A HISTORY

OF THE

MATHEMATICAL THEORY OF PROBABILITY

FROM THE TIME OF PASCAL TO THAT OF LAPLACE.

BY

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CHAPTER VII.

JAMES BERNOULLI.

92. We now propose to give an account of the *Ars Conjectandi* of James Bernoulli.

James Bernoulli is the first member of the celebrated family of this name who is associated with the history of Mathematics. He was born 27th December, 1654, and died 16th August, 1705. For a most interesting and valuable account of the whole family we may refer to the essay entitled *Die Mathematiker Bernoulli... von Prof. Dr. Peter Merian*, Basel, 1860.

93. Leibnitz states that at his request James Bernoulli studied the subject. Feu Mr. *Bernoulli* a cultivé cette matière sur mes exhortations; Leibnitii *Opera Omnia, ed. Dutens,* Vol. VI. part 1, page 217. But this statement is not confirmed by the correspondence between Leibnitz and James Bernoulli, to which we have already referred in Art. 59. It appears from this correspondence that James Bernoulli had nearly completed his work before he was aware that Leibnitz had heard any thing about it. Leibnitz says, page 71,

Audio a Te doctrinam de aestimandis probabilitatibus (quam ego magni facio) non parum esse excultam. Vellem aliquis varia ludendi genera (in quibus pulchra hujus doctrinae specimina) mathematice tractaret. Id simul amoenum et utile foret nec Te aut quocunque gravissimo Mathematico indignum.

James Bernoulli in reply says, page 77,

Scire libenter velim, Amplissime Vir, a quo habeas, quod Doctrina de probabilitatibus aestimandis a me excolatur. Verum est me a pluribus retro annis hujusmodi speculationibus magnopere delectari, ut vix putem, quemquam plura super his meditatum esse. Animus etiam erat, Tractatum quendam conscribendi de hac materia; sed saepe per integros annos seposui, quia naturalis meus torpor, quem accessoria valetudinis meae infirmitas immane quantum auxit, facit ut aegerrime ad scribendum accedam; et saepe mihi optarem amanuensem, qui cogitata mea leviter sibi indicata plene divinare, scriptisque consignare posset. Absolvi tamen jam maximam Libri partem, sed deest adhuc praecipua, qua artis conjectandi principia etiam ad civilia, moralia et oeconomia applicare doceo...

James Bernoulli then proceeds to speak of the celebrated theorem which is now called by his name.

Leibnitz in his next letter brings some objections against the theorem; see page 83: and Bernoulli replies; see page 87. Leibnitz returns to the subject; see page 94: and Bernoulli briefly replies, page 97,

Quod Verisimilitudines spectat, et earum augmentum pro aucto scil. observationum numero, res omnino se habet ut scripsi, et certus sum Tibi placituram demonstrationem, cum publicavero.

94. The last letter from James Bernoulli to Leibnitz is dated 3rd June, 1705. It closes in a most painful manner. We here see him, who was perhaps the most famous of all who have borne his famous name, suffering under the combined sorrow arising from illness, from the ingratitude of his brother John who had been his pupil, and from the unjust suspicions of Leibnitz who may be considered to have been his master:

Si rumor vere narrat, redibit certe frater meus Basileam, non tamen Graecam (cum ipse sit ἀναλφάβητος) sed meam potius stationem (quam brevi cum vita me derelicturum, forte non vane, existimat) occupaturus. De iniquis suspicionibus, quibus me immerentem onerasti in Tuis penultimis, alias, ubi plus otii nactus fuero. Nunc vale et fave etc.

95. The *Ars Conjectandi* was not published until eight years after the death of its author. The volume of the *Hist. de l'Acad....Paris* for 1705, published in 1706, contains Fontenelle's *Éloge* of James Bernoulli. Fontenelle here gave a brief notice, derived from Hermann, of the contents of the *Ars Conjectandi* then unpublished. A brief notice is also give in another *Éloge* of

James Bernoulli which appeared in the *Journal des Sçavans* for 1706: this notice is attributed to Saurin by Montmort; see his page IV.

References to the work of James Bernoulli frequently occur in the correspondence between Leibnitz and John Bernoulli; see the work cited in Art. 59, pages 367, 377, 836, 845, 847, 922, 923, 925, 931.

96. The *Ars Conjectandi* was published in 1713. A preface of two pages was supplied by Nicolas Bernoulli, the son of a brother of James and John. It appears from the preface that the fourth part of the work was left unfinished by its author; the publishers had desired that the work should be finished by John Bernoulli, but the numerous engagements of this mathematician had been an obstacle. It was then proposed to devolve the task on Nicolas Bernoulli, who had already turned his attention to the Theory of Probability. But Nicolas Bernoulli did not consider himself adequate to the task; and by his advice the work was finally published in the state in which its author had left it; the words of Nicolas Bernoulli are, Suasor itaque fui, ut Tractatus iste qui maxima ex parte jam impressus erat, in eodem quo eum Auctor reliquit statu cum publico communicaretur.

The *Ars Conjectandi* is not contained in the collected edition of James Bernoulli's works.

