SECTION 2

THOUGHT EXPERIMENTS IN LOGIC AND MATHEMATICS

THOUGHT EXPERIMENTS AND CONCEIVABILITY CONDITIONS IN MATHEMATICS

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I

THE issues one is faced with in dealing with the problem of thought experiments in a particular discipline can be separated into, at least, the following four categories: (a) The first includes issues related to the examination of what one would consider as specific examples of thought experiments in the area, which could be gathered and classified as such on a tentative basis. That is, gathered in a way which fits one's intuition of what could or should count as a thought experiment in the area. Such a gathering does not have to take place with, as its necessary presupposition, a preconceived theoretical model in mind of what sort of pattern or patterns thought experiments should follow. It is enough for the scientist or the philosopher who makes the attempt towards such a gathering to have a minimal and not necessarily very precise idea of what a thought experiment is. An essential characteristic of such an idea should, of course, be that a thought experiment is an exploratory ideal process meant to answer a theoretical or metatheoretical question in the general framework of the given discipline and carried out according to rules specified by logic and the particularities of the discipline itself. Such a process does not have to have an exactly isomorphic counterpart in the area of what one would call "genuine (laboratory-like) experiments" for at least two reasons. First, in the case of natural sciences (e.g., physics) a thought experiment most of the time involves a non-exactly imitable idealization of the actual conditions under which a certain phenomenon or a set of phenomena are taking place. Second, in the case of mathematics there are no such counterparts at all. Facing the danger of being called a dogmatist one could insist that the only "genuine experiments" in mathematics are the thought experiments. (b) The second category of issues one is faced with in dealing with the problem of thought experiments in a particular area is of a classificatory nature. One is faced, that is, with issues related to the classification of the gathered cases into groups according to common characteristics that some have and others don't. As a by-product of such a classification one could attempt to define or analytically describe what could count as a thought experiment in the particular field. A classification of this sort could again take place without a preconceived theoretical idea of how it should be carried out. It is more fruitful to start from the gathered material and be led by it to general classificatory schemata than the other way round, having in mind, of course, the obvious truism that such a process is never absolutely free or independent of one's already acquired conceptual background. That is, we never start from scratch and therefore our classificatory attempts are always in a sense biased. Yet it is correct to say that, starting from the gathered material and being led mainly and consciously by it to general classificatory schemata, we at least try to follow the path of the empiricist doing our best to avoid the sometimes overbiased rationalistic tendencies and the pitfalls to which these tendencies lead us whenever we attempt to move from the general to the particular. (c) The third category includes issues having to do with the specification of similarities and differences between thought experiments as they function in the theoretical framework of various scientific fields. A specification of this sort could be the product of a research program[1] conceived as such with the, quite possibly, unattainable goal in mind of forming a general theory of thought experiments which would cover experimentation of such a kind across the scientific board. (d) The fourth category deals with issues related to the conceivability conditions or constraints endemic in the specific area as they influence the practicing of thought experiments in it. Constraints of this sort play an important role especially whenever some kind of choice is involved concerning alternative paths which could be followed in the process of thought experimentation. Additionally "why" questions concerning, e.g., the choice of a conceptual framework over alternative ones could be legitimately asked and, may be, satisfactorily answered only in the light of a prior analysis of what would count as a conceivability condition or constraint and of a specification of such conditions as they refer to particular examples of thought experiments.

The aim of this paper is to examine some of the issues mentioned above in the case of thought experimentation in mathematics. The objective is to propose an open to further development classification of thought experiments in mathematics and to examine and classify the conceivability conditions related to them. Such classifications can be thought of as first approximations towards more general taxonomies concerning, first, the activity of producing or creating mathematics as a whole and, second, the conceivability conditions or constraints endemic in the process of such a production or creation[2].

II

I will start with the classification of thought experiments in mathematics that I have in mind. The proposed scheme consists of six different groups. Some of these groups overlap, but this is not necessarily a negative aspect of the proposed classification.

1. The first group includes the thought experiments usually performed in a conceptual framework which has not been yet theoretically solidified. Such experiments are taking place for the purpose of answering specific questions (usually posed in the form of a conjecture) as to whether something is the case or not. In the process of answering such a question, which process could include the attempts towards an answer by more than one individual and during more than one generation of mathematicians, the initial conjecture sometimes gets transformed into a new conjecture. The latter is usually better than the former, at least in the sense that the conceptual grid, upon which the latter is based, is better worked out and the boundaries of the domain of its application are less fuzzy than the ones of the domain of application of the former. The fact that the conceptual framework has not yet been theoretically fixed makes the history of the transformations of the initial conjecture quite interesting because such a history is usually immediately connected to the history of the successive transformations of this same conceptual framework towards the ideal of a theoretical fixation or solidification. It makes also the problem-solving activity relatively free from the strictures that a fixed theoretical framework would impose on it. A good grasp of the history of the transformations of an initial mathematical question during a certain period of time can help us to appreciate the role of informal mathematics and of its growth in the overall development of mathematics. Such transformations are the result of thought experiments performed, with the purpose in mind of answering the initial question.

