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ELIMINATIVE INDUCTION AS A METHOD OF DISCOVERY: HOW EINSTEIN DISCOVERED GENERAL RELATIVITY

One who has himself poked about so much in the chaos of possibilities can understand very well your fate. You haven't the faintest idea what I, as a mathematical ignoramus, had to go through until I entered this harbor.[1]

1. INTRODUCTION

So began Einstein's weary and consoling response of August 1915 to his correspondent Paul Hertz on a yet another proposal concerning the troubled and still incomplete general theory of relativity. Einstein had been working on the theory for eight years and within a few months would overcome his final obstacles, bringing to completion his greatest scientific achievement.[2] My concern in this paper is to establish two theses about Einstein's discovery of his general theory of relativity. The first concerns the heuristic methods he used to navigate the "chaos of possibilities"; the second concerns an important moral Einstein, the "mathematical ignoramus," drew from the experience:

1. In broad outline, Einstein discovered the theory through a sequence of eliminative inductions in which empirically based generalizations were used to eliminate theories from a universe of candidate theories with the goal of converging onto a unique theory.
2. Einstein's later and much celebrated fascination with a canon of mathematical simplicity in the quest for fundamental physical laws was derived in significant measure from his experience with the discovery of general relativity.

The first thesis describes what I shall call an "eliminative model of scientific discovery" and its treatment will attract the bulk of my efforts in this paper.

1.1. *On Eliminative Induction*

I shall construe eliminative inductions broadly as arguments with premises of two types:

(a) premises that define a universe of theories or hypotheses, one of which is posited as true; and

(b) premises that enable the elimination of members of this universe by either deductive or inductive inference. (These are called "eliminative principles" below.)

Jarrett Leplin (ed.), The Creation of Ideas in Physics, 29–69.

The goal of inference in an eliminative induction is to converge on the true theory in the universe specified by (a). However we shall see a case in which the elimination is too thorough and the premises of (b) enable elimination of all the theories of (a); we shall also see a case in which more than one of the theories of (a) remain after the elimination has proceeded.

Eliminative induction has long been a recognized form of inference. In recent years it has attracted little attention in the literature, a neglect which has been persuasively denounced recently by John Earman.[3] Mill's canons provide some of the best known examples of eliminative induction. He labelled them "methods of elimination" since they are intended to enable one to eliminate all but the true causes out of the range of possible causes for a given phenomenon.[4] A tradition of work in eliminative induction has amplified the basic methods Mill laid out.[5] It is easy to underestimate the power of eliminative induction, especially when so many of its examples are fairly unimpressive disjunctive syllogisms in which one is faced with the relatively easy task of eliminating all but one of a small number of candidate hypotheses. The true power of the argument form emerges, as we shall see below, when one considers cases in which the universe of theories or hypotheses in question is infinitely large, containing almost every conceivable possibility, and the elimination of theories is not effected singly but "wholesale" in infinite sets.

Although they bear the name "induction" in the literature eliminative inductions need not be ampliative. For example, an eliminative induction that has the form of a disjunctive syllogism is a deductively valid argument. However ampliative inferences usually wait in the wings. The establishment of the premises of an eliminative induction that is also a disjunctive syllogism will usually be ampliative or, in more complicated cases, the actual eliminations may be carried out by ampliative inductions. Since eliminative inductions are often demonstrative (i.e. non-ampliative), they are closely associated with so-called "demonstrative induction" and commonly cited examples of both could qualify as either. In demonstrative induction, premises of greater generality are combined deductively with premises of lesser generality to yield a conclusion of intermediate generality.[6] In an important sequence of papers, Jon Dorling has shown that demonstrative induction has played on important role in the history of theoretical physics, including the work of Einstein.[7]

1.2. *Amplification on the Theses*

The first thesis of this paper requires several amplifications: *The bulk of theories in the universe of theories of Einstein's eliminative inductions were unarticulated and remained so.*

That is, that universe simply consisted of the set of theories that Einstein could have chosen, or, as he put it above, his "chaos of possibilities".

Most of them were not chosen and typically not even formulated explicitly by him. To anticipate an objection, I stress that the thesis makes no more presumptions about the existence of some Platonic world of theories forming this universe than does everyday talk of unrealized choices or possibilities. We shall see that it is quite straightforward to define, even if loosely, a universe of theories and proceed to eliminate all but a few of them without ever articulating the bulk of the theories.[8]

As the eliminative induction proceeds, however, Einstein retains a smaller and smaller subset of uneliminated theories that are given more and more complete articulation by the actual process of elimination until the induction, if successful, concludes with the full articulation of the final theory. That is: *The carrying out of the eliminative induction is also the actual construction of the final theory.*

Further, since this induction is a rational process and, at the same time, a justification of the theory, we have: *The generation of the theory proceeded hand in hand with the development of its justification.*

Thus Einstein's later expositions of the theory often contains a recapitulation of steps taken in its discovery, offered as a partial justification of the theory for the reader. In particular, the eliminative principles – the "(b)" premises above – that were used in the generation of general relativity included the principles of general covariance and equivalence as well as the requirements of conservation of energy and momentum and of the appropriate Newtonian and special relativistic limit. With the discovery of the theory completed, these principles were retained for the theory's justification and took their place in the axiomatic foundations of Einstein's standard expositions of the theory. Finally: *With the possible exception of the principle of general covariance, these eliminative principles were empirically based.*

Thus the discovery process and the justification it spawned have substantial empirical foundations.

The second thesis affirms that Einstein, who had denounced the *a priori* in physics, did not himself pluck his later insistence on the decisive importance of mathematical simplicity from the "Olympus of the *a priori*."[9] Rather he derived the heuristic in the manner one would expect of any good empiricist, from experiences in scientific discovery. We shall see that prior to concluding his work on the theory he was indifferent or even hostile to such a canon, but he came to realize that adherence to this canon would have accelerated greatly his completion of the theory when his usual direct physical analysis actually turned out to be more of a hindrance.

1.3. *Preview*

The first thesis suggests that Einstein's actual process of discovery, at least at a broad level, admits quite simple, rational and even mechanical characterization. The contrary view that scientific discovery is not susceptible to logical analysis is very common. Popper, for example, advances it under the banner "Elimination of Psychologism" and quotes Einstein's own remarks for support.[10] Since Einstein's own remarks on scientific discovery are often cited as support of such views as Popper's, I shall briefly review in the following section what Einstein does say about the matter. I shall urge that Einstein's remarks do rule out mechanical characterization of a particular aspect of scientific discovery, but that they do not rule out and even invite the eliminative model of his own discovery process given in the first thesis above. In Section 3, I shall review Einstein's discovery of special relativity and argue that his distinction of constructive theories from theories of principle arose as a part of an application of the eliminative model and that this discovery was one of Einstein's early successes with the eliminative model.

In Sections 4, 5 and 6, I turn to general relativity and review three of the major decision points in its discovery, characterizing each as an attempt at theory construction by eliminative induction. The first concerns the starting point of Einstein's work on general relativity, his speculation in 1907 on how one might modify gravitation theory to bring it into accord with his 1905 special theory of relativity. This work led to the striking conclusion that no special relativistic theory of gravitation was acceptable, that a new theory of space, time and gravitation was needed and that this theory would extend the principle of relativity to accelerated motion. The second major decision point concerned the basic question of how gravitation was to be represented in the new theory. In the theory of static gravitational fields that Einstein developed in 1907 to 1912, he concluded that gravitation was to be represented by a variable speed of light. In 1912 and 1913 he combined the resulting program of work on gravitation with the four dimensional methods introduced to relativity theory by Hermann Minkowski five years before. With the assistance of his mathematician friend Marcel Grossmann, he arrived at essentially the complete general theory of relativity in which gravitation was represented by the metric of spacetime itself. The third decision point concerns the new theory's gravitational field equations. The equations Einstein and Grossmann constructed in 1913 were not generally covariant and Einstein even came to believe that generally covariant field equations would be physically unacceptable. Einstein struggled for nearly three more years with this problem until he returned to general covariance and brought the theory to its essentially final form in November 1915.

