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QUANTIFICATION AS AN ACT OF MIND*

This paper presents an algebraic treatment of logic, aiming to further the project of a "subjective semantics" for quantification, identity, and modality.

Semantics has largely concentrated on truth and reference, the relation of language to the world, and this is part of its usual definition. It has also been the discipline which offers theories to explain the logical behavior of various sorts of expressions traditionally studied in logic: connectors, modalities, quantifiers, abstractors, the identity predicate, and so forth. Recently a number of studies have concentrated on the relation between language and the states of mind (generally epistemic or doxastic attitudes) of its users, to provide such explanations, and the term "semantics" has also been used there.[1] The present study will be among these. Truth and reference will be eschewed. Intuitive descriptions of the framework will be given, albeit briefly, in terms of mental operations on propositions (regarded therefore as the sort of thing which we can vary or modify in imagination). Generality will be part of the aim; neutrality with respect to certain non-classical logics will be guarded. The Appendix will show how standard semantic analyses can fit into this general framework. The algebraic (lattice-with-transformations) analysis of modality, quantification and identity given here may therefore be of some interest outside 'subjective semantics' as well.

1. INTUITIVE DESCRIPTION OF ABSTRACTION

Contemplating the proposition that Socrates is mortal, we can *abstract* the property of (someone's) being mortal, and *generalize* to produce the proposition that everyone has this property. Alternative abstractions and generalizations are possible: the proposition eventually generalized to could be that all animals have this property, or that Socrates has every property, or, somewhat further afield, that Socrates was, is, and will be mortal. There

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must be many alternative abstraction operations and many associated universal quantifications.[2]

How does abstraction proceed? In one model of the process, the contemplated proposition is seen as complex, as having distinct ingredients (e.g. the individual concept of Socrates and general concept of mortality) and abstraction is a sort of deletion or separation. It will be a basic working hypothesis of this paper (suggested by difficulties encountered elsewhere) that propositions are *au fond* the only things the mind works with. There are no picture galleries of concrete and abstract individuals in addition to the propositions. (One reason: the idea of a picture without propositional content, say a picture of Socrates that is not of Socrates walking, or sitting, or standing, etc. does not make sense.) A second model of the process of abstraction utilizes the ideas of mental *variation* and invariance. Suppose I transform the initial proposition successively into: that Callias is mortal, that Gorgias is mortal, . . . The *abstract* (i.e. what is abstracted, what is 'common' to these propositions) can equally well be represented by the set of all these resultant propositions themselves. Generalization then concerns what remains invariant under these transformations.

What are such transformations like? In this example, they preserve logical structure: the inferential relations, conjunction and disjunction, perhaps much more (how much depends rather on how much logical structure the family of propositions has, a question on which our initial opinions ought to be minimal).

The example of variation I gave is a special one, and deceptively limited. It lends itself to exposition because we have the linguistic resources to describe it in simple terms. That cannot be the whole story. Contemplate a battle scene; after studying it for a moment you may conclude that all the soldiers are either fighting or wounded. You do not have a name for each one. Perhaps you went through it from left to right, saying to yourself " "This one is wounded, that one is fighting, this one is wounded but still fighting, . . ." Possibly you have the linguistic resources to describe each one uniquely in qualitative terms, and if you do, perhaps you have unconsciously utilized them. But possibly not. Possibly this was a real battle scene, and the causal chains between you and it constituted genuine reference; but possibly not. Possibly it was a painting or a hallucination, or the memory of part of Trajan's Column or just something you imagined. It does not seem to matter; you are able to arrive at the general proposition (whether as conclusion asserted or as possibility contemplated) in any case. My second working hypothesis will be that the process of abstraction and generalization can always be regarded as *like* the simple case, in that it can be represented by a set of transformations in the same way.

To conclude this intuitive description, I shall now give a preview of the theory to be developed here. The family of propositions will be assumed to be closed under arbitrary conjunction and disjunction (not necessarily finite) and hence carry an implication relation (in the familiar way: A implies B if and only if A is the same as the conjunction of A and B). The weakest proposition, implied by all, will be called K and referred to as the *tautology* or the *a priori.* (Necessity is an independent subject, with *it is necessary that* analyzed in the same way as a universal quantifier.)

Propositions will be taken as basic, treated as 'black boxes', attributed no internal structure. Nor will domains of discourse be utilized, nor anything else that exists independently of the propositions. It may be puzzling how such a theory can treat of identity and predication, so I shall give a preliminary sketch. Names (more generally, categorematic terms) appear simply as labels for abstractors. Of course if we express a proposition we must do so in words, so to explain facts about propositions, we always proceed in effect by presenting syntactic analogues. "Tom" is the label of the abstractor which, intuitively speaking, abstracts the property *being tall* from the proposition *that Tom is tall.* Two such abstractors may be correlated; intuitively, what the Tom abstractor does to *that Tom is tall* is just what the Harry abstractor does to *that Harry is tall.* To put it slightly more precisely, there is a *Harry-for-Tom* operation, which turns *that Tom is tall* into *that Harry is tall*, and this operation (I call it, naturally, a *variation*) allows us to express the relation between these two abstractors. Variations are transformations in the sense discussed above: they preserve logical structure.

For any operation f on propositions we define the *core* of f: the set of propositions it turns into the tautology. And then, if we define $I(f)$ to be the proposition which is the conjunction (possibly transfinite, of the core of f, we are able to deduce crude forms of Leibniz' law (roughly put, if f is an idempotent transformation, and the conditional operator $\supset$ has some familiar properties, then If implies $A \supset fA$ and also $fA \supset A$). If we now let f be a variation like *Harry-for-Tom*, then $I(f)$ has all the characteristics we expect of the proposition *that Harry = Tom.* And this proposition rarely has the character of a tautology or contradiction: even if true, even if necessary, it is not *a priori.*

Predication in general, including lambda-abstraction, is somewhat more complicated. The theory will use two mutually complementary guiding ideas: a predicate labels a search procedure, which locates propositions; equivalently, it labels a coordinate system which coordinatizes a 'local' family of propositions. The coordinates (or, search procedure inputs) are those abstractors which are labelled by names.

In this way all logical structure of propositions derives from relations among them and operations upon them. This study will be kept so general that no prior choice need be made between, for example, classical, intuitionistic, or quantum logic.

2. LOGICS AND ABSTRACTORS

In its most common sense, a logic is a system of rules for proof and derivation. There is however a respectable secondary use, common especially in writings on quantum logic, in which a logic is an algebra of propositions. One example of a logic in this sense, familiar outside quantum logic, is the Boolean algebra of sets of possible worlds in a modal model structure; for there a proposition is identified with the set of worlds in which it is true. In this example, as also in the standard model for quantum logic, the algebra of propositions is a complete lattice of sets. The Appendix will illustrate this further for classical and intuitionistic quantification theory.

Henceforth a *logic* will be a complete lattice L; its members will be called *propositions*, its partial ordering $\leqslant$ *implication*; its maximal element K the *a priori.* A function which maps L into itself, preserving K, $\leqslant$, complete meet $\bigwedge$ and complete join $\bigvee$, I shall simply call a *transformation* of L.

An *abstractor* on L is any set of transformations of L; if A is a proposition and b an abstractor then the *b-abstract* of A is $bA = \{gA: g \in b\}$. A note on notation: I shall use $A, B, C, \ldots$ for propositions, and $a, b, c, \ldots$ for abstractors, and $X, Y, Z, \ldots$ for sets of propositions. Later I shall use other capital letters for other sets (e.g. of abstractors). For transformations I shall use $g, h, \ldots$. The value of function x for argument A, I denote $x(A)$ or xA.

A (partial) operation φ on L will be called *abstractive* exactly if there exists an abstractor b and function φ^* such that $\varphi(A) = \varphi^*(bA)$. One special example is the *universal quantifier* associated with abstractor b, defined as $\forall_b A = \bigwedge (bA)$, that is, the meet of all the propositions gA with g in b.