97. The *Ars Conjectandi*, including a treatise on infinite series, consists of 306 small quarto pages besides the title leaf and the preface. At the end there is a dissertation in French, entitled *Lettre à un Amy, sur les Parties du Jeu de Paume* which occupies 35 additional pages. Montucla speaks of this letter as the work of an anonymous author; see his page 391: but there can be no doubt that it is due to James Bernoulli, for to him Nicolas Bernoulli assigns it in the preface to the *Ars Conjectandi*, and in his correspondence with Montmort. See *Montmort*, page 333.

98. The *Ars Conjectandi* is divided into four parts. The first part consists of a reprint of the treatise of Huygens *De Ratiociniis in Ludo Aleæ*, accompanied with a commentary by James Bernoulli. The second part is devoted to the theory of permutations and combinations. The third part consists of the solution

of various problems relating to games of chance. The fourth part proposed to apply the Theory of Probability to questions of interest in morals and economical science.

We may observe that instead of the ordinary symbol of equality, =, James Bernoulli uses ∞, which Wallis ascribes to Des Cartes; see Wallis's *Algebra*, 1693, page 138.

99. A French translation of the first part of the *Ars Conjectandi* was published in 1801, under the title of *L'Art de Conjecturer, Traduit du Latin de Jacques Bernoulli; Avec des Observations, Éclaircissemens et Additions. Par L. G. F. Vastel,...* Caen. 1801.

The second part of the *Ars Conjectandi* is included in the volume of reprints which we have cited in Art. 47; Maseres in the same volume gave an English translation of this part.

100. The first part of the *Ars Conjectandi* occupies pages 1—71; with respect to this part we may observe that the commentary by James Bernoulli is of more value than the original treatise by Huygens. The commentary supplies other proofs of the fundamental propositions and other investigations of the problems; also in some cases it extends them. We will notice the most important additions made by James Bernoulli.

101. In the Problem of Points with two players, James Bernoulli gives a table which furnishes the chances of the two players when one of them wants any number of points not exceeding nine, and the other wants any number of points not exceeding seven; and, as he remarks, this table may be prolonged to any extent; see his page 16.

102. James Bernoulli gives a long note on the subject of the various throws which can be made with two or more dice, and the number of cases favourable to each throw. And we may especially remark that he constructs a large table which is equivalent to the theorem we now express thus: the number of ways in which m can be obtained by throwing n dice is equal to the co-efficient of x^m in the development of $(x+x^2+x^3+x^4+x^5+x^6)^n$ in a series of powers of x. See his page 24.

103. The tenth problem is to find in how many trials one may undertake to throw a six with a common die. James Bernoulli gives a note in reply to an objection which he suggests might be urged against the result; the reply is perhaps only intended as a popular illustration: it has been criticized by Prevost in the *Nouveaux Mémoires de l'Acad....Berlin* for 1781.

104. James Bernoulli gives the general expression for the chance of succeeding m times at least in n trials, when the chance of success in a single trial is known. Let the chances of success and failure in a single trial be $\frac{b}{a}$ and $\frac{c}{a}$ respectively: then the required chance consists of the terms of the expansion of $\left(\frac{b}{a}+\frac{c}{a}\right)^n$ from $\left(\frac{b}{a}\right)^n$ to the term which involves $\left(\frac{b}{a}\right)^m\left(\frac{c}{a}\right)^{n-m}$, both inclusive. This formula involves a solution of the Problem of Points for two players of unequal skill; but James Bernoulli does not explicitly make the application.

105. James Bernoulli solves four of the five problems which Huygens had placed at the end of his treatise; the solution of the fourth problem he postpones to the third part of his book as it depends on combinations.

106. Perhaps however the most valuable contribution to the subject which this part of the work contains is a method of solving problems in chances which James Bernoulli speaks of as his own, and which he frequently uses. We will give his solution of the problem which forms the fourteenth proposition of the treatise of Huygens: we have already given the solution of Huygens himself; see Art. 34.

Instead of two players conceive an infinite number of players each of whom is to have one throw in turn. The game is to end as soon as a player whose turn is denoted by an odd number throws a six, or a player whose turn is denoted by an even number throws a seven, and such player is to receive the whole sum at stake. Let b denote the number of ways in which six can be thrown, c the number of ways in which six can fail; so that $b = 5$, and $c = 31$; let e denote the number of ways in which seven can be thrown, and f the number of ways in which seven can fail, so that $e = 6$, and $f = 30$; and let $a = b + c = e + f$.

Now consider the expectations of the different players; they are as follows:

I.	II.	III.	IV.	V.	VI.	VII.	VIII....
$\frac{b}{a}$,	$\frac{ce}{a^2}$,	$\frac{bcf}{a^3}$,	$\frac{c^2ef}{a^4}$,	$\frac{bc^2f^2}{a^5}$,	$\frac{c^3ef^2}{a^6}$,	$\frac{bc^3f^3}{a^7}$,	$\frac{c^4ef^3}{a^8}$....