The example of Einstein's generation of these field equations enables us to address a question concerning Einstein's procedures. Are the eliminative inductions I describe merely clever devices that happen to solve the problems which they address? Or are they applications of genuine methods? I shall take the distinction between these two options to be that a method supplies explicitly identifiable procedures that can be used to solve a range of problems, whereas a device can be used only in the one case in which it arises. At least in the instance of these field equations, we shall see that Einstein clearly thought his procedure was an application of a method. In fact the term "method" is his. He explains the method by showing how it could be applied to solve other problems, in this case the problem of generating the field equation of electrostatics, thereby demonstrating directly that it can solve more than one problem.

Finally, in Section 7, I reflect on the examples of eliminative induction of the preceding sections and then, in Section 8, I consider the heuristic that Einstein elected to assign a lesser importance in his search for the theory, the canon of mathematical simplicity, and show how he came to regret and reverse that decision.

2. EINSTEIN'S VIEWS ON SCIENTIFIC DISCOVERY

Of all of Einstein's pronouncements on scientific discovery, probably the best known are those that seek to deny that theories can be deduced from experience.[11] Typical of these pronouncements are the words of his 1918 address, 'Principles of Research', where he proclaims:

> The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them.[12]

Elsewhere, in his 1933 'On the Methods of Theoretical Physics', Einstein offers two related justifications for this "no logical path" claim. Excepting the constraint that the concepts and fundamental principles of a theory entail conclusions compatible with experience,

> . . . these latter [concepts and fundamental principles] are the inventions of the human intellect, which cannot be justified either by the nature of that intellect or in any other fashion *a priori*.[13]

Thus he continues on the following page to infer the erroneousness of

> . . . the idea that the fundamental concepts and postulates of physics were not in the logical sense free inventions of the human mind but could be deduced from experience by "abstraction" – that is by logical means.

This "free invention" view is in turn supported immediately by the observation that general relativity embraced an even wider range of empirical facts than Newtonian theory while using foundations quite different from those of Newtonian theory. This observation led to the general claim that

> . . . quite apart from the question of the superiority of one or the other, the fictitious character of fundamental principles is perfectly evident from the fact that we can point to two essentially different principles, both of which correspond with experience to a large extent; this proves at the same time that every attempt at logical deduction of the basic concepts and postulates of mechanics from elementary experiences is doomed to failure.

We now of course label the new claim introduced as the thesis of the underdetermination of theory by evidence or, more briefly, the underdetermination thesis.

While Einstein's views on "no logical path", "free invention" and the underdetermination thesis are widely known and often cited, there is a very significant set of qualifications to them which are less well reported. He continued the passage cited above from 'Principles of Research' by stressing that the underdetermination of theories by evidence does not arise in practice:

In this methodological uncertainty, one might suppose that there were any number of possible systems of theoretical physics all equally well justified; and this opinion is no doubt correct theoretically. But the development of physics has shown that at any given moment, out of all conceivable constructions, a single one has always proved itself decidedly superior to all the rest. Nobody who has really gone deeply into the matter will deny that in practice the world of phenomena uniquely determines the theoretical system, in spite of the fact that there is no logical bridge between phenomena and their theoretical principles;. . .

Einstein offers a similar qualification in his 1936 'Physics and Reality', where he writes:

The liberty of choice [of axioms]. however, is of a special kind; it is not in any way similar to the liberty of a writer of fiction. Rather, it is similar to that of a man engaged in solving a well-designed word puzzle. He may, it is true, propose any word as the solution; but there is only *one* word which really solves the puzzle in all its parts. It is a matter of faith that nature – as she is perceptible to our five senses – takes the character of such a well formulated puzzle. The successes reaped up to now by science do, it is true, give a certain encouragement for this faith.[14]

This latter set of views surely reflect the practical experiences of Einstein the working scientist. We shall see, for example, how his work on the special and general theories of relativity led him to quite definite theories, even if their elements were introduced as free inventions of his mind. This definiteness, the impossibility of adjustment of any of these elements, is what made the success of the general theory's prediction of the anomalous motion of Mercury so striking and brought to a triumphant close the eight years of his search for the theory.

How are we to reconcile these two groups of views? On the one hand Einstein insists that the concepts and fundamental principles of our theories are free inventions of our minds and underdetermined by experience. On the other hand, our choice of theory is actually determined by experience after all, either as a matter of practice or, more strongly, because nature is so constituted as to admit determinate theories.

The best reconciliation that I can offer of these views proceeds as follows. At any point in history, the scientist works within a universe of conceivable theories applicable to the problems at hand. The selection of the theories of this universe and the concepts and fundamental principles used to construct them, is an historically highly contingent matter, dependent on the creative thought and the conceptual and experiential resources of the scientists involved. Thus the universe of theories of space and time available to a Newton could not have included spacetime theories with variable curvature metrics. Our current universe of theories may not contain some to be conceived in the future. However once the universe of theories is defined, then nature is so constituted that experience enables selection of a single theory as the best theory from that universe. The indeterminateness however still remains in the sense that an expansion of the universe of theories may well change the theory which would be selected and, presumably, such an expansion is always possible.

For my purposes here, the crucial point of this reconciliation is this view: *once the universe of theories with attendant concepts and principles is specified. experience enables selection of a single theory as the best.* But how is experience to direct us to that theory? In his 'On the Methods of Theoretical Physics', Einstein gives one answer: "Our experience hitherto justifies us in believing that nature is the realization of the simplest conceivable mathematical ideas." (p. 274) In other words, in the universe of conceivable theories, experience directs us to prefer the mathematically simpler theory. However in his work towards general relativity, this heuristic was accorded less importance and to his cost. Instead, Einstein let experience guide him through a number of other devices. One of the most important had proved its worth in his work on special relativity.

3. THEORIES OF PRINCIPLE: THE LESSON OF THE SPECIAL THEORY OF RELATIVITY

Investigations in the history of special relativity indicate the existence of a crucial decision point in the development of the theory.[15] At this decision point, Einstein knew that the then current electrodynamics of Maxwell and Lorentz, unlike mechanics, required a preferred state of rest, but that this preferred state of rest seemed to escape all attempts at observational identification. What made the preferred state of rest all the more suspect was the fact that such escapes from observational verification seemed to be built into the deepest foundations of the theory, as Einstein's celebrated thought experiment of the magnet and conductor showed.[16] Moreover he believed that it was not feasible to modify electrodynamics to embody an emission theory of light. In such a theory, the velocity of light would depend on the motion of its source in the same way as in a mechanical-corpuscular theory, so that the need for a preferred state of rest is precluded. Finally the problem was complicated immeasurably by Einstein's knowledge from his investigation into the behavior of black body radiation and, in particular, its fluctuations, that Maxwell-Lorentz electrodynamics was not a correct theory.

At this point, Einstein may well also have suspected that the problem involved the theory of space and time and that this theory would have to be modified in a way to be revealed by electrodynamics. Such an insight however would surely have made the problem seem all the more intractable, for Einstein would have to contemplate not just the possibility of modifications to electrodynamics but also to the theory of space and time and thus other sciences such as mechanics that depended upon it. Presumably it was to the resolution of this crisis that Einstein referred in his *Autobiographical Notes*, when he wrote (p. 49):

Reflections of this type made it clear to me as long ago as shortly after 1900, i.e. shortly after Planck's trailblazing work, that neither mechanics nor electrodynamics could (except in limiting cases) claim exact validity. Gradually I despaired of the possibility of discovering the true laws by means of constructive efforts based on known facts. The longer and the more desperately I tried, the more I came to the conviction that only the discovery of a universal formal principle could lead us to assured results. The example I saw before me was thermodynamics. The general principle was there given in the theorem: The laws of nature are such that it is impossible to construct a *perpetuum mobile* (of the first and second kind). How, then, could such a universal principle be found?

