proof.

[1] The first equation in the set is not derived from the difference equation but is simply a rewriting of P_{ruin}(1) - P_{ruin}(0) = [P_{ruin}(1) - P_{ruin}(0)].

Probability of Ruin in a Fair Game (Again)

This probability can be derived from the difference equations above as long as we only use that portion that is compatible with a fair game, with $p = q = 1/2$, $r = 1$. We have from (9) that

$$P_{ruin}(S) - 1 = (1 + r + r^2 + \ldots + r^{S-1}) \,.\, [P_{ruin}(1) - 1] = S \,.\, [P_{ruin}(1) - 1]$$

To evaluate $[P_{ruin}(1) - 1]$, we use $P_{ruin}(M) = 0$. Substituting M in this last equation, we have

$$-1 = P_{ruin}(M) – 1 = M \,.\, [P_{ruin}(1) - 1]$$

so that $[P_{ruin}(1) - 1] = -1/M$. Substituting in the expression for $P_{ruin}(S) – 1$ we have

$$P_{ruin}(S) - 1 = S \,.\, [-1/M]$$

Hence

$$P_{ruin}(S) = 1 – S/M = (M-S)/M$$

which completed the proof.

Expected Duration in an Unfair Game

The expected duration is computed by relating the expected duration for a gambler who has a stake z and makes one bet. After that one bet, the gambler's stake has risen to z+1 or fallen to z-1, with probability p or q respectively. Allowing that this first bet adds one to the expected duration of play, we have, for z>0:

$$D(z) = 1 + p\, D(z+1) + q\, D(z-1)$$

We *assume* that the expectation is finite. Using $1 = p + q$, we can re-express $D(z)$ as:

$$D(z) = p.\, D(z) + q.\, D(z)$$

Subtracting, we have

$$p\, (D(z+1) - D(z)) = q\, (D(z) - D(z-1)) - 1$$

(12) $$D(z+1) - D(z) = (q/p)\, (D(z) - D(z-1)) – 1/p = r\, (D(z) - D(z-1)) – 1/p$$

To solve for $D(S)$, we set up the difference equations with the boundary conditions

$$D(M) = D(0) = 0$$

since, if the gambler's fortune rises to M or falls to 0, play stops. They are the zero duration cases.

$$D(2) – D(1) = r\, (D(1) - D(0)) – 1/p = rD(1) – 1/p$$

$$D(3) – D(2) = r\, (D(2) - D(1)) – 1/p$$

...

$D(S) – D(S\text{-}1) = r\,(D(S\text{-}1) - D(S\text{-}2)) – 1/p$

Sequentially substituting terms, we have

(13) $$D(2) - D(1) = rD(1) - \frac{1}{p} = rD(1) - \frac{1}{p}\frac{r-1}{r-1}$$

$$D(3) - D(2) = r^2D(1) - \frac{r}{p} - \frac{1}{p} = r^2D(1) - \frac{1}{p}\frac{r^2-1}{r-1}$$

$$D(4) - D(3) = r^3D(1) - \frac{r^2}{p} - \frac{r}{p} - \frac{1}{p} = r^3D(1) - \frac{1}{p}\frac{r^3-1}{r-1}$$

…

$$D(S) - D(S-1) = r^{S-1}D(1) - \frac{1}{p}\frac{r^{S-1}-1}{r-1}$$

The geometric series summations assume that r does not equal zero. Summing these S-1 equations, we have

$$D(S) - D(1) = D(1)[r + r^2 + \cdots + r^{S-1}]$$
$$-\frac{1}{p}.\frac{1}{r-1}[(r-1) + (r^2-1) + \cdots + (r^{S-1}-1)]$$

That is

(14) $$D(S) = D(1)[1 + r + r^2 + \cdots + r^{S-1}]$$
$$-\frac{1}{p}.\frac{1}{r-1}[(r-1) + (r^2-1) + \cdots + (r^{S-1}-1)]$$

Evaluating the two terms in (14), we have

$$D(1)[1 + r + r^2 + \cdots + r^{S-1}] = D(1)\frac{r^S-1}{r-1}$$

and

$$-\frac{1}{p}.\frac{1}{r-1}[(r-1) + (r^2-1) + \cdots + (r^{S-1}-1)]$$
$$=-\frac{1}{p}.\frac{1}{r-1}[r + r^2 + \cdots + r^{S-1} - (S-1)]$$
$$= -\frac{1}{p}.\frac{1}{r-1}[1 + r + r^2 + \cdots + r^{S-1} - S]$$
$$-\frac{1}{p}.\frac{1}{r-1}\left[\frac{r^S-1}{r-1} - S\right]$$

Combining, (14) becomes

(15) $$D(S) = D(1)\frac{r^S-1}{r-1} - \frac{1}{p}.\frac{1}{r-1}\left[\frac{r^S-1}{r-1} - S\right]$$
$$= \left[D(1) - \frac{1}{p}.\frac{1}{r-1}\right]\frac{r^S-1}{r-1} + \frac{1}{p}.\frac{S}{r-1}$$

To complete the evaluation, we use the boundary condition $D(M) = 0$. Substituting M into (15) gives

$$0 = D(M) = \left[D(1) - \frac{1}{p}.\frac{1}{r-1}\right]\frac{r^M-1}{r-1} + \frac{1}{p}.\frac{M}{r-1}$$

Since this term is zero, we can multiply by $\frac{r^S-1}{r^M-1}$ to recover

$$0 = D(M) = \left[D(1) - \frac{1}{p}.\frac{1}{r-1}\right]\frac{r^S-1}{r-1} + \frac{1}{p}.\frac{M}{r-1}\frac{r^S-1}{r^M-1}$$

Subtracting from (15) we have

$$D(S) = \frac{1}{p}.\frac{S}{r-1} - \frac{1}{p}.\frac{M}{r-1}\frac{r^S-1}{r^M-1} = \frac{S}{q-p} - \frac{M}{q-p}\cdot\frac{r^S-1}{r^M-1}$$

which completes the proof.

Expected Duration of Play in a Fair Game

The derivation proceeds as above, but with the assumption that $p = q = 1/2$ and $r = 1$. Then the difference equations simplify to

$$D(2) - D(1) = D(1) - \frac{1}{p}$$

$$D(3) - D(2) = D(1) - \frac{2}{p}$$

$$D(4) - D(3) = D(1) - \frac{3}{p}$$

…

$$D(S) - D(S-1) = D(1) - \frac{S-1}{p}$$

Summing these equations, we have

$$D(S) - D(1) = (S-1)D(1) - \frac{1}{p}(1 + 2 + \cdots + (S-1))$$

$$= (S-1)D(1) - \frac{1}{p}\frac{S(S-1)}{2}$$

Using p = 1/2 and rearranging, we have

(16) $$D(S) = S\,D(1) - S(S-1)$$

To evaluate $D(1)$, we use boundary condition $D(M) = 0$. Substituting M into (16) we have

$$0 = D(M) = M\,D(1) - M(M-1)$$

from which it follows that

$D(1) = (M\text{-}1)$

Substituting this value for $D(1)$ into (16), we recover

$$D(S) = S(M-1) - S(S-1) = S(M-S)$$

which completed the proof.

Notes

Through calculations not shown here, we can affirm that the expected limiting relations hold between the results for fair and unfair games. That is, the probability of ruin (4) for an unfair game approaches the probability of ruin (3) for a fair game in the limit of r goes to 1. The expected duration of play for an unfair game (6) approaches the expected duration of play for a fair game (5) in the limit of r goes to 1.

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