For it is obvious that $\frac{b}{a}$ expresses the expectation of the first player. In order that the second player may win, the first throw must fail and the second throw must succeed; that is there are ce favourable cases out of a^2 cases, so the expectation is $\frac{ce}{a^2}$. In order that the third player may win, the first throw must fail, the second throw must fail, and the third throw must succeed; that is there are cfb favourable cases out of a^3 cases, so the expectation is $\frac{bcf}{a^3}$. And so on for the other players. Now let a single player, A, be substituted in our mind in the place of the first, third, fifth,...; and a single player, B, in the place of the second, fourth, sixth.... We thus arrive at the problem proposed by Huygens, and the expectations of A and B are given by two infinite geometrical progressions. By summing these progressions we find that A's expectation is $\frac{ab}{a^2 - cf}$, and B's expectation is $\frac{ce}{a^2 - cf}$; the proportion is that of 30 to 31, which agrees with the result in Art. 34.

107. The last of the five problems which Huygens left to be solved is the most remarkable of all; see Art. 35. It is the first example on the *Duration of Play*, a subject which afterwards exercised the highest powers of De Moivre, Lagrange, and Laplace. James Bernoulli solved the problem, and added, without a demonstration, the result for a more general problem of which that of Huygens was a particular case; see *Ars Conjectandi* page 71.

Suppose A to have m counters, and B to have n counters; let their chances of winning in a single game be as a to b; the loser in each game is to give a counter to his adversary: required the chance of each player for winning all the counters of his adversary. In the case taken by Huygens m and n were equal.

It will be convenient to give the modern form of solution of the problem.

Let u_x denote A's chance of winning all his adversary's counters when he has himself x counters. In the next game A must either win or lose a counter; his chances for these two contingencies are $\frac{a}{a+b}$ and $\frac{b}{a+b}$ respectively: and then his chances of winning all his adversary's counters are u_{x+1} and u_{x-1} respectively. Hence

$$u_x = \frac{a}{a+b} u_{x+1} + \frac{b}{a+b} u_{x-1}.$$

This equation is thus obtained in the manner exemplified by Huygens in his fourteenth proposition; see Art. 34.

The equation in Finite Differences may be solved in the ordinary way; thus we shall obtain

$$u_x = C_1 + C_2 \left(\frac{b}{a}\right)^x,$$

where C_1 and C_2 are arbitrary constants. To determine these constants we observe that A's chance is zero when he has no counters, and that it is unity when he has all the counters. Thus u_x is equal to 0 when x is 0, and is equal to 1 when x is $m+n$. Hence we have

$$0 = C_1 + C_2, \quad 1 = C_1 + C_2 \left(\frac{b}{a}\right)^{m+n};$$

therefore
$$C_1 = -C_2 = \frac{a^{m+n}}{a^{m+n} - b^{m+n}}.$$

Hence
$$u_x = \frac{a^{m+n} - a^{m+n-x} b^x}{a^{m+n} - b^{m+n}}.$$

To determine A's chance at the beginning of the game we must put $x = m$; thus we obtain

$$u_m = \frac{a^n (a^m - b^m)}{a^{m+n} - b^{m+n}}.$$

In precisely the same manner we may find *B*'s chance at any stage of the game; and his chance at the beginning of the game will be

$$\frac{b^m(a^n-b^n)}{a^{m+n}-b^{m+n}}.$$

It will be observed that the sum of the chances of *A* and *B* at the beginning of the game is *unity*. The interpretation of this result is that one or other of the players must eventually win all the counters; that is, the play must terminate. This might have been expected, but was not assumed in the investigation.

The formula which James Bernoulli here gives will next come before us in the correspondence between Nicolas Bernoulli and Montmort; it was however first published by De Moivre in his *De Mensura Sortis*, Problem IX., where it is also demonstrated.

108. We may observe that Bernoulli seems to have found, as most who have studied the subject of chances have also found, that it was extremely easy to fall into mistakes, especially by attempting to reason without strict calculation. Thus, on his page 15, he points out a mistake into which it would have been easy to fall, *nisi nos calculus aliud docuisset.* He adds,

Quo ipso proin monemur, ut cauti simus in judicando, nec ratiocinia nostra super quâcunque statim analogiâ in rebus deprehensâ fundare suescamus; quod ipsum tamen etiam ab iis, qui vel maximè sapere videntur, nimis frequenter fieri solet.

Again, on his page 27,

Quæ quidem eum in finem hîc adduco, ut palàm fiat, quàm parùm fidendum sit ejusmodi ratiociniis, quæ corticem tantùm attingunt, nec in ipsam rei naturam altiùs penetrant; tametsi in toto vitæ usu etiam apud sapientissimos quosque nihil sit frequentius.

Again, on his page 29, he refers to the difficulty which Pascal says had been felt by M. de * * * *, whom James Bernoulli calls Anonymus quidam cæterà subacti judicii Vir, sed Geometriæ expers. James Bernoulli adds,

Hâc enim qui imbuti sunt, ejusmodi ἐναντιοφανείαι minimè morantur, probè conscii dari innumera, quæ admoto calculo aliter se habere comperiuntur, quàm initio apparebant; ideoque sedulò cavent, juxtà id quod semel iterumque monui, ne quicquam analogiis temerè tribuant.