We could admit that the usual process involved in the growth of informal mathematics follows more or less the pattern described by Lakatos in his well-known "Proofs and Refutations."[3] The element of thought experimentation is presented by him as the protagonist in the rational reconstruction he offers of the history in a distilled form of the attempts of the mathematical community to answer the question concerning the provability of the well-known formula relating the number of vertices, the number of edges and the number of faces of a regular polyhedron. According to Lakatos the process described in the case he chose to present can be generalized. That is, given an initial informal mathematical conjecture, one could look for a final proof experimenting with various alternative routes and performing the necessary proof-analysis on them. Such analysis, in the process of performing it, could possibly lead to the transformation of the initial conjecture or to the concretization of it by making it, that is, more rigorous and more conceptually legitimate to be asked via, for instance, appropriate definitional modifications of the involved concepts. Similarly one could look (depending always upon his intuitive predisposition towards the specific question he is supposed to answer) for a refutation of the conjecture via the construction of a counterexample or a reductio ad absurdum proof.

2. The second group includes the thought experiments in mathematics performed in a conceptual framework provided by a fixed mathematical theory. The sort of questions which are supposed to be answered in this case differ in a certain definite sense from the questions which trigger thought experiments classified in the first group. They are questions well-posed in the strict linguistic framework provided by the fixed theory. The interplay between posing such questions and answering them does not involve substantial alterations of the initial questions. New questions could arise or the old questions could split into subordinate ones but such a process is not oriented towards theory-building. The theoretical framework is given and therefore conceptual alterations are restricted to a bare minimum.

It is the case, of course, that sometimes a positive or negative answer to a well-posed question in the framework of a fixed mathematical theory can be attained only if the framework itself is altered. We have an example of such a case if we consider any undecidable well-formed-formula written in the first-order language of a consistent mathematical theory. The Axiom of Choice, for instance, is an undecidable well-formed-formula in the language of the Zermelo-Fraenkel set theory. Yet, the undecidability of the Axiom of Choice is a definite answer, though neither positive[4] nor negative[5], to the question: which one, the Axiom of Choice or its negation is provable from the axioms of the Zermelo-Fraenkel set theory?

Faced with an open question or conjecture in the fixed framework of a mathematical theory we are invited to explore alternative routes which could lead, in a mutually exclusive way, to one of the following results: (i) A positive answer in the sense of producing a proof of what is conjectured. (ii) A negative answer in the form of a proof of the negation of what is conjectured. Such a proof could sometimes (especially whenever what is conjectured is of a universal nature) take the form of the construction of an appropriate counterexample. (iii) An answer which is neither positive nor negative in the above sense, namely an answer which would establish the undecidability of what is conjectured in the framework of the given theory. Such an alternative is usually established by model-theoretic argumentation which does not take place in the framework of the given theory but at a metatheoretical level.

The element of experimentation in the process of establishing which of the (i), (ii) and (iii) is the case, consists in the possibility of following alternative lines of thought, for which we do not know at the time we choose to follow them whether they are going to lead us to an answer or not. That is not to say that we are faced with some kind of random choice whenever we attempt to find a solution to an open mathematical problem. The working mathematician, most of the time, prefers certain alternatives over others for reasons related to his intuitive predisposition towards them. It is the case, of course, that the existence of such predispositions does not alter the fact that following a certain line of thought we never know *a priori* whether or not it will lead us to the desired results. The mathematician, that is, led by the desire to answer a given open question in the framework of a fixed mathematical theory (no matter whether or not he is intuitively predisposed towards a certain line of thought) is, in any case, performing a thought experiment the final success of which is not by any means guaranteed to him at the beginning of the experimentation.