His solution lay in his distinction between two types of theory. Constructive theories, as he explained elsewhere, "attempt to build up a picture of the more complex phenomena out of the materials of a relatively simple formal scheme from which they start out."[17] His example was the kinetic theory of gases. Theories of principle are of the type of thermodynamics in which the entire theory is derived logically from a few empirically discovered principles. Einstein sought to let a few empirically discovered principles determine his selection of theory and thus resolve the crisis. The principles that Einstein chose are well known. The first was the principle of relativity of inertial motion. The second was the light postulate which encapsulated the contribution of electrodynamics to the new theory. In so far as the light postulate required the independence of the velocity of light from its source, the postulate summarized Einstein's doubts over an emission theory of light as an alternative to Maxwell-Lorentz electrodynamics.[18]

These two principles lead to the new kinematics of the special theory of relativity, which completely solves Einstein's original problem of the state of rest in Maxwell-Lorentz electrodynamics. If this electrodynamics is coupled without modification with the new kinematics, the electrodynamics immediately satisfies the principle of relativity and no longer requires a preferred state of rest. However, because of its ingenious means of construction, the new kinematics is not dependent on the complete truth of that electrodynamics, but only on a tiny part of the electrodynamics which seemed robust to Einstein and was expressed by the light postulate.

It is customary to portray the kinematics of special relativity as deduced essentially from the principle of relativity and the light postulate alone. This is a seriously misleading oversimplification both logically and historically and the generation of the final theory requires a further breakthrough. Einstein stressed that the final special theory of relativity, like the principles of thermodynamics, can only be used to *eliminate* possibilities:[19]

. . . the theory of relativity in no way hands out a means of deducing hitherto unknown laws from nothing. It provides only a criterion applicable everywhere which limits the possibilities; in this regard it is comparable with the energy principle or with the second law of thermodynamics.

Correspondingly, the two principles of the new theory can only be used to eliminate possibilities in kinematical theories and, unless one has a broad enough view of what these possibilities are, the principles might well be judged as "apparently irreconcilable," as Einstein himself remarked in his 'On the Electrodynamics . . .', so that nothing can be deduced through them. Einstein continued to note in his *Autobiographical Notes*, after recalling his revelation over theories of principle, that the theory could not be completed "as long as the axiom of the absolute character of time, or of simultaneity, was rooted in the unconscious" (p. 51). One of the final breakthroughs in Einstein's discovery of special relativity seems to have been the disclosure and rejection of this axiom so that Einstein was free to contemplate the possibility of a kinematics without absolute simultaneity.

At this point, Einstein had all the ingredients of a classic eliminative induction. The final insight about simultaneity had solved one of the most difficult problems in setting up an eliminative induction. It had directed him to a universe of kinematical theories that would include theories without absolute simultaneity and so was sufficiently large for the induction to proceed. The principle of relativity and the light postulate could then be applied as eliminative principles to this universe of kinematical theories and the kinematics of special relativity recovered. The precise steps that Einstein used to effect this inference for the first time remain a matter of historical debate. We now know very many ways that this inference can be carried out. The most pertinent example is Einstein's own of 1905 in §3 of his 'On the Electrodynamics of Moving Bodies'.

In that celebrated version of the argument, the universe of kinematical theories is characterized in a very simple manner. Einstein considers the familiar inertial coordinate systems (x, y, z, t) of a space and time, where the Cartesian spatial coordinates, x, y and z, are given directly by the usual measuring operations with rigid rods and the time coordinate t by measurements with clocks. A kinematics is defined by the group of transformations relating these inertial systems. Einstein's universe of kinematical theories contains all those for which the relevant group always consists of linear equations so that, in the case of coincident origins, the transformation relating two inertial coordinate systems (x, y, z, t) and (X, Y, Z, T) is given by

$$(1)\qquad \begin{aligned} X &= \alpha_{11}x + \alpha_{12}y + \alpha_{13}z + \alpha_{14}t \\ Y &= \alpha_{21}x + \alpha_{22}y + \alpha_{23}z + \alpha_{24}t \\ Z &= \alpha_{31}x + \alpha_{32}y + \alpha_{33}z + \alpha_{34}t \\ T &= \alpha_{41}x + \alpha_{42}y + \alpha_{43}z + \alpha_{44}t \end{aligned}$$

where the coefficients α are constants. Roughly speaking, this is the kinematics of a homogeneous and isotropic space and time.[20] The principle of relativity and the light postulate are then used to eliminate all but the group of the Lorentz transformation, thus arriving at the kinematics of special relativity. In summary form, the eliminative induction is:

Universe of Theories:	Kinematics of homogeneous and isotropic spaces and times as given by (1).
Eliminative Principle:	Principle of Relativity
Eliminative Principle:	Light Postulate
Conclusion:	Special relativistic kinematics (Lorentz transformation).

Einstein's discovery of special relativity illustrates a number of important aspects of the eliminative model of scientific discovery:

- Even though the universe of theories chosen is very large, the eliminative induction leads directly and perhaps even mechanically to the choice of a definite theory.
- The carrying out of the eliminative induction automatically involves construction of the final theory.
- The eliminative principles are at least indirectly empirically based. Einstein indicates in the opening paragraphs of 'On the Electrodynamics . . .' that the principle of relativity is based on "unsuccessful attempts to discovery any motion of the earth relatively to the "light medium" as well as internal evidence from electrodynamics, a theory which is itself based on numerous experiments. Similarly, the light postulate is derived from electrodynamics.[21]
- The justification of the theory develops alongside its discovery. The two eliminative principles which governed its discovery were offered by Einstein as the axiomatic foundations of the new theory and the justification of the theory could be reduced to the justification of the axioms. This justification was already partially in place because of the care exercised in the original choice of these principles.

In the following sections, I will urge that Einstein used essentially the same eliminative method in his discovery of the general theory of relativity and that all of the above points are of importance in this latter case as well. However in the case of special relativity, the procedure was applied at one decision point. In the case of general relativity the procedure was used repeatedly at a number of decision points. The earlier instances led to unsatisfactory or incomplete results. The later instances, designed specifically to remedy these early deficiencies, were dazzling successes.

4. THE FAILURE OF GRAVITATION WITHIN SPECIAL RELATIVITY

Einstein's first decision point in what became his work on general relativity came in 1907 when he was invited to write a review article on relativity theory for the *Jahrbuch der Radioaktivität und Elektronik*. As a part of this article he hoped to show how one had to modify Newtonian gravitation theory in order to bring it into accord with special relativity. The startling conclusion of these efforts was that there was no acceptable special relativistic gravitation theory.

This conclusion was derived by means of an eliminative induction. The eliminative induction was not carried out as tightly as the one described for special relativity. In this new case, theories were not eliminated by deductive inference from the eliminative principles but by far less sure inductive inference to the simplest or the most natural choice, excluding all others. The universe of theories contained all conceivable three space gravitation theories. By this I refer to gravitation theories formulated after the standard methods current in 1907. In particular the theories are *not* spacetime theories. Space and time are represented by different manifolds – space by a three dimensional manifold and time by a one dimensional manifold.