109. The second part of the *Ars Conjectandi* occupies pages 72—137: it contains the doctrine of Permutations and Combinations. James Bernoulli says that others have treated this subject before him, and especially Schooten, Leibnitz, Wallis and Prestet; and so he intimates that his matter is not entirely new. He continues thus, page 73,

...tametsi quædam non contemnenda de nostro adjecimus, inprimis demonstrationem generalem et facilem proprietatis numerorum figuratorum, cui cætera pleraque innituntur, et quam nemo quod sciam ante nos dedit eruitve.

110. James Bernoulli begins by treating on permutations; he proves the ordinary rule for finding the number of permutations of a set of things taken all together, when there are no repetitions among the set of things and also when there are. He gives a full analysis of the number of arrangements of the verse Tot tibi sunt dotes, Virgo, quot sidera cœli; see Art. 40. He then considers combinations; and first he finds the total number of ways in which a set of things can be taken, by taking them one at a time, two at a time, three at a time,...He then proceeds to find what we should call the number of combinations of n things taken r at a time; and here is the part of the subject in which he added most to the results obtained by his predecessors. He gives a figure which is substantially the same as Pascal's *Arithmetical Triangle;* and he arrives at two results, one of which is the well-known form for the nth term of the rth order of figurate numbers, and the other is the formula for the sum of a given number of terms of the series of figurate numbers of a given order; these results are expressed definitely in the modern notation as we now have them in works on Algebra. The mode of proof is more laborious, as might be expected. Pascal as we have seen in Arts. 22 and 41, employed without any scruple, and indeed rather with approbation, the method of induction: James Bernoulli however says, page 95,... modus demonstrandi per inductionem parùm scientificus est.

James Bernoulli names his predecessors in investigations on figurate numbers in the following terms on his page 95:

Multi, ut hoc in transitu notemus, numerorum figuratorum contemplationibus vacârant (quos inter Faulhaberus et Remmelini Ulmenses, Wallisius, Mercator in Logarithmotechniâ, Prestetus, aliique)...

111. We may notice that James Bernoulli gives incidentally on his page 89 a demonstration of the Binomial Theorem for the case of a positive integral exponent. Maseres considers this to be the first demonstration that appeared; see page 233 of the work cited in Art. 47.

112. From the summation of a series of figurate numbers James Bernoulli proceeds to derive the summation of the powers of the natural numbers. He exhibits definitely Σn, Σn^2, Σn^3,... up to Σn^{10}; he uses the symbol $\int$ where we in modern books use Σ. He then extends his results by induction without demonstration, and introduces for the first time into Analysis the coefficients since so famous as the *numbers of Bernoulli*. His general formula is that

$$\Sigma n = \frac{n^{c+1}}{c+1} + \frac{n}{2} + \frac{c}{2} A n^{c-1} + \frac{c(c-1)(c-2)}{2.3.4} B n^{c-3}$$

$$+ \frac{c(c-1)(c-2)(c-3)(c-4)}{2.3.4.5.6} C n^{c-5}$$

$$+ \frac{c(c-1)(c-2)(c-3)(c-4)(c-5)(c-6)}{2.3.4.5.6.7.8} D n^{c-7} + \dots$$

where $$A = \frac{1}{6},\ B = -\frac{1}{30},\ C = \frac{1}{42},\ D = -\frac{1}{30},\ \dots$$

He gives the numerical value of the sum of the tenth powers of the first thousand natural numbers; the result is a number with thirty-two figures. He adds, on his page 98,

E quibus apparet, quàm inutilis censenda sit opera Ismaëlis Bullialdi, quam conscribendo tam spisso volumini Arithmeticæ suæ Infinitorum impendit, ubi nihil præstitit aliud, quàm ut primarum tantum sex potestatum summas (partem ejus quod unicâ nos consecuti sumus paginâ) immenso labore demonstratas exhiberet.

For some account of Bulliald's *spissum volumen*, see Wallis's *Algebra*, Chap. LXXX.

113. James Bernoulli gives in his fourth Chapter the rule now well known for the number of the combinations of n things

taken c at a time. He also draws various simple inferences from the rule. He digresses from the subject of this part of his book to resume the discussion of the Problem of Points; see his page 107. He gives two methods of treating the problem by the aid of the theory of combinations. The first method shews how the table which he had exhibited in the first part of the *Ars Conjectandi* might be continued and the law of its terms expressed; the table is a statement of the chances of A and B for winning the game when each of them wants an assigned number of points. Pascal had himself given such a table for a game of six points; an extension of the table is given on page 16 of the *Ars Conjectandi*, and now James Bernoulli investigates general expressions for the component numbers of the table. From his investigation he derives the result which Pascal gave for the case in which one player wants one point more than the other player. James Bernoulli concludes this investigation thus; Ipsa solutio Pascaliana, quæ Auctori suo tantopere arrisit.

James Bernoulli's other solution of the Problem of Points is much more simple and direct, for here he *does* make the application to which we alluded in Art. 104. Suppose that A wants m points and B wants n points; then the game will certainly be decided in $m+n-1$ trials. As in each trial A and B have equal chances of success the whole number of possible cases is 2^{m+n-1}. And A wins the game if B gains no point, or if B gains just one point, or just two points,... or any number up to $n-1$ inclusive. Thus the number of cases favourable to A is

$$1+\mu+\frac{\mu(\mu-1)}{2}+\frac{\mu(\mu-1)(\mu-2)}{\underline{|3}}+\ldots+\frac{\mu(\mu-1)\ldots(\mu-n+2)}{\underline{|n-1}},$$

where $\mu=m+n-1$.