3. The third group includes thought experiments in mathematics performed fervently during and immediately after a foundational crisis. The overall activity during such periods is mainly centered around the construction of a new conceptual framework wherein the source of crisis in the old framework is hoped to be tamed. Usually the taming of the source of the foundational crisis has as its by-product the transformation of the source into a fountain of exceptional conceptual enrichment and creative innovation at the technical and methodological levels. What is important is such cases is that thought experiments of this sort are spurred not by an initial conjecture or a question posed in the framework of a given theory and asked to be answered during the normal exploratory periods of mathematical research but by a more or less sudden discovery of a troublesome and, from the point of view of the old framework, pathological counterexample. Sometimes more than one alternative new conceptual frameworks are proposed. Which one is going to win in the long run is difficult to assess at the very beginning of their formation.

Such activity during and immediately after periods of foundational crisis is characterized mainly as thought experimentation at the conceptual level. Appropriate modifications of problematic concepts are proposed and entirely new concepts emerge. The interesting thing is that such modifications do not come automatically and they are not ready-made. They come as the result of experimenting with various alternatives which emerge as the result of the creative activity of the mathematical (and sometimes the philosophical) community.

The best and the most well-known example of such creative activity spurred by a foundational crisis is the one occurred at the beginning of the 20th century during and after the appearance of the set-theoretic paradoxes. Various proposals concerning the modification of the naive Cantorian concept of the set were put forward, with most notable among them those of the Russellian theory of types, of the Zermelo-Fraenkel theory of sets and of the Gödel-Bernays set theory. Other proposals advocated not simply a modification of the concept but a completely new approach to the totality of the mathematical activity. An example of this sort is the intuitionistic proposal according to which definite constructivistic strictures should be imposed on the mathematical activity and the notion of the set as based upon its extension should be abolished. All these proposals not only *started* as thought experiments but they *were* thought experiments in the sense of open-ended exploratory attempts in a playful and in a state of crisis conceptual universe, which, so to speak, was set free in motion by the emergence of the paradoxes.

4. According to our classification the fourth group of thought experiments in mathematics contains those spurred by the intrinsic impossibility of proving or disproving a postulate or a sentence, which mathematicians under certain circumstances come to think as basic, from the axioms of a more or less given mathematical theory. An example of this sort of experimentation one can find in the relatively recent development of the research and the corresponding literature around the large cardinals axioms in the framework of Zermelo-Fraenkel set theory. Another beautiful example we can find in the history of the parallel-postulate. For almost 2000 years mathematicians tried to prove that the postulate did not deserve its celebrated status. That is, they tried to prove it from the rest of what today[6] we call the axioms of Euclidean geometry. In vain, though, with sometimes quite painful results as the following excerpt from a letter of the father Wolfgang Bolyai to his son Johann shows:

> "It is unbelievable that this stubborn darkness, this eternal eclipse, this flaw in geometry, this eternal cloud on virgin truth can be endured."

Again as the father is horrified by the thought that his son is attracted by the problem of parallels he writes to him:

> "You must not attempt this approach to parallels. I know this way to its very end. I have traversed this bottomless night, which extinguished all light and joy of my life. I entreat you, leave the science of parallels alone.... I thought I would sacrifice myself for the sake of the truth. I was ready to become a martyr who would remove the flaw from geometry and return it purified to mankind. I accomplished monstrous, enormous labors; my creations are far better than those of others and yet I have not achieved complete satisfaction. For here it is true that si paullum a summo discessit, vergit ad imum. I turned back when I saw that no man can reach the bottom of this night. I turned back unconsoled, pitying myself and all mankind."

And yet again:

> "I admit that I expect little from the deviation of your lines. It seems to me that I have been in these regions; that I have traveled past all reefs of this infernal Dead Sea and have always come back with broken mast and torn sail. The ruin of my disposition and my fall date back to this time. I thoughtlessly risked my life and happiness-aut Caesar aut nihil"[7].

The appearance of non-Euclidean geometries at the beginning of the 19th century was a bold thought experiment towards the opposite direction. The pioneers Gauss, Johann Bolyai and Lobachevski, independently from each other, started working with the negation of the parallel-postulate together with the rest of the axioms of Euclidean geometry. They started producing theorems of a new geometry without knowing, though, whether or not they would stumble sooner or later upon an inconsistency. We know today, especially after the work of Klein and Poincaré, that they *would not*. We know, that is, that if the new geometry they initiated was inconsistent then the Euclidean geometry would be inconsistent too. The model-theoretic result that the new geometry could not be inconsistent unless the Euclidean one was, gave to the initiative of Gauss, Johann Bolyai and Lobachevski un *a posteriori* safe base for its conceptually innovative character. The element of ignorance on the part of the pioneers of the new subject about the safety of their approach, on the other hand, gave to their initiative its bold experimental character.