The induction began with the application of the first eliminative principle, the requirement that special relativity hold, which led Einstein to eliminate all but field theories of gravitation as his most natural choice. In his later recollections, Einstein reconstructed his argument:

I first came a step nearer to the solution of the problem when I attempted to deal with the law of gravity within the framework of the special theory of relativity. Like most writers at the time, I tried to frame a *field-law* for gravitation, since it was no longer possible, at least in any natural way, to introduce direct action at a distance owing to the abolition of the notion of absolute simultaneity.[22]

The second eliminative principle was the requirement of compatibility with Newtonian gravitation theory in some suitable limiting case, so that the new theory could agree with Newtonian theory in the empirical domain in which Newtonian theory had been verified. The simplest way of achieving this was to ensure that the new theory had the same general form as Newtonian theory. So Einstein eliminated all but those theories patterned after Newtonian theory in which the gravitational field is represented by a scalar field potential and whose interaction with masses is governed by a field equation and a force law. He could not use the Newtonian field equation and force law because they were incompatible with the requirement of special relativity. So he sought their simplest relativistic generalization. Thus Einstein continued:

The simplest thing was, of course, to retain the Laplacian scalar potential of gravity, and to complete the equation of Poisson in an obvious way by a term differentiated with respect to time in such a way that the special theory of relativity was satisfied. The law of motion of the mass point in a gravitational field had also to be adapted to the special theory of relativity. The path was not so unmistakably marked out here, since the inert mass of a body might depend on the gravitational potential. In fact, this was to be expected on account of the principle of the inertia of energy.

If φ is the scalar gravitational potential and we adopt a standard coordinate system (x, y, z, t) of special relativity, where x, y and z are Cartesian spatial coordinates and t the time coordinate, then the transition referred to is from the Newtonian field equation

$$(1) \qquad \Delta\varphi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi = 4\pi G\rho,$$

to its Lorentz covariant extension

$$(3) \qquad \Box\varphi = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)\varphi = -4\pi G\rho,$$

where G is the universal constant of gravitation and ρ the rest mass density of matter. Thus Einstein could readily find a field law compatible with special relativity and, because of the large size of c, probably also with the requirement concerning the limiting Newtonian case. The case of the force law was not quite so straightforward. Here the Newtonian force law was

$$f_x = -m\ \partial\varphi/\partial x,$$

where f_x is the force on a body of rest mass m in the x direction and there are similar expressions for f_y and f_z. The adaptation of this law to special relativity cannot be effected merely by the insertion of the expected relativistic contraction or dilation factors. Further modifications are needed and one can choose between at least two ways in which they can be carried out. The first involves the assumption that the rest mass m varies with the gravitational potential; the second allows the addition of extra field terms to the right hand side of the equations.[23] Einstein's remarks above suggest that he followed the former route and, it would seem from the comments that followed, that it immediately led to dubious conclusions. He continued:

> These investigations, however, led to a result which raised my strong suspicions. According to classical mechanics, the vertical acceleration of a body in the vertical gravitational field is independent of the horizontal component of its velocity. Hence in such a gravitational field the vertical acceleration of a mechanical system or of its center of gravity works out independently of its internal kinetic energy. But in the theory I advanced, the acceleration of a falling body was not independent of its horizontal velocity or the internal energy of the system.
>
> This did not fit with the old experimental fact that all bodies have the same acceleration in a gravitational field. This law, which may also be formulated as the law of equality of inertial and gravitational mass, was now brought home to me in all its significance.

The result that Einstein outlines holds whether one modifies the force law in the first or second way outlined above. Take the case of a gravitational field whose gradient is non-vanishing only in the the x-direction, the "vertical" direction. The vertical x-component of the gravitational force acting on a body with velocity v but, at that instant, vanishing vertical velocity, is:

$$f_x = -m\sqrt{1 - v^2/c^2}\ \partial\phi/\partial x,$$

Thus the greater the horizontal velocity v, the smaller the vertical gravitational force on the body. It now follows that a spinning body or a gas – any body whose internal energy is in part due to the kinetic energy of motion in the horizontal direction – will fall slower than a simple body with no horizontal motion. As Einstein notes, this result is incompatible with the equality of inertial and gravitational mass.

This equality was introduced as the third eliminative principle of the induction which now yielded a quite definite result: all the theories in the universe of theories in question had been eliminated. Einstein's further work on gravitation presumed that there could be no acceptable special relativistic gravitation theory. To summarize the argument:

Universe of Theories:	Three space gravitation theories.
Eliminative Principle:	Special relativity holds.
Eliminative Principle:	Newtonian gravitation theory holds in a suitable limiting case.
Intermediate Conclusion:	In the uneliminated theories, the acceleration of fall of a body depends on its horizontal velocity or internal energy.
Eliminative Principle:	Equality of inertial and gravitational mass: the acceleration of fall of a body is independent of its horizontal velocity or internal energy.
Conclusion:	All theories are eliminated.

The heuristic path that Einstein followed to this conclusion was soon converted into its justification. This conversion can be seen clearly in a vitriolic dispute in which Einstein engaged with Max Abraham in 1912 over Einstein's new work in gravitation theory. To justify his conclusion, Einstein allowed that gravitation could be represented in the then current theory of special relativity either as a "four-vector" or a "six-vector", using the terminology then current.[24] The scalar field of his 1907 speculations was an instance of the four-vector case; its spacetime gradient formed the four vector in question. The six-vector case corresponded to gravitation theories modelled after Maxwell electrodynamics; the Maxwell field tensor is a second rank antisymmetric tensor with six independent components and was then called a six-vector. In either case, Einstein urged, extending the compass of this 1907 reasoning, the transformational behavior of these quantities would lead to a violation of equality of inertial and gravitational mass.[25]

5. THE INCORPORATION OF GRAVITATION INTO THE STRUCTURE OF SPACE AND TIME

5.1. *The Scalar Theory of Static Gravitational Fields of 1907–1912*

If Einstein could not have a special relativistic theory of gravitation, he was certainly not going to give up on the problem of gravitation and relativity. He expanded his compass to include gravitation theories not compatible with special relativity except in the limiting case of a vanishing gravitational field. Once again, Einstein faced an overwhelming diversity of possible theories. From this diversity, he plucked a theory in which the now variable speed of light c was to represent the scalar gravitational potential. This was the theory that he unveiled in the final Part V of the 1907 review article whose invitation had triggered the entire investigation.[26] The theory was developed further in a sequence of papers published in 1911 and 1912.[27] The 1911 paper is the best known of the series through its republication in the volume *The Principle of Relativity*,[28] but the 1912 papers give a far more fully developed version of the theory.

Einstein arrived at this theory by means of an eliminative induction. That induction was a modification of his original eliminative induction which had yielded no theories at all. He retained the same universe of all conceivable three space gravitation theories. However he weakened the first eliminative principle, which required satisfaction of special relativity, to require just the satisfaction of special relativity in the limiting case of a vanishing gravitational field. In order to ensure convergence of the induction to a definite theory, he needed to match this weakening with a strengthening elsewhere. He focussed on the third eliminative principle. This was the remarkable fact of experience that he felt largely responsible for the failure of special relativistic gravitation theories, the equality of inertial and gravitational mass. Convinced of its decisive importance, he strengthened and generalized it to the eliminative principle that would dominate the construction of the new theory. As he continued to explain in his 'Notes on the Origin of the General Theory of Relativity' (p. 287):

> The principle of the equality of inertial and gravitational mass could now be formulated quite clearly as follows: In a homogeneous gravitational field all motions take place in the same way as in the absence of a gravitational field in relation to a uniformly accelerated coordinate system. If this principle held good for any events whatever (the "principle of equivalence"), this was an indication that the principle of relativity needed to be extended to coordinate systems in non-uniform motion with respect to each other, if we were to reach a natural theory of the gravitational fields.