Pascal had in effect advanced as far as this; see Art. 23: but the formula is more convenient than the *Arithmetical Triangle*.

114. In his fifth Chapter James Bernoulli considers another question of combinations, namely that which in modern treatises is enunciated thus: to find the number of homogeneous products of the r^{th} degree which can be formed of n symbols. In his sixth Chapter he continues this subject, and makes a slight reference to

the doctrine of the number of divisors of a given number; for more information he refers to the works of Schooten and Wallis, which we have already examined; see Arts. 42, 47.

115. In his seventh Chapter James Bernoulli gives the formula for what we now call the number of permutations of n things taken c at a time. In the remainder of this part of his book he discusses some other questions relating to permutations and combinations, and illustrates his theory by examples.

116. The third part of the *Ars Conjectandi* occupies pages 138—209; it consists of twenty-four problems which are to illustrate the theory that has gone before in the book. James Bernoulli gives only a few lines of introduction, and then proceeds to the problems, which he says,

...nullo ferè habito selectu, prout in adversariis reperi, proponam, præmissis etiam vel interspersis nonnullis facilioribus, et in quibus nullus combinationum usus apparet.

117. The fourteenth problem deserves some notice. There are two cases in it, but it will be sufficient to consider one of them. A is to throw a die, and then to repeat his throw as many times as the number thrown the first time. A is to have the whole stake if the sum of the numbers given by the latter set of throws exceeds 12; he is to have half the stake if the sum is equal to 12; and he is to have nothing if the sum is less than 12. Required the value of his expectation. It is found to be $\frac{15295}{31104}$, which is rather less than $\frac{1}{2}$. After giving the correct solution James Bernoulli gives another which is plausible but false, in order, as he says, to impress on his readers the necessity of caution in these discussions. The following is the false solution. A has a chance equal to $\frac{1}{6}$ of throwing an ace at his first trial; in this case he has only one throw for the stake, and that throw may give him with equal probability any number between 1 and 6 inclusive, so that we may take $\frac{1}{6}(1+2+3+4+5+6)$, that is $3\frac{1}{2}$, for his mean throw. We may observe that $3\frac{1}{2}$ is the Arith-

metical mean between 1 and 6. Again A has a chance equal to $\frac{1}{6}$ of throwing a two at his first trial; in this case he has two throws for the stake, and these two throws may give him any number between 2 and 12 inclusive; and the probability of the number 2 is the same as that of 12, the probability of 3 is the same as that of 11, and so on; hence as before we may take $\frac{1}{2}(2+12)$, that is 7, for his mean throw. In a similar way if three, four, five, or six be thrown at the first trial, the corresponding means of the numbers in the throws for the stake will be respectively $10\frac{1}{2}$, 14, $17\frac{1}{2}$, and 21. Hence the mean of all the numbers is

$$\frac{1}{6}\{3\tfrac{1}{2}+7+10\tfrac{1}{2}+14+17\tfrac{1}{2}+21\},\text{ that is } 12\tfrac{1}{4};$$

and as this number is greater than 12 it might appear that the odds are in favour of A.

A false solution of a problem will generally appear more plausible to a person who has originally been deceived by it than to another person who has not seen it until after he has studied the accurate solution. To some persons James Bernoulli's false solution would appear simply false and not plausible; it leaves the problem proposed and substitutes another which is entirely different. This may be easily seen by taking a simple example. Suppose that A instead of an equal chance for any number of throws between one and six inclusive, is restricted to one *or* six throws, and that each of these two cases is equally likely. Then, as before, we may take $\frac{1}{2}\{3\frac{1}{2}+21\}$, that is $12\frac{1}{4}$ as the mean throw. But it is obvious that the odds are against him; for if he has only one throw he cannot obtain 12, and if he has six throws he will not necessarily obtain 12. The question is not *what is the mean number* he will obtain, but *how many throws* will give him 12 or more, and *how many* will give him less than 12.

James Bernoulli seems not to have been able to make out more than that the second solution must be false because the first is unassailable; for after saying that from the second solution we might suppose the odds to be in favour of A, he adds, Hujus autem contrarium ex priore solutione, quæ sua luce radiat, apparet; ...

The problem has been since considered by Mallet and by Fuss, who agree with James Bernoulli in admitting the plausibility of the false solution.

118. James Bernoulli examines in detail some of the games of chance which were popular in his day. Thus on pages 167 and 168 he takes the game called *Cinq et neuf*. He takes on pages 169—174 a game which had been brought to his notice by a stroller at fairs. According to James Bernoulli the chances were against the stroller, and so as he says, istumque proin hoc aleæ genere, ni præmia minuat, non multum lucrari posse. We might desire to know more of the stroller who thus supplied the occasion of an elaborate discussion to James Bernoulli, and who offered to the public the amusement of gambling on terms unfavourable to himself.