Proposition A is called *b-invariant* exactly if $bA = \{A\}$ and the abstractor b is called *destructive* if gA is b-invariant for each g in b and each A in L. Notice that if b is destructive, all its members must be idempotent. After all these definitions, though they are motivated by Section 1, an example will help. Suppose that the transformation g turns any proposition of the type *that Tom is F* into the corresponding one of type *that Harry is F*, and does nothing else. Then $ggA = gA$, so the very little abstractor $tom = \{g\}$ is destructive.

3. THE SINGLE UNIVERSAL QUANTIFIER (NECESSITY OPERATOR)

Let b be an abstractor on L and $\forall$ its associated universal quantifier (suppressing the subscript). Given that L is a logic, as defined, we deduce

(3-1) $\forall K = K$; if $A \leqslant B$ then $\forall A \leqslant \forall B$

(3-2) if $\bigwedge X \leqslant B$ then $\bigwedge \{\forall A : A \in X\} \leqslant \forall B$

(3-3) $\forall A \leqslant gA$ if $g \in b$

(3-4) $\forall A \leqslant \forall\forall A$ if for each g and g' in b there exists g'' in b such that $gg'A = g''A$.

The proviso of (3-4) is satisfied for instance if b is destructive. Proofs of these results, which are recognizable as the basic laws of quantification and of necessity (in normal modal logic) are immediate from the definitions.

4. CORRELATED UNIVERSAL QUANTIFIERS

In familiar logical systems we see many universal quantifiers, not just a single one. In Fitch's perspicuous symbolism, $[a/Fa]$ is the property that a must have in order for Fa to be true, and $U[a/Fa]$ the assertion that this property is universal: everything has it. Clearly $[b/Fb] = [a/Fa]$ and $U[b/Fb] = U[a/Fa]$. The reason is that Fa and Fb are two propositions which are 'congruent' in a certain sense; each can be turned into the other by a simple variation.

Given abstractors $a, b, c, \ldots$ there may exist transformations

$(a-b)$, $(a-c)$, $(b-c)$, . . . such that if $A = (a-b)B$ and $B = (b-a)A$ then $aA = bB$ and hence also $\forall_a A = \forall_b B$. I shall call $(a-b)$ a *variation* and read it as "a for b"; it must itself be a transformation. These transformations, if they exist, *correlate* the abstractors, which then form a *correlated abstractor system.*

To preserve a contrast in the terminology, I shall call the transformations which are members of the abstractors, *instantiations.* In the case of such correlation, the abstractors $a, b, c, \ldots$ must have corresponding members $g^a, g^b, g^c, \ldots$ such that g^a does to A exactly what g^b does to B, when $A = (a-b)B$ and $B = (b-a)A$. The cross classified set $\{g^a, g^b, g^c, \ldots\}$ picks out one member from each abstractor. To put it differently, the correlation establishes an equivalence relation among instantiations: each equivalence class is a choice set selecting one instantiation from every abstractor.

What is this equivalence relationship? Think of each abstractor b as pertaining to a single parameter – which may as well also be called b – of which the proposition is a function, and whose value is "fixed" in different ways by different instantiations. There are then three ways to set parameters a and b equal to the same fixed value:

vary b to a, fix a:	$g^a(a-b)$
vary a to b, fix b:	$g^b(b-a)$
fix a, fix b the same way:	$g^b g^a$.

The identity of these three procedures I propose as the way to single out the correlation relationship. I shall lead up to the definition of a correlated abstractor system slowly, by postulating conditions one by one.

Let an *abstractor system* be any set G of destructive abstractors which do not interfere with each other, in the following sense:

I. (*Non-interference*) If h and h' belong to different abstractors in G then $hh' = h'h$.

Considering the properties of destructivity within abstractors in G and non-interference between them, we deduce

d(4-1) Abstractors in an abstractor system are disjoint.

The notation "d" is used with an eye on future generalization, to signify that destructivity is used in the proof. Next we introduce the variations:

(4-2) A *variation* from abstractor a to abstractor b is a function σ from L into L such that for all h in a there exists a member h' of b such that $hh' = h\sigma$.

An abstractor system G will be called *correlatable* if for each a and b in G there exists a variation $(a-b)$ from a to b such that principles II, III, IV below also hold.

II. $(a-b)$ is idempotent; $(a-a)$ is the identity on L

III. $\forall_a A \leqslant (b-a)A$.

To state IV we need a further definition applicable in this context.

(4-3) Instantiations h and h' in $\cup G$ are called *associated* in G (by the variations $(c-d)$: $c, d \in \cup G$) exactly if there are abstractors a, b in G such that $hh' = h(a-b)$.

IV. Association in G is an equivalence relation on $\cup G$ each of whose equivalence classes contains exactly one member of each abstractor in G.

The following definition and result will now allow the perspicuous representation used in the intuitive discussion above.

(4-4) If G is a set of abstractors then a *correlator* of G is a set G^* of functions mapping G into $\cup G$ for which there exists a set VAR of transformations such that for all a, b in G:
(4-41) VAR contains exactly one variation (denoted $(a-b)$) from a to b
(4-42) $a = \{g(a) : g \in G^*\}$
(4-43) $g(a)g(b) = g(a)(a-b) = g(b)(b-a)$.

(4-5) CORRELATOR THEOREM If an abstractor system is correlatable then it has a correlator.

To prove this define for each h in $\cup G$ the function g_h on G such that $g_h(a)$ is in a and associated with h, for each a in G. By IV this is well-defined. By the reflexivity of association, (4-42) holds; by its definition and symmetry, (4-43) as well.

Henceforth, the discussion will pertain to a correlatable abstractor system G with set VAR of variations and correlator G^* as described. For perspicuity I shall write $g(a)$ as g^a, and we can then write:

I*. $g_1^a g_2^b = g_2^b g_1^a$ if $a \neq b$

IV*. $g^a g^b = g^a(a-b) = g^b(b-a)$

(4-6) $aA = \{g^a A : g \in G^*\}$

(4-7) $\forall_a A = \bigwedge \{g^a A : g \in G^*\}$.

(4-8) CORRELATION THEOREM If $A = (a-b)B$ and $B = (b-a)A$ then $aA = bB$ and $\forall_a A = \forall_b B$.

The second part of (4-8) follows immediately from the first part, which is proved: $g^a A = g^a(a-b)B = g^b(b-a)B = g^b(b-a)(b-a)A = g^b(b-a)A = g^b B$ by appeal to IV* and II (idempotency); and finally (4-42).

For terminological contrast I shall call g^a, g^b .... *instantiations.* Intuitively, g^a fixes the value of a at g, "sets a equal to g". Note that the variations are also naturally grouped into abstractors:

(4-9) $a^+ = \{(b-a) : b \in G\}$

with associated universal quantifier $\forall_a^+$. This is reminiscent of the substitution interpretation of quantifiers, with $\forall_a A \leqslant \forall_a^+ A$ in view of III. The abstractors $a^+, b^+, \ldots$ are also destructive, but the Non-Interference principle cannot be expected to hold: $(a-c)$ will not generally commute with $(c-b)$. The *variational abstraction* introduced by (4-9) will play an important role in the analysis of predication.

5. IMPLICATION OPERATORS AND METHODOLOGY[3]

Henceforth G, VAR, and G^* will be assumed to be as described above, but we shall introduce a new assumption. I shall call logic L *pre-implicative* exactly if it has an operator $\supset$("ply") such that $A \supset B = K$ if and only if $A \leqslant B$. Even in quantum logic such an operator exists (the Sasaki hook). Given that L is pre-implicative, a transformation will be called *normal* if it preserves $\supset$, and an abstractor *normal* exactly if all its members are. The new assumption, made henceforth, is that L is pre-implicative and all members of G normal.

Note that if we define $(A \equiv B) = (A \supset B) \wedge (B \supset A)$ as usual then $(A \equiv B) = K$ iff $A = B$. This helps us to prove a result which is of great use in subsequent proofs.

(5-1) *Methodological Lemma.* Let τ, ρ be functions defined by composition of variations and/or instantiations. If $g^a\tau = g^a\rho$ for all g in G^* then $\tau = \rho$.