This principle of equivalence became the launching point for his 1907–1912 theory and its satisfaction would clearly ensure that the weaker equality of inertial and gravitational mass would also be satisfied automatically.[29]

The principle of equivalence provided an immediate answer to the question of what structures are to represent gravitation in the new theory.

The customary answer up to that time had been that one must define further structures in space and time, such as a scalar field, and these structures would represent the gravitational field. The principle of equivalence forced a completely novel answer to the question: the supposedly gravitation free space and time of special relativity *already* contained the structures needed to represent gravitation. For the principle asserted that the case of a uniformly accelerating frame in the supposedly gravitation free spaces of special relativity was in fact fully equivalent to – I would say, "just the same as" – the presence of a homogeneous gravitational field in an unaccelerated frame. Thus whatever structures were already present in special relativity as a theory of space and time had also to be able to represent this special case of a gravitational field, the homogeneous field, and thus presumably more general cases as well.

The principle of equivalence enabled Einstein to eliminate virtually all the theories of the new universe in favor of those that required only the structures of space and time already in special relativity. What remained was the task of determining which structures of space and time were to represent the gravitational field and what were their properties. It was here that he could apply the remaining eliminative principle, the requirement of an appropriate Newtonian limit. In the uniformly accelerating frame, free bodies move with uniform acceleration in a given direction. This acceleration is interpreted as the action of a homogeneous gravitational field, for which one must seek a structure approximating a scalar field which varied linearly in the direction of the motion. This structure was not hard to find. The scalar c, the now variable speed of light, varied linearly with the direction of acceleration, so that it could be chosen naturally as the gravitational potential.

Using fairly natural inductive arguments of this type, the eliminative principles introduced so far were sufficiently powerful to enable construction of a quite definite theory of static gravitational fields. The principles needed only to be supplemented by a further eliminative principle, the retention of the usual conservation laws of energy and momentum, including the equality of action and reaction. The principle of equivalence gave a special case of the gravitational field, a homogeneous field. Its properties could be investigated minutely and then extended to the more general static case in a manner compatible with the other constraints. As Einstein put it in his 1907 review article (p. 454):

> The heuristic value of the assumption lies in the fact that it allows replacement of a homogeneous gravitational field by a uniformly accelerated reference system, the latter case being amenable to theoretical treatment up to a certain degree.

To begin, Einstein arrived directly at the two best known conclusions of the theory. It turned out that the rate of natural clocks slowed in proportion to the magnitude of c in the homogeneous case; it was assumed that this would continue to hold in the inhomogeneous case. Similarly the connection between the speed of light c and the gravitational potential would be assumed to remain in the inhomogeneous case as well, so that Einstein could infer that a light beam propagating past the sun would be bent, an effect capable of experimental test, as he urged in the introductory paragraph of his 1911 paper. In the course of developing the theory in the 1912 papers, Einstein arrived at a series of results for the general case of inhomogeneous but static fields, which included a law for the gravitational force on a body, a gravitational field equation as well as theories of electrodynamics and thermodynamics in which the action of the gravitational field c was incorporated.

The case of the gravitational field equation illustrates how the eliminative induction forced him to a quite definite result, even to the point that he was dissatisfied but could not revoke the outcome. In the homogeneous case, the gravitational potential c varies linearly with the direction of the field – call it the x direction – so that it satisfies

$$c = c_0 + ax,$$

for a and c_0 constants. This was a solution of the equation

$$\Delta c = 0,$$

which, under correspondence with the Newtonian case, was the obvious choice for the source free field equation in the inhomogeneous case. The natural choice for the case with sources would then be

$$\Delta c = kc\rho,$$

where k is a constant and ρ the rest mass density. (The factor of c on the right hand side was introduced to ensure that the field equation defined c up to a multiplicative rather than additive constant.) These at least were the results offered in the first of the 1912 papers. The ease with which the equations arise belie the degree to which their content is determined by the eliminative principles. In the second of the 1912 papers, in its closing section 4, Einstein revealed that all was not well with the theory. It was incompatible with the equality of action and reaction (and thus the conservation laws), for he could show that it entailed that masses connected to a massless rigid frame would set themselves into motion under the action of their own gravitational field. More formally, we might note, the problem was that the theory did not admit the defining of a gravitational field stress tensor.

Einstein proceeded to investigate the possible modifications to the theory, an explicit exploration of other members of the universe of theories. The violation might be avoided if one allowed gravitation to act on the members of the rigid frame because they were stressed, even though they were massless. The failure of this modification was shown by two examples, one of electromagnetic radiation enclosed in a box with mirrored walls and the other of a monatomic gas enclosed in a box. A second attempt involved modification of his earlier law for the gravitational force on a body by the insertion of further terms. The failure of this second attempt was demonstrated through an eliminative induction of classic form. Its main features were as follows.

Einstein's original expression for the gravitational force R_s on a body of mass m moving with velocity q was

$$(4) \qquad R_s = -\frac{m \text{ grad } c}{\sqrt{1 - q^2/c^2}}\,.$$

He was prepared to entertain almost any rival to (4), but he had to eliminate all but those that differed from (4) by a multiplicative factor because of two requirements:

- In the case of a space of constant c, the force law had to yield results compatible with those of special relativity.
- In the case of a body at rest ($q = 0$), the expression for the force had to reduce to $-m$ grad c.

In conjunction with dimensional requirements, he concluded that the universe of possible force expressions was reduced to

$$(5) \qquad R_s = -\frac{m \text{ grad } c \cdot c^\beta}{\sqrt{1 - q^2/c^2}} \cdot \text{const.}$$

where "const." and β were constants. Finally, he reported that the constants in (5) had to have the values of the original formula 4 in order that

- The inner product of (non-gravitational force) · (velocity) is the time derivative of a quantity.[30]

He concluded that he had to retain the original expression for gravitational force, "if one did not want to give up the whole theory (determination of the static gravitational field by c)" (p. 455). This is a classic illustration of the power of an eliminative induction to force a quite definite result on the basis of very general premises.

Thus Einstein was driven to his final option, a modification of his original field equations. The modification that would do the job could be arrived at quite straightforwardly. One had to adopt the new field equation

$$\Delta c = k\{c\rho + (1/2k)\,(\text{grad}^2\, c/c)\}.$$

The new equation even had a natural interpretation. Its second term represented the energy of the gravitational field and it seemed appropriate that this energy also be a source for the field. Einstein explained, however, that he adopted this new equation reluctantly for it no longer admitted a linear dependence of the potential c on distance. Thus the homogeneous gravitational field of the principle of equivalence could no longer be included in the theory. This meant that the principle of equivalence could only be retained in the theory for the case of infinitesimally small regions, even though it was already limited to the case of homogeneous gravitational field. This result puzzled Einstein and he noted in his polemical exchange with Abraham that he knew of no satisfying reason for this limitation.[31]

This episode reveals two important aspects of Einstein's heuristic methods. First it shows us that the principle of equivalence did genuinely function as an aid to discovery, a heuristic principle. It was not an inviolable requirement. In this case it was clearly subordinate to the requirement of the conservation of energy and momentum, since Einstein gave it up partially in order to protect the conservation law. Notice that the principle takes on its later inviolable character only after it has made the transition to the justification of the theory.[32] Second we see just how powerfully the eliminative induction forces a quite definite result. Often this power is masked by the naturalness of the result arrived at so that Einstein need not advance arguments detailing just how restricted the choice of results has become. He need merely parade the result as the most natural or simplest. This case is an exception and gives us a glimpse of the "behind the scenes" considerations which presumably guide much of scientific discovery.