James Bernoulli then proceeds to a game called *Trijaques*. He considers that, it is of great importance for a player to maintain a serene composure even if the cards are unfavourable, and that a previous calculation of the chances of the game will assist in securing the requisite command of countenance and temper. As James Bernoulli speaks immediately afterwards of what he had himself formerly often observed in the game, we may perhaps infer that *Trijaques* had once been a favourite amusement with him.

119. The nineteenth problem is thus enunciated,

In quolibet Aleæ genere, si ludi Oeconomus seu Dispensator (*le Banquier du Jeu*) nonnihil habeat prærogativæ in eo consistentis, ut paulo major sit casuum numerus quibus vincit quàm quibus perdit; et major simul casuum numerus, quibus in officio Oeconomi pro ludo sequenti confirmatur, quàm quibus œconomia in collusorem transfertur. Quæritur, quanti privilegium hoc Oeconomi sit æstimandum?

The problem is chiefly remarkable from the fact that James Bernoulli candidly records two false solutions which occurred to him before he obtained the true solution.

120. The twenty-first problem relates to the game of *Bassette*;

James Bernoulli devotes eight pages to it, his object being to estimate the advantage of the banker at the game. See Art. 74.

The last three problems which James Bernoulli discusses arose from his observing that a certain stroller, in order to entice persons to play with him, offered them among the conditions of the game one which was apparently to their advantage, but which on investigation was shewn to be really pernicious; see his pages 208, 209.

121. The fourth part of the *Ars Conjectandi* occupies pages 210—239; it is entitled *Pars Quarta, tradens usum et applicationem præcedentis Doctrinæ in Civilibus, Moralibus et Oeconomicis.* It was unfortunately left incomplete by the author; but nevertheless it may be considered the most important part of the whole work. It is divided into five Chapters, of which we will give the titles.

I. *Præliminaria quædam de Certitudine, Probabilitate, Necessitate, et Contingentia Rerum.*

II. *De Scientia et Conjectura. De Arte Conjectandi. De Argumentis Conjecturarum. Axiomata quædam generalia huc pertinentia.*

III. *De variis argumentorum generibus, et quomodo eorum pondera æstimentur ad supputandas rerum probabilitates.*

IV. *De duplici Modo investigandi numeros casuum. Quid sentiendum de illo, qui instituitur per experimenta. Problema singulare eam in rem propositum, &c.*

V. *Solutio Problematis præcedentis.*

122. We will briefly notice the results of James Bernoulli as to the probability of arguments. He distinguishes arguments into two kinds, *pure* and *mixed.* He says, *Pura* voco, quæ in quibusdam casibus ita rem probant, ut in aliis nihil positivè probent: *Mixta,* quæ ita rem probant in casibus nonnullis, ut in cæteris probent contrarium rei.

Suppose now we have three arguments of the *pure* kind leading to the same conclusion; let their respective probabilities be

$1-\frac{c}{a}$, $1-\frac{f}{d}$, $1-\frac{i}{g}$. Then the resulting probability of the conclusion is $1-\frac{cfi}{adg}$. This is obvious from the consideration that any one of the arguments would establish the conclusion, so that the conclusion fails only when all the arguments fail.

Suppose now that we have in addition two arguments of the *mixed* kind: let their respective probabilities be $\frac{q}{q+r}$, $\frac{t}{t+u}$. Then James Bernoulli gives for the resulting probability

$$1-\frac{cfiru}{adg\,(ru+qt)}.$$

But this formula is inaccurate. For the supposition $q=0$ amounts to having one argument *absolutely decisive against* the conclusion, while yet the formula leaves still a certain probability *for* the conclusion. The error was pointed out by Lambert; see Prevost and Lhuilier, *Mémoires de l'Acad....Berlin* for 1797.

123. The most remarkable subject contained in the fourth part of the *Ars Conjectandi* is the enunciation and investigation of what we now call *Bernoulli's Theorem.* It is introduced in terms which shew a high opinion of its importance:

Hoc igitur est illud Problema, quod evulgandum hoc loco proposui, postquam jam per vicennium pressi, et cujus tum novitas, tum summa utilitas cum pari conjuncta difficultate omnibus reliquis hujus doctrinæ capitibus pondus et pretium superaddere potest. *Ars Conjectandi,* page 227. See also De Moivre's *Doctrine of Chances,* page 254.

We will now state the purely algebraical part of the theorem. Suppose that $(r+s)^{nt}$ is expanded by the Binomial Theorem, the letters all denoting integral numbers and t being equal to $r+s$. Let u denote the sum of the greatest term and the n preceding terms and the n following terms. Then by taking n large enough the ratio of u to the sum of all the remaining terms of the expansion may be made as great as we please.

If we wish that this ratio should not be less than c it will be sufficient to take n equal to the greater of the two following expressions,

$$\frac{\log c + \log (s-1)}{\log (r+1) - \log r}\left(1+\frac{s}{r+1}\right) - \frac{s}{r+1},$$

and
$$\frac{\log c + \log (r-1)}{\log (s+1) - \log s}\left(1+\frac{r}{s+1}\right) - \frac{r}{s+1}.$$

James Bernoulli's demonstration of this result is long but perfectly satisfactory; it rests mainly on the fact that the terms in the Binomial series increase continuously up to the greatest term, and then decrease continuously. We shall see as we proceed with the history of our subject that James Bernoulli's demonstration is now superseded by the use of Stirling's Theorem.