5. There is another group of thought experiments in mathematics which deserves our attention. They are those triggered and influenced—among other things—by philosophical considerations of a quite revolutionary nature. They can be either thought experiments concerning the construction of quite peculiar, and out of the main stream of mathematical research, theoretical framework, taking place in times of foundational crisis, or thought experiments leading to the reconceptualization of a given area and taking place in times which could not be characterized as such.

A good example of the first sort we have if we consider the history of the birth and the subsequent development of Intuitionism. Thought experiments of the second kind take place sometimes in a way that is characterized by the element of surprise. They occur in normal periods of mathematical research and they are either global (i.e., if we consider the totality of mathematical research at a time) or local (i.e., if we consider the mathematical research carried out in a specific area, e.g. mathematical analysis). During such periods there is not any obvious need traceable in the overall mentality of the members of the mathematical community for some sort of conceptual innovation. It is the case then that attempts of this kind, which lead to the reconceptualization of a given area, come almost out of the blue and are due to the philosophical worries or curiosity of exceptional individuals and not to an appropriately conceptually saturated prior period of mathematical research. It is of course a truism to say that periods of the last sort are characterized, among other things, by the dialectical interplay between mathematicians in the strict sense and mathematicians with philosophical worries or philosophers technically equipped to understand mathematics.

A good example of a thought experiment of the second kind is the one performed by A. Robinson[8] when he successfully introduced a technically articulated new approach to the old subject of mathematical analysis. In his non-standard analysis the viciously attacked (especially by Russell), almost discarded and nearly forgotten Leibnizian infinitesimals were appropriately rehabilitated. They were given a rigorous treatment and the old standard analysis was seen in a conceptually new light.

6. The last group of thought experiments in mathematics according to

our classification includes those induced by the need to specify a new conceptual framework easier to work with in a specific area. Such a kind of experimentation is marked by the introduction of stronger axioms or more powerful tools for the exploration of the given area. An example is that of the relatively recent adoption of infinitary combinatorics and of philosophically controversial set-theoretic axioms, like the Continuum Hypothesis or its negation, large cardinal axioms, Martin's axiom and the like, for fruitful research in classical areas like topology, descriptive set theory and functional analysis.

The above proposed classification of thought experiments in mathematics is not exhaustive by any means. It indicates though that thought experimentation in mathematics is (a) a quite rich phenomenon and (b) it includes almost every aspect of creative mathematics in the sense of being an epistemologically open-ended dialogue between the mathematician and the hidden aspects of the conceptual structures he is creating. A dialogue which has to be carried out, of course, with the only safe means the mathematician has at his disposal, that is with what characterizes him as a thinker, i.e. his logic.

III

The activity of thought experimentation in mathematics is taking place between certain boundaries which are dependent upon what I call *conceivability conditions or constraints in mathematics*. Instead of defining what counts as a conceivability condition or constraint in mathematics I will give a list of such conditions for three reasons: First, because I believe that a more or less open-ended[9] list of conceivability conditions has an obvious priority over a definition, which could be final only if such a list was complete. Second, because a list could show in a descriptive way the connection between such conditions and thought experimentation in mathematics. Third, because the listing of conceivability conditions in mathematics has to come first and certainly before any serious attempt to answer "why" questions concerning the mathematical activity in general is made. In what follows I will neglect any direct reference to such questions, as they are beyond the scope of this paper. I will restrict myself instead to the description of eight different groups of conceivability conditions in mathematics adding only the following explanatory note: Among those groups conceivability conditions of a standard logical nature are not included because such conditions are not dependent upon any particular scientific discipline.

(a) *Simplicity conditions or constraints*. Such conditions are not exactly definable. It is nevertheless the case that everybody recognizes the existence of certain inexact boundaries beyond which humans are unable to process the information available to them. For instance, it is beyond the capacity of humans to work with extremely long and complex formulae or proofs. The tendency shared by most mathematicians is, given a basic mathematical intuition or a set of basic intuitions to work with the simplest possible concepts and axioms and attempt to give the simplest possible proofs which capture such an intuition or a set. An example of mathematical thought experimentation induced by such a tendency is the Scott-Solovay Boolean-valued theoretic approach of Cohen's forcing method of producing consistency and independence results. Yet "simple" means sometimes cumbersome and mathematicians have their own aesthetic values which tell them that what is cumbersome is not elegant. What is elegant for them is not only what is clever but what is clear as well. Clarity, on the other hand, is a characteristic connected with the sense of familiarity one has with a certain set of concepts or operations. This leads us to the second group, the:

(b) *Familiarity constraints*. Such conditions or constraints are responsible for the adoption of certain systems of concepts and operations with which the mathematical community feels familiar. Consider, e.g., the system of logical connectives which mathematicians or logicians use (negation, conjunction, disjunction, implication and equivalence) as opposed to alternative adequate systems with fewer connectives which they could use. Or consider the basic operations of addition and multiplication in number theory which are definable in terms of the successor operator. It is always of theoretical interest to find the simplest possible system of concepts—if such a system is possible to be found—for building up a certain mathematical theory, but it does not mean that the system of primitive concepts finally adopted is the simplest possible one. Most of the time, the system finally adopted is the one which appears to be both simple enough and natural. A similar situation obtains in the case of the choice between proposed correct proofs of the same theorem. Familiarity paired with simplicity in a coordinated manner pave the way of mathematical research. There is one more point which must be emphasized though. New concepts, especially in times of mathematical breakthroughs, do not have to be familiar to the majority of mathematicians. It is nevertheless the case that, most of the time, the proposal of a new cumbersome system of concepts is followed by research leading to the final adoption of a simpler and more familiar one.

(c) The third group of conceivability conditions or constraints is the one we could call *plausibility conditions or constraints*. They do not play as important a role in mathematical research as the ones played by the first two groups. They are nevertheless connected with a certain unexpressed need for easier communication between the members of the mathematical community and they have to do with the conservative element that is hidden behind any fixed system of intersubjectively shared mathematical intuitions.

(d) The fourth group is that of *efficiency constraints*. Under this title we subsume constraints which mainly have to do with tendencies, concerning our need for a reasonable computational apparatus connected with the proof-theoretic machinery used and for conceptual relevance with respect to what is sought after.

(e) Another group is that of conditions or constraints related to the success of a proposed reconceptualization; that is, of conceptual transformation or translation. As an example of conceptual transformation or translation we can consider, for instance, the algebraization by Descartes of Euclidean geometry. Under such a translation certain parts of the old theory become more accessible from the point of view of mathematical research. Relevant conservative translation metatheorems are established which guarantee that what is provable in the old framework is provable in the new one as well and that to the proof of any sentence in the new framework which is the translation of a sentence in the old one corresponds a proof in the old framework of the initial sentence. As other examples of reconceptualization one could consider the translatability of standard analysis into the framework of A. Robinson's non-standard analysis or the translatability of Cohen's initial version of the forcing method into the framework of the Boolean-valued model-theoretic method of Solovay and Scott. The success of the proposed reconceptualization is measured, first, by the existence of relevant conservative metatheorems and, second, by the higher workability of the new conceptual framework as it is contrasted to the old one.

(f) Another group is that of conditions or constraints induced by specific philosophical ideas and positions. Examples of such constraints one can see pretty easily as existing in the cases, for instance, of Intuitionism, Logicism, in the case of Hilbert's program as well as in the cases of any philosophically influenced mathematical research program.

(g) The last two groups of conceivability conditions are those of *specificity* and *generality or unification constraints*. The first group includes constraints concerned with the construction of theories for the description of a given mathematical object like the real line, the domain of natural numbers, etc. The old dream for the categorical efficient description of such objects has, of course, collapsed after Gödel's incompleteness results. Yet the need for specificity in their axiomatic description is still with us. The second group includes constraints which concern the construction of axiomatic theories capturing only the general and absolutely necessary characteristics of large areas of mathematical objects. Examples of such theories are, for instance, the theory of groups, the theory of fields, etc.

NOTES

1. In the standard or even the Lakatosian sense.

2. I use the words "production" and "creation" disjunctively in order to emphasize that, no matter what ontology one prefers to adopt in the case of mathematics, the proposed classifications are neutral with respect to them.

3. See Imre Lakatos, "Proofs and Refutations: The Logic of Mathematical Discovery," in John Worral and Elie Zahar, (eds.) (Cambridge: Cambridge University Press, 1977).

4. In the sense that the Axiom of Choice is provable from the axioms of the Zermelo-Fraenkel set theory.

5. In the sense that the negation of the Axiom of Choice is provable from the axioms of the Zermelo-Fraenkel set theory.

6. That is, after Hilbert's rigorous axiomatization of Euclidean geometry.

7. See Herbert Meschkowski, "Non-Euclidean Geometry" (Tr. by A. Shenitzer) (New York: Academic Press, 1964), pp. 31-32.

8. See A. Robinson, "Non-standard Analysis" (Amsterdam: North-Holland, 1951).

9. Like the one proposed by me.