The eliminative induction can be summarized as:

Universe of Theories:	Three space gravitation theories.
Eliminative Principle:	Special relativity holds in the limiting case of a vanishing gravitational field.
Eliminative Principle:	Principle of Equivalence.
Intermediate Conclusion:	Eliminate all gravitation theories which represent gravitation by more structure than is already present in the space and time of special relativity.
Eliminative Principle:	Newtonian gravitation theory holds in a suitable limiting case.
Eliminative Principle:	Conservation of energy and momentum
Conclusion:	Gravitation theory in which: – the variable speed of light c is the gravitational potential: – clocks are slowed and light beams bent in a gravitational field. Gravitational field equation. Gravitational force law.

Einstein could not rest contentedly with his 1912 theory, for it was not yet sufficiently general and it still fell far short of meeting the expectations he had entertained in 1907. To begin, of course, the theory dealt only with static gravitational fields and he surely planned a theory that would deal with dynamically varying fields. The theory did not even deal adequately with time-independent fields. There was the gravitational field produced by uniform rotation in special relativity and this field had no home in Einstein's 1912. He described this case to Ehrenfest as analogous to a magnetostatic field, whereas the fields of the 1912 theory were analogous to electrostatic fields.[33] Further, he had made clear from the very start in the 1907 review article that he expected the new theory to implement an extension of the principle of relativity. In so far as this extension required an expansion of the covariance of the theory beyond Lorentz covariance, there was clearly more to be expected. While the 1912 theory might extend the Lorentz covariance of special relativity to include transformations to uniformly accelerating systems, it still excluded many others, most notably transformations to rotating systems.

Einstein had enjoyed some success in his program of extending the principle of relativity. Part of that program, as he recalled later,[34] involved implementation of the idea, which he attributed to Mach, that inertia arose entirely from an interaction between bodies. He was able to show that weak field effects compatible with this view were derivable from his 1912 theory: an accelerating shell of matter would tend to drag along with it test masses placed inside and the presence of the shell would increase the inertia of the bodies placed within it.[35]

However by July 1912, Einstein had clearly reached another crisis point. He described in his reply to Abraham of that month[36] how the equivalence principle raises the possibility of a theory of relativity containing gravitation which would be invariant under acceleration and rotational transformations. But he had to admit that the path to this goal would be very difficult and that he did not know what form the general spacetime transformation equations would take. We can gauge his feeling of desperation from the fact that he concluded with a plea quite uncharacteristic of his writings: "I would like to ask all colleagues to apply themselves to this important problem."

5.2. *The 'Entwurf' Theory*

However grim things may have seemed in July 1912, Einstein was soon able to solve the problem himself with dazzling success. The papers of 1911 and 1912 on gravitation theory were prepared in Prague. Within months of his August 1912 move to Zurich, Einstein had in hand essentially all the crucial elements of the final general theory of relativity. In particular he had a spacetime theory in which gravitation was incorporated into a variable curvature, symmetric, Lorentz signature spacetime metric tensor. The resulting theory, called here the 'Entwurf' theory after the title of the paper in which it was published, lacked only the gravitational field equations of the final general theory of relativity.[37]

The crucial element in the transition came with Einstein's move to Zurich in August 1912, There he put to his mathematician friend, Marcel

Grossmann, the question of whether there were mathematical methods which would enable the formulation of a theory covariant under arbitrary coordinate transformations. Grossmann informed Einstein of the so called "absolute differential calculus" of Ricci and Levi-Civita which enabled just such a formulation and, with Grossmann as mathematical advisor and co-author, these methods were used to write the theory of the 'Entwurf' paper. We know that the transition had already commenced prior to Einstein's exposure to the absolute differential calculus. For he recalled in the introduction to the Czech edition of his popular text *Relativity*:[38]

I first had the decisive idea of the analogy of the mathematical problems connected with the theory and Gauss's theory of surfaces in 1912 after my return to Zurich without knowing at that time Ricmann's and Ricci's or Levi-Civita's work.

We do not know the full story of this transition. Stachel has posed this as a problem of a "missing link". He urges convincingly that Einstein's investigation of the non-Euclidean geometry of a uniformly rotating disk in special relativity provided this missing link.[39] That problem, which had been considered in the first of the 1912 papers on the theory of static gravitational fields, alerted Einstein to the relevance of the infinitesimal geometry of curved surfaces in which arbitrary coordinate systems were routinely employed. He may even have recalled this latter practice from lectures given at the Zurich ETH by C. F. Geiser on the subject and for which Einstein had registered. Moreover, the rotating disk provided Einstein with a case in which the coordinate system used lost its direct metrical significance, a result that had to be accepted to complete the transition to a generally covariant theory. This result had also arisen automatically for the time coordinate in the 1912 theory. Einstein used this time coordinate example, rather than the rotating disk, as support for the result in the relevant Section 3 of his part of the 'Entwurf' paper.[40]

In terms of the eliminative model, the transition represented a move to a new universe of theories in an attempt to find a theory that would satisfy all of Einstein's desiderata. The new universe contained four dimensional spacetime theories of the type of those given by Minkowski, but now formulated using the the generalized four dimensional vector analysis that resulted from combining the four dimensional vector analysis of Minkowski with the absolute differential calculus of Ricci and Levi-Civita.[41]

We do not know the details of the circumstances surrounding Einstein's crucial transition to this new universe of theories because of the scarcity of historical resources for this episode. Perhaps further arguments of the type advanced by Einstein in 1912 convinced him that he would not find the more general theory sought in the older universe of theories. Or perhaps the brilliance of his above mentioned "decisive idea" of the analogy to Gauss' theory of surfaces simply outshone his earlier efforts so that he abandoned them. Either way, the nature of the expansion was almost dictated by his "decisive idea". In Gauss's theory of surfaces, one can expand the analysis of the geometry of a two dimensional Euclidean surface from the Cartesian coordinates (x, y), in which the line element has the form

(6) $$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2,$$

to an arbitrary coordinate system (u, v) where the line element now takes on the form

(7) $$\mathrm{d}s^2 = \mathrm{E}\mathrm{d}u^2 + 2\mathrm{F}\mathrm{d}u\mathrm{d}v + \mathrm{G}\mathrm{d}v^2,$$

for E, F and G functions of u and v. In Minkowski's formulation of special relativity, the fundamental quantity was the invariant interval which, in infinitesimal form in a standard coordinate (x, y, z, t) system, was

(8) $$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2.$$

Presumably the analogy Einstein perceived was between line elements (6) and (8) and his hope was that arbitrary coordinate systems could be introduced for line element (8) analogously to the transition from (6) to (7). What may have contributed to the perception of this analogy was the content of a remarkable addendum on the final page of Einstein's second 1912 paper on the theory of static gravitational fields.[42] There he concluded that the theory's equations of motion for a free point mass could be written as the variational principle

$$\delta\{\int\sqrt{c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2}\} = 0,$$

which is just the equation of a geodesic in a spacetime with the line element (8), where now c is allowed to vary as a function of x, y and z.

Finally there is a quite prosaic reason that might in itself have been sufficient to lead to the expansion of the universe of theories to include those formulated four dimensionally. As late as 1912, some five years after their introduction, Einstein was still not using the new four dimensional methods of Minkowski, Sommerfeld and Laue, even though they were becoming widespread. Perhaps he simply decided it was time to overcome his legendary early reticence over adopting fancy mathematical techniques, to accept the inevitable and start using the new methods.

Once the new universe of theories was adopted, Einstein again faced an overwhelming diversity of theories. But the same set of eliminative principles came into play as were applied in the 1907–1912 search and they immediately reduce the universe of theories. In particular, the principle of equivalence once again enables the conclusion that gravitation is not to be represented by a new field structure defined in spacetime, but that the existing spacetime structure of special relativity is already sufficient. This conclusion is arrived at somewhat indirectly in two steps. First, it turns out that Einstein's 1912 theory of static gravitational fields corresponds to the theory of a spacetime with line element (8) but in which c is a function of x, y and z. Thus such a spacetime corresponds to a spacetime with a static gravitational field of the type of the 1907–1912 theory. The result is mentioned without fuss in the second section of Einstein's part of the 'Entwurf' paper (p. 229), referring back to the summary of the 1912 theory given in the first section.