124. Let us now take the application of the algebraical result to the Theory of Probability. The greatest term of $(r+s)^{nt}$, where $t=r+s$ is the term involving $r^{nr}s^{ns}$. Let r and s be proportional to the probability of the happening and failing of an event in a single trial. Then the sum of the $2n+1$ terms of $(r+s)^{nt}$ which have the greatest term for their middle term corresponds to the probability that in nt trials the number of times the event happens will lie between $n(r-1)$ and $n(r+1)$, both inclusive; so that the ratio of the number of times the event happens to the whole number of trials lies between $\frac{r+1}{t}$ and $\frac{r-1}{t}$. Then, by taking for n the greater of the two expressions in the preceding article, we have the odds of c to 1, that the ratio of the number of times the event happens to the whole number of trials lies between $\frac{r+1}{t}$ and $\frac{r-1}{t}$.

As an example James Bernoulli takes

$$r=30, \quad s=20, \quad t=50.$$

He finds for the odds to be 1000 to 1 that the ratio of the number of times the event happens to the whole number of trials shall lie between $\frac{31}{50}$ and $\frac{29}{50}$, it will be sufficient to make 25550 trials; for the odds to be 10000 to 1, it will be sufficient to make 31258 trials; for the odds to be 100000 to 1, it will be sufficient to make 36966 trials; and so on.

125. Suppose then that we have an urn containing white balls and black balls, and that the ratio of the number of the former to the latter is *known to be* that of 3 to 2. We learn from the preceding result that if we make 25550 drawings of a single ball, replacing each ball after it is drawn, the odds are 1000 to 1 that the white balls drawn lie between $\frac{31}{50}$ and $\frac{29}{50}$ of the whole number drawn. This is the *direct* use of James Bernoulli's theorem. But he himself proposed to employ it *inversely* in a far more important way. Suppose that in the preceding illustration we do not know anything beforehand of the ratio of the white balls to the black; but that we have made a large number of drawings, and have obtained a white ball R times, and a black ball S times: then according to James Bernoulli we are to infer that the ratio of the white balls to the black balls in the urn is approximately $\frac{R}{S}$. To determine the precise numerical estimate of the probability of this inference requires further investigation: we shall find as we proceed that this has been done in two ways, by an inversion of James Bernoulli's theorem, or by the aid of another theorem called Bayes's theorem; the results approximately agree. See Laplace, *Théorie...des Prob....* pages 282 and 366.

126. We have spoken of the *inverse* use of James Bernoulli's theorem as the most important; and it is clear that he himself was fully aware of this. This use of the theorem was that which Leibnitz found it difficult to admit, and which James Bernoulli maintained against him; see the correspondence quoted in Art. 59, pages 77, 83, 87, 94, 97.

127. A memoir on infinite series follows the *Ars Conjectandi*, and occupies pages 241—306 of the volume; this is contained in the collected edition of James Bernoulli's works, Geneva, 1744: it is there broken up into parts and distributed through the two volumes of which the edition consists.

This memoir is unconnected with our subject, and we will therefore only briefly notice some points of interest which it presents.

128. James Bernoulli enforces the importance of the subject in the following terms, page 243,

Cæterum quantæ sit necessitatis pariter et utilitatis hæc serierum contemplatio, ei sane ignotum esse non poterit, qui perspectum habuerit, ejusmodi series sacram quasi esse anchoram, ad quam in maxime arduis et desperatæ solutionis Problematibus, ubi omnes alias humani ingenii vires naufragium passæ, velut ultimi remedii loco confugiendum est.

129. The principal artifice employed by James Bernoulli in this memoir is that of subtracting one series from another, thus obtaining a third series. For example,

let $$S = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n+1},$$

then $$S = \quad 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \frac{1}{n+1};$$

therefore $$0 = -1 + \frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} + \frac{1}{3\,.\,4} + \ldots + \frac{1}{n(n+1)} + \frac{1}{n+1},$$

therefore $$\frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} + \frac{1}{3\,.\,4} + \ldots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}.$$

Thus the sum of n terms of the series, of which the r^{th} term is $\frac{1}{r(r+1)}$, is $\frac{n}{n+1}$.

130. James Bernoulli says that his brother first observed that the sum of the infinite series $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is infinite; and he gives his brother's demonstration and his own; see his page 250.

131. James Bernoulli shews that the sum of the infinite series $\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ is finite, but confesses himself unable to give the sum. He says, page 254, Si quis inveniat nobisque communicet, quod industriam nostram elusit hactenus, magnas de nobis gratias feret. The sum is now known to be $\frac{\pi^2}{6}$; this result is due to Euler: it is given in his *Introductio in Analysin Infinitorum*, 1748, Vol. I. page 130.