An example of τ would be $g^b(b-a)(b-a)$, which occurred in the proof of (4-8); the abstractor a in the Lemma need not be specially chosen, it may appear in the description of τ or ρ but need not. To prove the Lemma, note that $g^a\tau = g^a\rho$ means that for all A in L, $g^a\tau A = g^a\rho A$, and therefore $(g^a\tau A \equiv g^a\rho A) = K$. By normality of G, $g^a(\tau A \equiv \rho A) = K$. If that is so for all g in G^* then $\forall_a(\tau A \equiv \rho A) = K$; hence $(\tau A \equiv \rho A) = K$ as well by III and II. But then $\tau A = \rho A$. That being so for all A in L, $\tau = \rho$.

This is at first sight a surprising result; the proof establishes along the way that if $g^aA = g^aB$ for all g in G^* then $A = B$. For a putative counter-example, let $G = \{a\}$ and $a = \{h\}$. Surely we cannot conclude from $hA = hB$ that $A = B$? But III requires then that $hA = \forall_a A \leqslant (a-a)A = A$, so we have $hA \leqslant A$ for all A in L. From $hA = hB$ we can conclude $(hA = hB) = K$, and by normality, $h(A \equiv B) = K$, but by the preceding observation then $K \leqslant (A = B)$, so $A = B$. Hence h must be one-to-one. Thus we see that III, which looks at first like an *ad hoc* addition to the theory, actually plays a crucial role in determining what a correlated abstractor system is like.

6. INTERACTION AND VARIATION EQUALITIES

Although substitution is a ubiquitous operation in logical theory, its semantic analysis is, as far as I know, found only in bits and pieces scattered through the literature.[4] Using the Methodological Lemma, I shall here take up all questions whose answers are needed to calculate the effect of any composition of instantiations and/or variations.

(6-1) *Interaction Equalities*
d(6-11) $g^b(a-b) = (a-b)$ if $a \neq b$
d(6-12) $(a-b)g^b = g^b$
(6-13) $g^c(a-b) = (a-b)g^c$ if $c \neq a, c \neq b$.

The first two are reminiscent of Destructivity (and proved by appeal to it), the third of Non-Interference. Here is the first proof. For g_1 in G^* and $a \neq b, g_1^a g^b(a-b)A = g^b g_1^a(a-b)A = g^b g_1^a g_1^b A = g^b g_1^b g_1^a A = g_1^b g_1^a A = g_1^a(a-b)A$ by successive appeals to I*, IV*, I*, Destructivity, and IV*. The Methodological Lemma (5-1) now entails (6-11). The proofs of the others are similar.

(6-2) *Variation Equalities*

- d(6-21) *Triangle Equality* $(a-b)(b-c) = (a-b)(a-c)$
- (6-22) *Special Commutation* $(a-b)(c-d) = (c-d)(a-b)$ if a, b, c, d all distinct
- d(6-23) *Triangle Commutation* $(a-b)(a-c) = (a-c)(a-b)$
- d(6-24) *Non-Aberration* $(a-c)(d-c) = (d-c)$.

The Triangle Equality, which is the most useful, must be proved for two subcases. If $a \neq c$, we deduce $g^a(a-b)(b-c) = g^a g^b(b-c) = g^a g^b g^c = g^b g^a g^c = g^b g^a(a-c) = g^a g^b(a-c) = g^a(a-b)(a-c)$ for arbitrary g in G^* by appeal to IV*, I*. By the Methodological Lemma, (6-21) follows. If $a = c$, we claim that $(a-b)(b-a) = (a-b)$. We deduce $g^a(a-b)(b-a) = g^a g^b(b-a) = g^a g^b g^a = g^a g^a g^b = g^a g^b = g^a(a-b)$ by appeal to IV*, I*, II, for all g in G^*; and the result by the Methodological Lemma. The proofs for the others are similar. Together they cover all combinations which can be reduced. The principle of Non-Aberration says in effect that the variational abstractor c^+ is destructive; but I could not resist the mnemonic name.

7. PARAMETER DEPENDENCE AND SIMULTANEOUS VARIATION[5]

One guiding idea of the present theory is that the apparent parameter *Tom* in *that Tom is happy* should be identified with an abstractor which turns that proposition into the property *being happy*. It is heuristically useful, however, to keep the alternative picture of parameter dependence, to think of the abstractor as a parameter on which this proposition depends, and the members of the abstractor as different ways of fixing the value of the parameter. (Read "g^a" as "set the value of a equal to g".) In that case, we may call a proposition which is b-invariant, one for which b is an *irrelevant* parameter. Thus b is *relevant to A* exactly if $g^a A \neq A$ for some g in G^*.

(7-1) *Irrelevance Conditions*

d(7-11) If $a \neq b$, then a is irrelevant to $(b-a)A$

d(7-12) a is irrelevant to $g^a A$ and $\forall_a A$

(7-13) If a is irrelevant to A then:

(7-131) a is irrelevant to $(b-c)A$ provided $a \neq b$

d(7-132) a is irrelevant to $g^b A$ and $\forall_b A$

(7-133) $(b-a)A = A$

(7-134) $(b-a)(a-c)A = (b-c)A$

(7-135) $\forall_a(a-b)A = \forall_b A$

(7-14) If b is relevant to A then a is relevant to $(a-b)A$.

The first of these is implied directly by the first Interaction Equality, the second by destructivity. The third (7-131) is implied by the third Interaction Equality, the next by Non-Interference. The Methodological Lemma can be used to prove (7-133) and (7-134) – consider that if a is irre-irrelevant to A then $g^b(b-a)A = g^b g^a A = g^b A$ for all g in G^*, for example. Finally, the last one can be generalized from the fact that $g^a(a-b)A = g^a g^b A = g^b g^a A = g^b A$ given that A is a-invariant.

The proof of (7-14) is interesting. Suppose a is not relevant to $(a-b)A$. Then $(b-a)(a-b)A = (a-b)A$ by (7-133). But also $(b-a)(a-b)A = (b-b)A = A$ by (7-134) and II. Hence $(a-b)A = A$. But then $g^b A = g^b(a-b)A = (a-b)A$ by Interaction Equality (6-11), provided $a \neq b$; and hence $g^b A = A$. Thus b is irrelevant to A. Note that (7-14) also holds trivially for $a = b$.

A quick final note on why $(a-b)$ will not generally commute with $(b-c)$ or with $(c-a)$. For suppose that b but not a is relevant to A. Then $(a-b)(c-a)A = (a-b)A$ by Irrelevance Condition (7-133). But $(c-a)(a-b)A = (c-b)A$ by (7-134). Since a is relevant to $(a-b)A$ by (7-14) and not relevant to $(c-b)A$ by (7-131), always provided these parameters are distinct, we conclude that the two resulting propositions are not the same, so the operations do not commute. Similar remarks apply to $(a-b)(b-c)$.

(7-2) *Simultaneous Variation* If $a_1, \ldots, a_n$ are distinct abstractors, then $(b_1-e_1) \ldots (b_n-e_n)(e_n-a_n) \ldots (e_1-a_1)A$ is the same proposition for any choice of distinct parameters $e_1, \ldots, e_n$ which are irrelevant to A, and distinct from $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$.

The resultant proposition may be thought of as produced by a simultaneous variation of a_i to b_i $(i = 1, \ldots, n)$. The analogue for syntactic substitution is well-known.