In a less formal 1913 exposition of the physical foundations of his new theory, Einstein summarized the line of reasoning developed in greater detail in the 'Entwurf' paper and which leads to identification of the spacetime structures that are to represent gravitation. In particular, he makes clear the role of the principle of equivalence:[43]

According to the usual theory of relativity, an isolated material point moves uniformly in a straight line according to the equation

$$\delta(\int ds) = 0,$$

where

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2dt^2,$$

and c is the (constant) speed of light. The equivalence hypothesis now admits the conclusion that a material point in a *static* gravitational field (of a special kind) moves according to the above equation, where however c is a function of place and determined by the gravitational potential.

Unlike the case of the 1912 theory, there was no doubt about how the theory was to be further extended to include more general gravitational fields than these special static gravitational fields. The line element (8) with c varying linearly with the spatial coordinates had been arrived at by transforming to a uniformly accelerated coordinate system. If we now allowed transformations to arbitrary coordinate systems, as one ought to if the theory were to be generally covariant, then the line element (8) would adopt the form

$$ds^2 = \sum_{ik} g_{ik} dx_i dx_k, \tag{9}$$

where x_i with $i = 1, 2, 3, 4$ are the new spacetime coordinates and summation extends over $i, k = 1, 2, 3, 4$. The matrix g_{ik} of metrical coefficients revealed by this process is the structure that would represent the gravitational field. More precisely, the presence of a gravitational field would be associated with a non-constancy of these coefficients. Thus Einstein continued:

From this special case of the gravitation field one can arrive at a more general field in any case by changing to moving coordinate systems through coordinate transformation. (Footnote: Thereby we postulate that we arrive at an equally justified description of processes, in that we relate it to an appropriately moving coordinate system; thus we adhere to the fundamental idea of the theory of relativity.) On this path one recognizes that the only generalization, sufficiently broad from the invariant theoretic point of view, of the above equation of motion consists in assuming a "line element ds" of the form

$$ds^2 = \sum_{ik} g_{ik} dx_i dx_k \qquad (i, k = 1, 2, 3, 4)$$

where the g_{ik} are functions of x_1, x_2, x_3 and x_4 and the first three coordinates characterize place and the last time and that the equation of motion ought again to have the form

$$\delta(\int ds) = 0.$$

What Einstein did not make very clear in this summary is that the matrix gik produced by transformation from the line element (8) can only represent a special case of the gravitational field, for this matrix can always be transformed over some neighborhood to the diagonal matrix corresponding to line element (8); that is, the metric is flat. It is an additional but very natural assumption that the matrix gik relieved of the flatness condition can represent arbitrary gravitational fields.[44]

The eliminative induction can be given in summary form as follows. Notice the structural similarity between this induction and the corresponding induction of the earlier theory of static fields, the most prominent difference being the differing choice of universe of theories:

Universe of Theories:	Spacetime theories of gravitation.
Eliminative Principle:	Special relativity holds in the limiting case of a vanishing gravitational field.
Eliminative Principle:	Principle of Equivalence.
Intermediate Conclusion:	Eliminate all gravitation theories which represent gravitation by more structure than is already present in the spacetime structure of a Minkowski spacetime.
Eliminative Principle:	Newtonian gravitation theory holds in a suitable limiting case.
Eliminative Principle:	Principle of general covariance.
Conclusion:	The matrix of metrical coefficients g_{ik} represents the gravitational field.

This conclusion is actually still restricted to a Minkowski spacetime. As I have noted above, that it also applies to other more general spacetimes is a natural generalization. I postpone further discussion of the point until the discussion of the gravitational field equations in which the nature of the more general spacetimes becomes the basic issue.

I have refrained from pointing out that all the eliminative principles used so far have obvious empirical foundations. The above argument contains the only problematic case, the principle of general covariance, but I must leave open here the question of whether the principle has an empirical foundation. On the one hand, the principle is introduced sometimes as a purely mathematical requirement. On the other, Einstein sometimes offers the principle as the generalization of the principle of relativity of special relativity, but then it is still supported at least partially by *epistemological* as opposed to empirical grounds.[45]

How important was the reasoning of the above argument to Einstein's actual discovery process? It would seem to have been of "decisive importance" if we believe that Einstein's remarks to Laue, in letters of September 12, 1950 and January 16, 1951, were autobiographical.[46] Laue had objected to Einstein's conclusion that the presence of a gravitational field coincides with the non-constancy of the metrical coefficient g_{ik} (which coincides with the non-vanishing of the Christoffel symbols Γ^{l}_{ik}). Laue preferred to identify the presence of a gravitational field with a non-vanishing curvature tensor R_{iklm}. Einstein replied in the September letter:

> . . . what characterizes the existence of a gravitational field from the empirical standpoint is the non-vanishing of the Γ^{l}_{ik}, not the non-vanishing of the R_{iklm}. If one does not think intuitively in such a way, one cannot grasp why something like a curvature should have anything at all to do with gravitation. In any case, no reasonable person would have hit upon such a thing. The key for the understanding of the equality of inertial and gravitational mass is missing.

Laue persisted, noting that the coordinate transformation might only alter the spatial components and thus have nothing to do with a gravitational field producing transformation. Einstein's reply in the January letter contained the remarks:

> Heuristically, the interpretation of the field existing relative to a system, parallel accelerated against an inertial system (Equivalence principle) was naturally of decisive importance, since this field is equivalent to a Newtonian gravitational field with parallel lines of force. In this case, the Newtonian field strengths are equal to the spatial derivatives of the g_{44}.

The above line of reasoning indicates how the principles of equivalence and general covariance enabled Einstein to select from the universe of theories in question a quite particular class of theories in which gravitation is represented by a Lorentz signature, symmetric tensor. Once again, Einstein's pathway to this discovery provided him with an argument that could be used as a later justification for the theory. Thus the above line of reasoning is often recapitulated in Einstein's development of the theory. Perhaps the most interesting instance lies in a letter to J. Becquerel of August 16, 1951.[47] Einstein is assisting Becquerel to convince a sceptic who accepts special but not general relativity. Einstein carefully lays out a step by step pathway from special to general relativity with the purpose of convincing the sceptic or at least laying out the assumptions made in the transition. The path contains exactly the eliminative reasoning laid out above, but now used as justification for the theory.

With the basic structure of the theory decided, the remaining components of the theory are determined almost completely by the larger list of eliminative principles: the principle of equivalence, the generalized principle of relativity (which took the form of a principle of general covariance), the requirement of an appropriate Newtonian limit and the laws of conservation of energy and momentum. The most interesting of these components are the gravitational field equations, which posed unexpected problems and will be looked at in the next section.

Other components were selected quite naturally. For example the equation of motion of a point mass in a general gravitational field was taken to be the same as the corresponding equation of special relativity.[48] That is, the trajectory of a free particle in both cases is a geodesic of the metric satisfying

(10) $\delta\int ds = 0$

Einstein could confirm that such a choice was compatible with the Newtonian limit and with the conservation laws. Note that the introduction of such an equation of motion as a "natural" choice tends to obscure the lack of viable alternatives.