132. James Bernoulli seems to be on more familiar terms with infinity than mathematicians of the present day. On his page 262 we find him stating, correctly, that the sum of the infinite series $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\ldots$ is infinite, for the series is greater than $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ He adds that the sum of all the odd terms of the first series is to the sum of all the even terms as $\sqrt{2}-1$ is to 1; so that the sum of the odd terms would appear to be *less* than the sum of the even terms, which is impossible. But the paradox does not disturb James Bernoulli, for he adds,

...cujus ἐναντιοφανείας rationem, etsi ex infiniti natura finito intellectui comprehendi non posse videatur, nos tamen satis perspectam habemus.

133. At the end of the volume containing the *Ars Conjectandi* we have the *Lettre à un Amy, sur les Parties du Jeu de Paume,* to which we have alluded in Art. 97.

The nature of the problem discussed may be thus stated. Suppose A and B two players; let them play a set of games, say five, that is to say, the player gains the set who first wins five games. Then a certain number of sets, say four, make a match. It is required to estimate the chances of A and B in various states of the contest. Suppose for example that A has won two sets, and B has won one set; and that in the set now current A has won two games and B has won one game. The problem is thus somewhat similar in character to the Problem of Points, but more complicated. James Bernoulli discusses it very fully, and presents his result in the form of tables. He considers the case in which the players are of unequal skill; and he solves various problems arising from particular circumstances connected with the game of tennis to which the letter is specially devoted.

On the second page of the letter is a very distinct statement of the use of the celebrated theorem known by the name of Bernoulli; see Art. 123.

134. One problem occurs in this *Lettre à un Amy*... which it may be interesting to notice.

Suppose that A and B engage in play, and that each in turn by the laws of the game has an advantage over his antagonist. Thus suppose that A's chance of winning in the 1st, 3rd, 5th... games is always p, and his chance of losing q; and in the 2nd, 4th, 6th... games suppose that A's chance of winning is q and his chance of losing p. The chance of B is found by taking that of A from unity; so that B's chance is p or q according as A's is q or p.

Now let A and B play, and suppose that the stake is to be assigned to the player who first wins n games. There is however to be this peculiarity in their contest: If each of them obtains $n-1$ games it will be necessary for one of them to win two games in succession to decide the contest in his favour; if each of them wins one of the next two games, so that each has scored n games, the same law is to hold, namely, that one must win two games in succession to decide the contest in his favour; and so on.

Let us now suppose that $n=2$, and estimate the advantage of A. Let x denote this advantage, S the whole sum to be gained.

Now A may win the first and second games; his chance for this is pq, and then he receives S. He may win the first game, and lose the second; his chance for this is p^2. He may lose the first game and win the second; his chance for this is q^2. In the last two cases his position is neither better nor worse than at first; that is he may be said to receive x.

Thus $$x = pq\,S + (p^2+q^2)\,x;$$

therefore $$x = \frac{pq\,S}{1-p^2-q^2} = \frac{pq\,S}{2pq} = \frac{S}{2}.$$

Hence of course B's advantage is also $\frac{S}{2}$. Thus the players are on an equal footing.

James Bernoulli in his way obtains this result. He says that *whatever* may be the value of n, the players are on an equal footing; he verifies the statement by calculating numerically the chances for $n=2$, 3, 4 or 5, taking $p=2q$. See his pages 18, 19.

Perhaps the following remarks may be sufficient to shew that whatever n may be, the players must be on an equal footing. By the peculiar law of the game which we have explained, it follows that the contest is not decided until one player has gained at least n games, and is at least two games in advance of his adversary.

Thus the contest is either decided in an *even* number of games, or else in an odd number of games in which the victor is at least three games in advance of his adversary: in the last case no advantage or disadvantage will accrue to either player if they play one more game and count it in. Thus the contest may be conducted without any change of probabilities under the following laws: the number of games shall be *even*, and the victor gain not less than n and be at least two in advance of his adversary. But since the number of games is to be *even* we see that the two players are on an equal footing.

135. Gouraud has given the following summary of the merits of the *Ars Conjectandi*; see his page 28:

> Tel est ce livre de l'*Ars conjectandi*, livre qui, si l'on considère le temps où il fut composé, l'originalité, l'étendue et la pénétration d'esprit qu'y montra son auteur, la fécondité étonnante de la constitution scientifique qu'il donna au Calcul des probabilités, l'influence enfin qu'il devait exercer sur deux siècles d'analyse, pourra sans exagération être regardé comme un des monuments les plus importants de l'histoire des mathématiques. Il a placé à jamais le nom de Jacques Bernoulli parmi les noms de ces inventeurs, à qui la postérité reconnaissante reporte toujours et à bon droit, le plus pur mérite des découvertes, que sans leur premier effort, elle n'aurait jamais su faire.

This panegyric, however, seems to neglect the simple fact of the date of *publication* of the *Ars Conjectandi*, which was really subsequent to the first appearance of Montmort and De Moivre in this field of mathematical investigation. The researches of James Bernoulli were doubtless the earlier in existence, but they were the later in appearance before the world; and thus the influence which they might have exercised had been already produced. The problems in the first three parts of the *Ars Conjectandi* cannot be considered equal in importance or difficulty to those which we find investigated by Montmort and De Moivre; but the memorable theorem in the fourth part, which justly bears its author's name, will ensure him a permanent place in the history of the Theory of Probability.