For the proof suppose that $e_1, \ldots, e_n$ are distinct and irrelevant to A; and $e'_1, \ldots, e'_n$ also form a set of n distinct abstractors irrelevant to A, and that $a_1, \ldots, a_n$ are distinct from each other and from e_i and e'_i $(i = 1, \ldots, n)$ and $b_1, \ldots, b_n$ are distinct from e_i and e'_i $(i = 1, \ldots, n)$. For easy reference consider:

$$\beta = (b_1 - e'_1) \ldots (b_n - e'_n)$$

$$\gamma = (e'_n - e_n) \ldots (e'_1 - e_1)$$

$$\delta = (e_1 - a_1) \ldots (e_n - a_n).$$

Our aim will be to prove both:

(7-21) $\beta\gamma\delta A = (b_1 - e'_1) \ldots (b_n - e'_n)(e'_n - a_n) \ldots (e'_1 - a_1)A$

(7-22) $\beta\gamma\delta A = (b_1 - e_1) \ldots (b_n - e_n)(e_n - a_n) \ldots (e_1 - a_1)A$

whence the result follows. To prove the first note that

(7-23) $\gamma\delta A = (e'_n - e_n) \ldots (e'_1 - e_1)(e_1 - a_1) \ldots (e_n - a_n)A$
$= (e'_n - e_n) \ldots (e'_2 - e_2)(e'_1 - a_1)(e_2 - a_2) \ldots (e_n - a_n)A$

by Irrelevance Condition (7-134). Now e'_1 may be identical with e_2; nevertheless, we may commute $(e'_1 - a_1)$ with $(e_2 - a_2)$, either by Special Commutation or by Triangle Commutation. Similarly for $e_3, \ldots, e_n$, so $(e'_1 - a_1)$ can moved all the way to the right hand side just before A:

(7-231) $\gamma\delta A = (e'_n - e_n) \ldots (e'_2 - e_2)(e_2 - a_2) \ldots (e_n - a_n)(e'_1 - a_1)A.$

We now simply repeat this whole process until we have

(7-24) $\gamma\delta A = (e'_n - a_n) \ldots (e'_1 - a_1)A$

and now (7-21) follows at once.

Next we consider $\beta\gamma(\delta A)$. We note that since $e'_1, \ldots, e'_n$ were irrelevant to A, e'_i can be relevant to δA only if $e_i = e_j$ for some j. But if $e'_n = e_1$, for example, e'_n will be irrelevant to $(e'_1 - e_1)\delta A$ unless $e'_1 = e_1$ as well, which is impossible since $e'_1 \neq e'_n$. Similarly for $2, 3, \ldots, n-1$. So e'_1 is irrelevant to $(e'_{n-1} - e_{n-1}) \ldots (e'_1 - e_1)\delta A$. Thus we have

(7-25) $\beta\gamma(\delta A) = (b_1{-}e'_1)\ldots(b_n{-}e'_n)(e'_n{-}e_n)\ldots(e'_1{-}e_1)\,A$
$= (b_1{-}e'_1)\ldots(b_{n-1}{-}e'_{n-1})(b_n{-}e_n)(e'_{n-1}{-}e_{n-1})\ldots$
$\ldots(e'_1{-}e_1)\,A$

by Irrelevance Condition (7-134) again. Now $(b_n{-}e_n)(e'_{n-1}{-}e_{n-1})$ commutes: $e_n \neq e_{n-1}$; if $e_n = e'_{n-1}$ then by Triangle Equality and Triangle Commutation; if $e_n \neq e'_{n-1}$ then either by Special or Triangle Commutation. And so forth for $n-2, n-3, \ldots, 1$. Hence $(b_n{-}e_n)$ can be moved all the way to the right, before δA:

(7-26) $\beta\gamma\delta A = (b_1{-}e'_1)\ldots(b_{n-1}{-}e'_{n-1})(e'_{n-1}{-}e_{n-1})\ldots$
$\ldots(e'_1{-}e_1)(b_n{-}e_n)\delta A.$

Because b_n is also distinct from all the e'_j, we can repeat the whole process for e'_{n-1}, and so forth; and so we arrive at:

(7-27) $\beta\gamma\delta A = (b_1{-}e_1)\ldots(b_n{-}e_n)\delta A.$

Finally, δ can be commuted at will by Special Commutation; hence (7-27) implies the desired (7-22).

The theorem allows the introduction by definition of partial functions that effect simultaneous variation of parameters. Let a *replacement* be any map σ of G into G; there will then be for each s in G^n a corresponding simultaneous variation operator σ^s which varies $s(i)$ to $\sigma s(i)$, for $i = 1, \ldots, n$.

(7-3) If σ is a replacement on G and s in G^n then σ^s is the (partial) operator on L such that $\sigma^s A = (\sigma s(1){-}e_1)\ldots(\sigma s(n){-}e_n)(e_n{-}s(n))\ldots(e_1{-}s(1))A$ where $e_1, \ldots, e_n$ are parameters irrelevant to A and distinct from each other and from $\sigma s(1), \ldots, \sigma s(n), s(1), \ldots, s(n)$; and $\sigma^s A$ is undefined if no such parameters are available.

That σ^s is well-defined follows from the preceding theorem. The following theorem converts suitable sequences of ordinary variations into a single simultaneous variation.

(7–4) *Variation Conversion* If $a_1, \ldots, a_n$ are distinct then there exists a replacement σ such that $(b_n{-}a_n)\ldots(b_1{-}a_1)A = \sigma^{(a_1,\ldots,a_n)}A$ for every proposition A in L for which there exist n parameters irrelevant to A and distinct from each other and from $a_1, \ldots, a_n, b_1, \ldots, b_n$.

Let us suppose that $e_1, \ldots, e_n$ are irrelevant to A and distinct from $a_1, \ldots, a_n, b_1, \ldots, b_n$. Let $\alpha = (b_n - a_n) \ldots (b_1 - a_1)$.

First of all we check whether $b_1 = a_2$. If it is we replace b_1 by b_2, by Triangle Equality, and then commute by Triangle Commutation. If not, we move $(b_1 - a_1)$ to the left, by Special or Triangle Commutation; then ask whether $b_1 = a_3$ and repeat the process. Eventually $(b_1 - a_1)$ is on the far left, if it survives at all. We now repeat the entire process, until we have turned α into

$$\alpha' = (b_1' - a_1) \ldots (b_n' - a_n)$$

where $b_i' = a_j$ only if $i = j$, which will happen in a finite number of steps; moreover, $\alpha A = \alpha' A$ for all propositions A. Consider now, for suitably chosen e_i:

$$\beta = (b_1' - e_1) \ldots (b_n' - e_n) \qquad \gamma = (e_n - a_n) \ldots (e_1 - a_1).$$

We assert that $\alpha A = \alpha' A = \beta\gamma A$; moreover, by definition, $\beta\gamma = \sigma^s$ where $\sigma(a_i) = b_i'$ for $i = 1, \ldots, n$. Since e_n is distinct from $e_{n-1}, \ldots, e_1$ it follows by an Irrelevance Condition and its irrelevance to A, that e_n is irrelevant to $(e_{n-1} - a_{n-1}) \ldots (e_1 - a_1)A$; so we can collapse $(b_n' - e_n)(e_n - a_n)$ into $(b_n' - a_n)$ by an Irrelevance Condition. This commutes now with $(e_{n-1} - a_{n-1})$ by the fact that $b_n' \neq a_{n-1}$ and $a_n \neq e_{n-1}$; similarly for the others, so $(b_n' - a_n)$ can be moved all the way to the right in $\beta\gamma A$. We now repeat the whole process with $(b_{n-1}' - e_{n-1})$, and go on to the next. Hence we see that the Variation Equalities and Irrelevance Conditions allow us to equate $\alpha' A$ with $\beta\gamma A$ as required.

With these results we have achieved our main objectives in the theory of variation (substitution). Further results would be desirable. I conjecture that if σ and ρ are two replacements and there are at least $2n$ parameters irrelevant to A, then $(\sigma\rho)^s A = \sigma^{\rho s}\rho^s A$, as the correct theorem for composition of simultaneous variations.

8. THE SINGLE IDENTITY

Names are labels of abstractors. Which proposition is the proposition *that Cicero = Tully*? Let us put it another way: identity theory has certain laws

– can these laws be deduced for a suitably chosen identification of the identity propositions?

Suppose that f is any function mapping L into L. With a nod in the direction of algebraic kernels, let us define the *core* of f to be the set $\{A \in L : fA = K\}$ and the *identity proposition* of f, call it $I(f)$, to be the meet of its core:

(8-1) $\quad If = \bigwedge\{A \in L : fA = K\}.$

Thus $I(f)$ is, intuitively, the conjunction of all the propositions which f turns into the *a priori*.

Recall that a transformation is called *normal* if it preserves the ply operation. If f is a normal, idempotent transformation we can argue: $f(A \supset fA) = (fA \supset ffA) = (fA \supset fA) = K$, hence $I(f) \leqslant A \supset fA$ which already looks like one quarter of Leibniz's law.

(8-2) *Single Identity Theorem* If f is an idempotent normal transformation of L then
(8-21) $I(f) \leqslant A$ iff $f(A) = K$
(8-22) $I(f) \leqslant A \supset fA; I(f) \leqslant fA \supset A$
(8-23) provided $K \supset A = A$ for all A in L,
$I(f) = \bigwedge\{A \equiv fA : A \in L\}$
(8-24) provided $A \wedge (A \supset B) \leqslant B$ for all A, B in L,
$I(f) \wedge A \leqslant fA;\ I(f) \wedge fA \leqslant A.$

One half of (8-21) follows at once from (8-1). Suppose then that $I(f) \leqslant A$; then $f(I(f)) \leqslant fA$. But $f(I(f)) = f\bigwedge\{A : fA = K\} = K$, because f is a transformation. The first half of (8-22) was proved above; similarly $f(fA \supset A) = ffA \supset fA = fA \supset fA = K$ so by (8-21) the second half follows.

Half of the identity in (8-23) follows from (8-22). To prove the other half, it suffices to show that if B is in the core of f, then some proposition $(A \equiv fA)$ implies B. But $f(I(f)) = K$ by (8-21), so $(I(f) \equiv f(I(f))) = (I(f) = K) = I(f)$ by the assumed properties of ply; and if B is in the core of f, this does indeed imply B.

By (8-22), $I(f) \leqslant A \supset fA$, so $I(f) \wedge A \leqslant (A \supset fA) \wedge A \leqslant fA$ by the assumed property of ply; similarly for the other half of (8-24).

Again, even the Sasaki hook of quantum logic has the properties assumed in the provisos; on the other hand the Sasaki hook and the classical counterfactual conditional both lack the intuitionistic property that $A \leqslant B \supset C$ if and only if $A \wedge B \leqslant C$, which I have therefore not assumed anywhere.

9. CORRELATED IDENTITIES

To prove the Methodological Lemma we assumed that all instantiations are normal. Henceforth we shall also assume that all variations are normal, and abbreviate "$I(a-b)$" to "Iab". Intuitively speaking we wish to establish that Iab can be thought of as a proposition asserting that a certain equivalence relationship holds between abstractors a and b. This relationship should be so tight that Iab implies, and is implied by, the 'indiscernability' of a and b – the equivalence of any proposition with the result of varying a to b on that proposition. In addition, we do not want Iab to be *a priori* except when a, b really are one and the same abstractor.

(9-1) $Iab \leqslant A$ iff $(a-b)A = K$

(9-2) $Iac \leqslant (c-b)Iab$

(9-3) · If a, b, c are distinct then $Iac \leqslant B$ only if $Iac \leqslant \forall_b B$

d(9-4) $(c-b)Iab \leqslant Iac$ provided $a \neq b$.

These results show that Iab really can be thought of as a proposition truly dependent on parameters a and b. We do not want (9-4) to hold if $a = b$, for Ibb ought not generally to imply Iac. Here (9-1) merely restates (8-21). This entails that (9-2) requires only that $(a-c)(c-b)Iab = K$, which follows if $(a-c)(c-b)A = K$ whenever $(a-b)A = K$. By the Triangle Equality, $(a-c)(c-b)A = (a-c)(a-b)A$, and transformations turn K into K, so if $(a-b)A = K$ then so is $(a-c)(a-b)A$; hence the result.

To prove (9-3), which we need as a Lemma for (9-4), note that when a, b, c are distinct, $(a-c)B = K$ entails $(a-c)g^bB = g^b(a-c)B = g^bK = K$ by the Interaction Equalities, and use (9-1).

The proof of (9-4) is somewhat more interesting. This is trivial if c is the same as a or b, so assume they are all distinct. We must show that

$$\bigwedge\{(c-b)A : (a-b)A = K\} \leqslant \bigwedge\{B : (a-c)B = K\}$$

which will follow if for each B in the core of $(a-c)$ there exists a proposition B^* such that $(a-b)B^* = K$ and $(c-b)B^* \leqslant B$.

So suppose that $(a-c)B = K$ and set $B^* = (b-c)\forall_b B$. Then $(c-b)B^* = (c-b)(b-c)\forall_b B = \forall_b B \leqslant B$ by the Irrelevance Conditions and III. Secondly, $(a-b)B^* = (a-b)(b-c)\forall_b B = (a-b)(a-c)\forall_b B$ by the Triangle Equality. But because $(a-c)B = K$, so is $(a-c)\forall_b B$ by (9-3) above, hence

$(a-b)B^* = (a-b)K = K$. The Irrelevance Conditions used in the proof required destructivity.

Now we can turn to some more familiar properties:

(9-5) $Iaa = K$

(9-6) $Iab = Iba$

d(9-7) $Iab \leqslant (Ibc \supset Iac)$

(9-8) provided $A \wedge (A \supset B) \leqslant B$ for all A, B in L: $Iab \wedge Ibc \leqslant Iac$

(9-9) If $Iab = K$ then a and b are the same abstractor.

For the first note that only K belongs to the core of $(a-a)$. To prove the second it suffices to show that $(a-b)$ and $(b-a)$ have the same core. Suppose that $(a-b)A = K$. Then $g^b(b-a)A = g^a(a-b)A = g^aK = K$ by IV*, for any g in G^*. Hence $\forall_b(b-a)A = K$; so also by III, $(b-a)A = K$.

By the Single Identity Theorem, $Iab \leqslant (Ibc \supset (a-b)Ibc) \leqslant (Ibc \supset Iac)$ provided $a \neq b$, by (9-2), d(9-4), and (9-6). We conclude (9-7) which holds trivially when a and b are the same. Finally, using the Single Identity Theorem and the proviso of (9-8) we deduce $Iab \wedge Ibc \leqslant (a-b)Ibc \leqslant Iac$ when $a \neq b$; and conclude (9-8) which again holds trivially if a, b are the same.

To prove (9-9), we appeal to the disjointness of distinct abstractors in a correlated abstractor system. If $Iab = K$, then $(A \equiv (a-b)A) = K$ for all A in L (using only (8-22) of the Single Identity Theorem). Hence $(a-b)A = A$ for all A in L. Because $Iab = Iba$ we conclude similarly that $(b-a)A = A$ for all A in L. Thus $g^aA = g^a(a-b)A = g^b(b-a)A = g^bA$, for any g in G^* and A in L.

Hence the abstractors a and b have the same members and so are identical.

Results (11-10)–(11-12) will give some more information about identity.

10. PREDICATION: LOCAL COORDINATES AND SEARCH PROCEDURES

When A is not b-invariant, we can also think of abstractor b as a parameter (whose variation affects A) and call A dependent on b or b relevant to A. Of course, A may be dependent on two, three, or more parameters. The proposition *Peter is older than Paul* depends on the parameters *Peter* and *Paul*, and so does *that Paul is older than Peter.* There is no asymmetry in this remark; nor in any relations between the two propositions. Nor is there, in our theory, any internal structure attributed to the propositions which could explain the apparent asymmetry in this description. Hence we must locate the source of the apparent asymmetry solely in the method of description.

As analogy, consider the search procedure: *start at the intersection of the equator and the Greenwich meridian, go miles north and miles west.* Different points are reached depending on whether I apply input (5, 3) or input (3, 5). Of course there is no asymmetry either in the points themselves or in the relations between them. *A predicate has as semantic correlate such a search procedure.* It will be clear from the example that under suitable circumstances, a search procedure is essentially the same thing as an assignment of coordinates. *A predicate has as semantic correlate a local coordinate system for propositions.* The proposition usually designated as *Rab* is the one which has coordinates (a, b) in coordinate system R. I shall now make this analogy precise.

Recall the variational abstract $a^+A = \{(b-a)A : b \in G\}$. You can get to an any proposition in that family, from A, by means of a single variation. In case b is irrelevant to A, then $(b-a)A$ has the same sort of privileged positio position in the abstract, and $a^+A = b^+(b-a)A$. Let us call such a variational abstract a *local family* (of propositions) *of complexity 1.* It will be the range of something associated with a predicate of degree 1, what I shall call a *selector* of degree 1. This is a function which, if you give it input b, produces $(b-a)A$. It coordinatizes its range by means of the family G of abstractors: $(b-a)A$ is the proposition in its range which has coordinate b.

Consider now the family of propositions we can get from A by varying a or b or both. This should be called a local family of complexity 2 (provided both a and b are relevant to A). Since the variation of a and b may be simultaneous (from $f(a, b)$ to $f(b, a)$ for example) the description cannot look as simple as a^+b^+A (see the Simultaneous Variation Theorem 7-2). When X is a set of propositions a^+X will be the set $\{(b-a)A : A \in X$ and $b \in G\}$. Now we can define:

(10-1) Class X of propositions is a *local family of complexity n* exactly if there exists a proposition A and distinct abstractors $a_1, \ldots, a_n$ all relevant to A such that $X = \cup\{e_1^+ \ldots e_n^+ a_n^+ \ldots a_1^+(A) : e_1, \ldots, e_n$ distinct and irrelevant to $A\}$.

If X is as described, a typical member looks like this, with $e_1, \ldots, e_n$ distinct from $a_1, \ldots, a_n$ (because of the irrelevance):

(10-2) $(b_1-e_1)\ldots(b_n-e_n)(c_n-a_n)\ldots(c_1-a_1)A = B.$

In this case we shall say that A *determines* X (or, is a *determinant* of X) *via* $a_1, \ldots, a_n$. When at least one abstractor is irrelevant to A, I shall call $\{A\}$ a local family of complexity *zero*. Finally, it will be clear that the existence of a local family of complexity n requires G to have at least $2n$ members.

Let us now take a look at the selectors, whose ranges these local families are meant to be. A selector f of degree 3, for example, will map triples of abstractors in G into propositions; I shall write $f(a, b, c)$ for its value at triple (a, b, c). We then want the proposition $f(a', b, c)$ to be $(a'-a)f(a, b, c)$ but more generally, $f(a', b', c')$ the simultaneous variation of a, b, c to a', b', c'. Of course the triple (a', b', c') itself is produced from (a, b, c) by a simple pointwise *replacement*, that is, a function σ of G into itself for which we define $\sigma((a, b, c)) = (\sigma(a), \sigma(b), \sigma(c))$. So if s is an n-ary sequence of abstractors, and f a selector of degree n, we want $f(\sigma s)$ to be produced from $f(s)$ by a simultaneous variation which 'duplicates' replacement σ.

(10-4) Function f of G^n into L is a *selector of degree n* exactly if there exists a sequence $s = (a_1, \ldots, a_n)$ in G^n (an *origin* of f) such that $a_1, \ldots, a_n$ are all distinct and for all replacements σ on G, $f(\sigma s) = \sigma^s f(s)$.

If not all members of s are relevant to $f(s)$ then f will have the same range as (and be definable from) a selector of lesser degree (see below). The equation at the end of (10-4) is stipulated to hold only if both sides are defined.

(10-5) *Selector Abstraction Definition.* If $a_1, \ldots, a_n$ are all distinct then $\lambda a_1 \ldots a_n . A$ is the selector f of degree n with origin $(a_1, \ldots, a_n)$ such that $f(a_1, \ldots, a_n) = A$.

The definitions are such that the existence of selectors in general is obvious, and the existence of $\lambda a_1 \ldots a_n A$ also quickly established under suitable conditions. The claim that a selector coordinates its range, a local family of related complexity, depends on the relative completeness of our theory of variations.

(10-6) *Selector Abstraction.* If $a_1, \ldots, a_n$ are distinct and relevant to A, and there are at least $2n$ distinct parameters irrelevant to A, then $\lambda a_1 \ldots a_n A$ exists, and its range is the local family of complexity n determined by A via $a_1, \ldots, a_n$.

If we define the function $f(s) = A$ and $f(\sigma s) = \sigma^s f(s)$, with $s = (a_1, \ldots, a_n)$, and the assumptions of the theorem hold, then f is well-defined on the whole of G^n, into L. Hence f is selector with origin $(a_1, \ldots, a_n)$, and by (10-5) this is $\lambda a_1 \ldots a_n A$. The local family in question is

$$X = \cup\{e_1^+ \ldots e_n^+ a_n^+ \ldots a_1^+ A : e_1, \ldots, e_n \text{ distinct and irrelevant to } A\}.$$

To begin, $f(\sigma s) = \sigma^* f(s) = (\sigma s(1) - e_1) \ldots (\sigma s(n) - e_n)(e_n - a_n) \ldots (e_1 - a_1) f(s)$ and $f(s) = A$, so this has form (10-2) and belongs to X. Hence the range of f is included in X. Secondly, suppose B is as described in (10-2). We have to show that $B = \sigma^s A$ for some replacement σ; then $B = f(\sigma s)$ and so in the range of f. Let us write

$$\alpha = (c_n - a_n) \ldots (c_1 - a_1)$$
$$\beta = (b_1 - e_1) \ldots (b_n - e_n).$$

Note that because of their irrelevance, $e_1, \ldots, e_n$ are distinct from $a_1, \ldots, a_n$.

The clue to what $\beta\alpha A$ is, is that the e_i can have become relevant to αA only if there are some of them among the c_j. Hence we proceed as follows: Check $(b_n - e_n)$. If $e_n \neq c_j$ for any j, then e_n is irrelevant to αA, and so by an Irrelevance Condition, $(b_n - e_n)$ may be discarded. If $e_n = e_j$ for some j, move $(b_n - e_n)$ in $\beta\alpha$ to the right until it reaches the first such case, by Special and Triangle Commutation, so that $\beta\alpha$ has been turned into

$$\ldots (b_n - e_n)(e_n - a_j)(c_{j-1} - a_{j-1}) \ldots (c_1 - a_1).$$

By Triangle Equality $(b_n - e_n)(e_n - a_j)$ becomes $(b_n - e_n)(b_n - a_j)$, which commutes, so we move $(b_n - e_n)$ still further to the right, repeating this process. Finally $(b_n - e_n)$ is at the very far right; now e_n is clearly irrelevant to A, and so at this point $(b_n - e_n)$ may be discarded altogether. To put it briefly: $(b_n - e_n)$ may be moved to the right, turning each c_j which equals e_n into b_n, and then finally discarded. Now we have a new expression:

$$\beta_1 \alpha_1 = (b_1 - e_1) \ldots (b_{n-1} - e_{n-1})(c_n^1 - a_n) \ldots (c_1^1 - a_1).$$

We simply repeat the whole process with $(b_{n-1} - e_{n-1})$, and so forth. Thus in n moves, we have come to

$$\alpha_n = (c_n^n - a_n) \ldots (c_1^n - a_1)$$

That $\alpha_n A = \sigma^{(a_1, \ldots, a_1)} A$ for some σ follows now by Variation Conversion given that $a_1, \ldots, a_n$ are relevant to A and there exist at least $2n$ parameters irrelevant to A, hence at least n such which are distinct from $c_1^n, \ldots, c_n^n$ as well. We conclude that X is the range of f.

(10-7) *Selector Collapse.* If f is a selector of degree n and origin $s = (a_1, \ldots, a_n)$ and a_i is irrelevant to $f(s)$, then there exists a selector of degree $n - 1$ with the same range as f.

It is not difficult to prove in addition a sort of converse to (10-7): new selectors of higher degree can be manufactured by selector abstraction using irrelevant parameters.

To prove this theorem note that the existence of f entails that for each $b_1, \ldots, b_n$ there exist parameters $e_1, \ldots, e_n$ irrelevant to $f(a_1 \ldots, a_n)$ and distinct from $a_1, \ldots, a_n, b_1 \ldots, b_n$. Since possibly $b_i = e_i$, there exist at least $2n$ parameters irrelevant to $f(a_1, \ldots, a_n) = f(s)$. Assume that a_i is irrelevant to it as well, and define f' to be the selector of degree $n - 1$ with origin $s_0 = (a_1, \ldots, a_i, a_{i+1}, \ldots, a_n)$ which has value $f(s)$ at that origin. Because of the availability of irrelevant parameters, the conditions defining f' are well-defined themselves, so f' exists. We claim now that it has the same range as f. This will clearly be the case if for all σ, $\sigma^s f'(s_0) = \sigma^s f(s)$, i.e. $\sigma^{s_0} f(s) = \sigma^s f(s)$. Hence we need the lemma:

(10-71) If a_i is irrelevant to A; $e_1, \ldots, e_n$ are distinct and irrelevant to A; $a_1, \ldots, a_n$ are mutually distinct; and $e_1, \ldots, e_n$ are distinct from $a_1, \ldots, a_n, b_1, \ldots, b_n$ then $(b_1 - e_1) \ldots (b_n - e_n)(e_n - a_n) \ldots (e_1 - a_1)A = (b_1 - e_1) \ldots (b_{i-1} - e_{i-1})(b_{i+1} - e_{i+1}) \ldots (b_n - e_n)(e_n - a_n) \ldots (e_{i+1} - a_{i+1})(e_{i-1} - a_{i-1}) \ldots (e_1 - a_1)A$.

Given the distinctness of the parameters and initial irrelevance, a_i is still irrelevant to $(e_{i-1} - a_{i-1}) \ldots (e_1 - a_1)A$; hence $(e_i - a_i)$ can be discarded without effect by an Irrelevance Condition. Having done that we note that e_i is distinct from all the other parameters involved and initially irrelevant to A. Hence it is still irrelevant to

$$(b_{i+1} - e_{i+1}) \ldots (b_n - e_n)(e_n - a_n) \ldots (e_{i+1} - a_{i+1})(e_{i-1} - a_{i-1}) \ldots (e_1 - a_1)A$$

and so $(b_i - e_i)$ can also be discarded. Thus our conclusion follows.

These results establish the main claims concerning coordinatization of local families of propositions by selectors. Note in passing that if the conjecture at the end of Section 7 is correct then, if f is a selector, $f(\sigma s) = \sigma^s(f(s))$ for all sequences s, in its domain, not just its designated origin. (In that case, every sequence of the right length is an origin of f.) The dependence of these results on the availability of a sufficient number of irrelevant parameters (correlated abstractors) is worrying in the absence of embedding results, but there is not much point in proving them at our present level of generality.

11. FIRST-ORDER SYNTAX WITH IDENTITY

Let us consider a syntax with connectives &, $\supset$, predicates F of various finite degrees, sentence constants $t, f, p, q, \ldots$; individual constants $\bar{a}$, $\bar{b}$, $\bar{c}$, ...; variables (for binding) $x, y, z, \ldots$; universal quantifiers (x), (y), (z), ...; special identity predicate =. Given a logic with a correlated abstractor system G (with correlator G^*) and ply operator $\supset$, we hope to interpret the syntax by assigning each well-formed expression E a semantic value $|E|$, as follows.

(11-1) $|t| = K$; $|f| = \Lambda$; $|p| \in L$; ...

(11-2) $|A \,\&\, B| = |A| \wedge |B|$; $|A \supset B| = |A| \supset |B|$

(11-3) $|\bar{a}|$ is a member of G (abbreviate $|\bar{a}|$ to a)

(11-4) $|F|$ is a selector on G with the same degree as predicate F, which is origin-normal (see (11-8) below)

(11-5) $|\bar{a} = \bar{b}| = Iab$

(11-6) $|F\bar{a}_1 \ldots \bar{a}_n| = |F|(a_1, \ldots, a_n)$

(11-7) $|(x)(x/\bar{a})A| = \forall_a |A|$.

With respect to (11-7) a consistency question arises: for $(x)(x/\bar{a})F\bar{a}$ is the same expression as $(x)(x/\bar{b})F\bar{b}$, so we need to be guaranteed that $\forall_a |F\bar{a}| = \forall_b |F\bar{b}|$. Related to this is the question what the semantic value of, say $(x)(x{=}\bar{a} \supset \bar{a}{=}x)$ is; obviously there will in general exist a selector i such that

$i(a, b) = Iab$. The following definition and two simple results, in addition to all our general results on variation, will bear on this:

(11-8) Selector f with origin $s = (a_1, \ldots, a_n)$ is *origin normal* exactly if at most $a_1, \ldots, a_n$ are relevant to $f(s)$.

(11-9) If a, b, c are distinct and f is an origin normal selector of degree 2 then $g^a f(a, b) = g^c f(c, b)$ and $\forall_a f(a, b) = \forall_c f(c, b)$.

(11-10) If a and b are distinct, then $g^a Iab = Ig^b$.

Note that (11-10) entails also the special case of (11-9) with $f(a, b) = Iab$.

To prove these, suppose without real sacrifice of generality that (e, b) is the origin of f in (11-8); then $g^a f(a, b) = g^a(a-e)f(e, b) = g^a g^e f(e, b) = g^e f(e, b)$ because, unless $a{=}e$, it is not relevant to $f(e, b)$, by origin normality, hence not relevant to $g^e f(e, b)$. Secondly, we prove for (11-10) as first lemma:

$$\begin{aligned}(11\text{-}11)\quad g^a Iab &= g^a \bigwedge\{A \equiv (a-b)A : A \in L\}\\ &= \bigwedge\{g^a A \equiv g^a(a-b)A : A \in L\}\\ &= \bigwedge\{g^a A \equiv g^a g^b A : A \in L\}\\ &= g^a \bigwedge\{A \equiv g^b A : A \in L\}\\ &= g^a I g^b\end{aligned}$$

whence the conclusion follows via the second lemma:

(11-12) If a and b are distinct then a is irrelevant to Ig^b.

Assume $a \neq b$ and suppose $Ig^b \leqslant B$, hence $g^b B = K$. Then $g^b g^a B = g^a g^b B = g^a K = K$, so $Ig^b \leqslant g^a B$. We conclude specifically that $Ig^b \leqslant g^a Ig^b$. Secondly, suppose $Ig^b \leqslant A$, so $g^b A = K$; but $g^b \forall_a A = g^b \bigwedge\{g_1^a A : g_1 \in G^*\} = \bigwedge\{g^b g_1^a A : g_1 \in G^*\} = \bigwedge\{g_1^a g^b A : g_1 \in G^*\} = \bigwedge\{g_1^a K : g_1 \in G^*\} = K$; hence $Ig^b \leqslant \forall_a A$. But since g^a is a transformation, it follows then that $g^a Ig^b \leqslant g^a \forall_a A = \forall_a A \leqslant (a-a)A = A$. We conclude specifically that $g^a Ig^b \leqslant Ig^b$. Combining the two conclusions, we find that Ig^b is a-invariant.

As an illustration consider the sentence $(x)(x = \bar{b} \supset Fx)$. Provided $\bar{b}$ and $\bar{a}$ are distinct constants, this sentence is identical with $(x)(x/\bar{a})(\bar{a} = \bar{b} \supset F\bar{a})$. Upon interpretation it has semantic value $\forall a(Iab \supset f(a)) = \bigwedge\{g^a(Iab \supset f(a)) : g \in G^*\} = \bigwedge\{Ig^b \supset g^a f(a) : g \in G^*\}$ via our just obtained results. In view of (11-9), we could abbreviate "$g^a f(a)$", "$g^b f(b)$", etc. uniformly to "$f(g)$". So it would not be unnatural to introduce yet one more picture (incompatible but complimentary to the abstractors and parameters pictures) in which the variables $x, y, \ldots$ stand for the functions g in the correlator G^*. At the moment our interpretation of the syntax does not extend to formulas with free variables; but this suggestion would lead to the interpretation of $(x = \bar{b})$ as Ig^b, $(y = \bar{b})$ as Ig_1^b, Fz as $g_2^c f(c)$, or more perspicuously $f(g_2)$; and so forth. It is clear that if we did want to give some interpretation to free variables, it would be quite different from that for names. It should also be remarked that different names could be assigned the same abstractor $(|\bar{a}| = |\bar{b}|)$, in which case the corresponding identity statement would have K as semantic value. But the leeway provided by the present theory is that different names can be assigned different semantic values without prejudice to the truth value of the identity statement. The fact that it is true (perhaps even necessary) but not *a priori* that Cicero is Tully would be reflected in the fact that the filter of true propositions in L includes (I |"Cicero"| |"Tully"|), although |"Cicero"| and |"Tully"| are different abstractors, so that the identity proposition would still not have the status of the *a priori*.

APPENDIX: TRANSFORMATION SEMANTICS AT AN INTERMEDIATE STAGE OF ABSTRACTNESS

Models and model structures encountered in familiar semantic analyses of modal logic and of classical and intuitionistic quantifier logic, furnish examples of the logics (i.e. proposition algebras) with abstractors studied in this paper. These examples are not simplified by looking upon them this way, but they guarantee a certain kind of completeness for us.

A1. *Normal Modal Logic.* This example can also be found, with some elaboration, in my *Formal Semantics and Logic*, Ch. V, section 2b.

An alethic possible world model structure (briefly, ms) is a couple $M = \langle K, R\rangle$ with K a non-empty set (the *worlds*) and R a reflexive relation on K (the *access* or *relative possibility* relation). A *proposition* of M is any set of worlds, hence the propositions form the complete lattice (Boolean algebra) of subsets of K. This 'logic' (of propositions in M) has the special operations $\sim A = K - A$; $A \supset B = \sim A \cup B$; $\Box A = \{\alpha \in K : R(\alpha) \subseteq A\}$ where $R(\alpha)$, the access sphere of α, is the set $\{\beta \in K : \alpha R \beta\}$. Conjunction and disjunction of propositions are of course set intersection and union.

Let us now see if there exists an abstractor m on this logic such that $\Box A = \forall_m A$. First of all define a *point transformation* on K to be a function f of K into K such that $\alpha Rf(\alpha)$ for all α in K. Because R is reflexive, it follows that $R(\alpha) = \{f(\alpha) : f \text{ is a point transformation on } K\}$.

For each point transformation f define the function f^* on propositions by the condition $f^*A = \{\alpha \in K : f(\alpha) \in A\}$. Then f^* is a transformation of the logic, in our sense, because if $A \subseteq B$ then $f^*A \subseteq f^*B$; $f^*(\cap X) = \cap\{f^*A : A \in X\}$ and $f^*(\cup X) = \cup\{f^*A : A \in X\}$. In addition $f^*(\sim A) = \{\alpha \in K : f(\alpha) \notin A\} = K - f^*A = \sim f^*A$; and hence also $f^*(A \supset B) = f^*A \supset f^*B$, so these transformations are also ply-normal.

Let the abstraction m be the set of all these functions f^*. Then $\forall_m A = \bigwedge\{f^*A : f \text{ a point transformation}\}$. Hence $\alpha \in \forall_m A$ iff for each f, $\alpha \in f^*A$, i.e. for each f, $f(\alpha) \in A$, and therefore, by our previous remarks, iff $R(\alpha) \subseteq A$. Hence $\forall_m A = \Box A$ as claimed.

A2. *Algebraic Treatment of Quantification over a Domain.* Here I shall consider a semantic analysis of quantifiers in logics, such as classical and intuitionistic logic, whose models can be regarded as complete lattices of propositions, but utilizing variables ranging over a universe or domain of discourse.

We begin with a complete lattice L_0 of *simple propositions.* In the classical case, this is a Boolean algebra, in the intuitionistic case a Heyting algebra. Secondly we specify a domain D (a non-empty set), thirdly a set VAR of variables (intuitively, they take values in D). Now the propositions are the family L of functions which map D^{VAR} into L_0. The explanation of this scheme in the classical case is that a sequence or function s which gives value $s(b)$ to variable b in domain D, may *satisfy* or not satisfy a proposition A. In this case $L_0 = \{T, F\}$ and "s satisfies A" is equated with "$A(s) = T$". The simple propositions form the two-element Boolean algebra in this case. Generalizing this, we think of proposition A as an entity dependent on certain variables; when the values of those variables are fixed by s, then A becomes the simple proposition $A(s)$. Thus a simple proposition is one which is not dependent on what values the variables take. In the intuitionistic case, with Kripke's semantic analysis, L_0 is the family of all sets X of worlds such that, if $\alpha \in X$ and $\alpha R\beta$, then β is also in X (R-closed set). When R is reflexive and transitive, those R-closed sets form a Heyting algebra.

What operations are there on L? To begin with we can 'lift' operations from L_0 to L. Thus we set $(\bigwedge A_i)(s) = \bigwedge A_i(s)$ and $(\bigvee A_i)(s) = \bigvee A_i(s)$ and $A \leqslant B$ iff $A(s) \leqslant B(s)$ for all s. Next, if L_0 has an n-ary operation φ (such as $\sim$ or $\supset$) we define similarly $\varphi(A_1, \ldots, A_n)(s) = \varphi(A_1(s), \ldots, A_n(s))$. In this way L receives a structure which is the same as L_0 in all those respects that affect logic. (So if a given deductive logic was sound and complete with respect to L_0 – or to class $\{L_0^1, L_0^2, \ldots\}$ – it will also be sound and complete with respect to L – or respectively, to class $\{L^1, L^2, \ldots\}$.

Next we can introduce special operations on L, namely quantifiers. The most obvious one is the universal quantifier: for $a \in \mathrm{VAR}$ we define

$$(\forall aA)(s) = \bigwedge\{A(e \parallel a)s : e \in D\}$$

where $(e \parallel a)s = s^1$ iff $s^1(a) = e$ and $s^1(b) = s(b)$ for all variables b other than a. But now we also define an operation on propositions:

$$((e \mid a)A)(s) = A((e \mid a)s)$$

and define the abstractor $\hat{a} = \{(e \mid a) : e \in D\}$, whereupon the universal quantifier can be redefined in two steps:

$$\hat{a}A = \{(e \mid a)A : e \in D\} \qquad (\forall aA) = \bigwedge \hat{a}A.$$

It may be noted at once, in the terminology of the body of this paper, that a is a destructive abstractor and if $a \neq b$ then a does not interfere with b. To study their correlation, we introduce the function:

$$\binom{a}{b}s = s^1 \text{ iff } s^1(b) = s^1(a) \text{ and } s^1(c) = s(c) \text{ for } c \neq b.$$

$$((a-b)A)(s) = A(\binom{a}{b}s)$$

It is easy to see that $(e \mid a)(a-b)A(s) = A(\binom{a}{b}(e \parallel a)s)$ and that the sequence $\binom{a}{b}(e \parallel a)s$ is just the result of changing both $s(a)$ and $s(b)$ to e. Hence we have

$$(e \mid a)(a-b) = (e \mid b)(b-a) = (e \mid a)(e \mid b)$$

which is the required instance of postulate IV*. Verification that both $(e-a)$ and $(a-b)$ are transformations which preserve all the structure of L which was lifted from L_0, is routine.

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[1] Probabilistic semantics provides probably the best worked out examples of a subjective semantics. See Van Fraassen (1981) and references therein.

[2] Restricted and other generalized quantifiers need to be explored, and also higher order quantification; it appears so far that the present treatment can provide a natural setting for their study.

[3] In this section, to prove the Methodological Lemma, I use the assumption that there exists an operator $\supset$ such that $A \supset B = K$ if and only if $A \leqslant B$. In discussion with Nuel Belnap it became clear that a weaker assumption suffices: that there exists in L a proposition T (intuitively, the truth, the logically strongest true proposition) such that $T \leqslant A \supset B$ if and only if $A \leqslant B$. This bears on the question whether relevance logics can be accommodated within this approach.

[4] Substitution (of which variation is the algebraic counterpart here) has always been a difficult subject, which one is content to explore no further than necessary for the purpose at hand. For various approaches, see Curry, Halmos (especially pp. 104–108), and Van Fraassen (1971; especially Ch. 2 section 2 and Ch. 4, sections 1 and 5).

[5] The reader is requested to refrain *pro tem* from any opinions as to what a proposition *is*. The notion of a 'general' object 'dependent' on parameters does not have any status in the standard ontology of today. Compare Locke's 'general' triangle which 'becomes' a 'specific' triangle when two sides and their inscribed angle are specified, or the lizard, which is to be found in the Sahara and the Gobi, though no single lizard is.

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