6. THE GRAVITATIONAL FIELD EQUATIONS

Einstein had concluded that a Minkowski metric represented a special case of the gravitational field. Upon this conclusion he based the generalization that arbitrary gravitational fields were to be represented by a Lorentz signature metric. The task of the gravitational field equations of his theory was to pick out which metrics were to be associated with which matter distributions. It is well known that the requirements of correspondence with the Newtonian case and satisfaction of the conservation of energy and momentum direct selection of quite definite gravitational field equations for the theory. In Section 5 of his part of the 'Entwurf' paper, Einstein commenced development of those equations in a manner typical of the arguments commonly used in developments of general relativity. He sought field equations of the form[49]

(11) $\kappa\Theta_{\mu\nu} = \Gamma_{\mu\nu}$,

where the stress-energy tensor $\Theta_{\mu\nu}$ represents the sources of the field, $\Gamma_{\mu\nu}$ is the gravitation tensor and κ a constant. That the field term $\Gamma_{\mu\nu}$ must be a tensor is directed, of course, by the requirement of general covariance. The form of the law (11) and the value of κ is directed by analogy with the corresponding Newtonian law (2). In particular, since the metrical coefficients $g_{\mu\nu}$ correspond to gravitational potentials, it is natural to require that $\Gamma_{\mu\nu}$ be constructed from $g_{\mu\nu}$ and its first and second derivatives and be linear in the latter.

At this point, later readers know that his requirements have already restricted the choice of gravitation tensor $\Gamma_{\mu\nu}$ to a very few tensors. The selection is expected to be finalized quite routinely by the requirement of conservation of energy and momentum. This requirement is expressible as the vanishing of the covariant divergence of the stress-energy tensor, which we would write in modern notation as

(12) $\nabla_i T^i_k = 0.$

One would then substitute the gravitation tensor G^i_k for T^i_k in this equation to recover

(13) $\nabla_i G^i_k = 0,$

which must hold identically. The fact that the gravitation tensor must satisfy (13) identically determines that it must be the so-called "Einstein tensor" (up to an additive cosmological term linear in g_{ik}) so that the choice of field equations is essentially fully determined. This most standard of derivations of the gravitational field equations of general relativity is an eliminative induction which can be given in summary form as:

Universe of Theories:	Field equations of form (11).
Eliminative Principle:	Principle of general covariance: $\Gamma_{\mu\nu}$ is a generally covariant tensor.
Eliminative Principle:	Requirement of Newtonian limit: $\Gamma_{\mu\nu}$ is composed of first and second derivatives of $g_{\mu\nu}$ and is linear in the latter.
Eliminative Principle:	Conservation of energy-momentum (applied via identity (13)).
Conclusion:	$\Gamma_{\mu\nu}$ is the Einstein tensor or Einstein tensor with cosmological term linear in $g_{\mu\nu}$.

However Einstein fails to meet the expectations of his later readers and does not produce this argument or even the same tensor. Instead he informs them of the surprizing conclusion that it has proved impossible to find a generally covariant tensor $\Gamma_{\mu\nu}$ which is the appropriate generalization of the Newtonian expression $\Delta\varphi$ of (2). The principal discussion of this problem is carried by Grossmann in Section 4.2 of his part of the paper, where he indicates that the Ricci tensor, the second rank contraction of the Riemann curvature tensor, is the obvious choice of gravitation tensor. Here Einstein and Grossmann stand on the threshold of the final theory, for the selection of the Ricci tensor would at least give the gravitational field equations of the final 1915 generally covariant theory in the source free case. "However," Grossmann wrote, "it turns out that this tensor does *not* reduce to the expression $\Delta\varphi$ in the special case of an infinitely weak, static gravitational field." Einstein and Grossmann then dispense with the requirement of general covariance and proceed to seek gravitational field equations which need not be generally covariant.

We see eliminative induction at work here, displaying its power to reduce the universe of theories. Unfortunately something went very wrong, for the the final theory adopted in November of 1915 has been eliminated as well. This disaster does reveal something interesting. Just as we saw in 1912 that the principle of equivalence was a heuristic principle and not inviolable, we see here that the principle of general covariance was, in 1913, functioning as a heuristic principle that could be dispensed with, should the expediencies of the search call for it. It is only when both are transferred to justifications of the final theory that they become asserted as inviolable premises.

Elsewhere, with the assistance of an unpublished notebook of calculations by Einstein from this period, I have analyzed in detail where I believe Einstein and Grossmann's application of the requirement of the Newtonian limit went astray.[50] Very briefly, the principal problem lay in two interlocking beliefs incompatible with the final theory. First, on the basis of his 1912 theory and the principle of equivalence, Einstein assumed that static fields in the new theory ought to have a line element of the form (8) in a suitably chosen coordinate system, where c varies as a function of x, y and z. It turns out that the final theory does not admit such spacetimes except in trivial cases; in even a quite simple weak field case, the coefficients of dx, dy and dz will be variable. Unfortunately for Einstein this first belief was compatible with a second that is also ruled out by the final theory. Pursuing an analogy with equation (3), Einstein assumed that the field equations of the new theory must reduce to

$$(14) \qquad \Box\gamma\mu\nu = \sum_{\alpha\beta} \frac{\partial}{\partial x_\alpha}\left(\gamma\alpha\beta\frac{\partial\lambda_{\mu\nu}}{\partial x_\beta}\right) = \kappa'\Theta_{\mu\nu},$$

in the weak field case, for κ' a constant. Einstein's final field equations of November 1915 do not reduce to this form. However, Equation (14) can be solved to yield a weak static field of form (Equation 8).

The bulk of the discussion concerning gravitational field equations in the 'Entwurf' paper focuses on the selection of a gravitation tensor $\Gamma_{\mu\nu}$ where this quantity would not transform as a tensor under arbitrary transformations but only under some subgroup. It turns out that the method Einstein used is essentially analogous to the modern method described above, in which the conservation law plays the crucial role in determining the field equation. It is also interesting to note that the field equations admit construction of a gravitational field stress-energy tensor. It was precisely the failure of his first field equation of 1912 to admit defining of a gravitational field stress tensor that had forced the embarrassing modification required in the second 1912 paper and even dictated its character.

Einstein went to considerable pains in his Section 5 of the 'Entwurf' paper to explain to the reader the heuristic method used to arrive at his field equations. In doing so, he made clear that he conceived his procedure to be more than just a clever trick that happened to work in this case. It was an application of a genuine method of discovery. Here I take the distinguishing characteristic of a method to be that it supplies explicitly identifiable procedures that can be applied in multiple cases. "So that the *method* [my emphasis] used stands out clearly," Einstein wrote, "I now want to apply it to a generally known example." He proceeded to show that this method was capable of discovering the field equation of electrostatics

$$(15) \qquad \sum_{\nu} \frac{\partial^2 \varphi}{\partial x_{\nu^2}} = -\rho,$$

where φ is the electrostatic potential, ρ the electric charge density and summation for ν extends over 1, 2 and 3. The crucial restriction in the procedure was the assumption that the theory of electrostatics is compatible with the conservation laws which were expressed in the theory by the requirement

Divergence of electric field stress tensor	=	force density on charge density ρ

This condition is given mathematical form as

$$(16) \qquad \sum_{\mu} \frac{\partial}{\partial x_{\mu}} \left(\frac{\partial \varphi}{\partial x_{\nu}} \frac{\partial \varphi}{\partial x_{\mu}} \right) - \frac{\partial}{\partial x_{\nu}} \left(\frac{1}{2} \sum_{\mu} \left(\frac{\partial \varphi}{\partial x_{\mu}} \right)^2 \right) = -\frac{\partial \varphi}{\partial x_{\nu}} \rho,$$

since the electrostatic field stress tensor is

$$\left(\frac{\partial \varphi}{\partial x_{\nu}} \frac{\partial \varphi}{\partial x_{\mu}} \right) - \delta_{\mu\nu} \left(\frac{1}{2} \sum_{\alpha} \left(\frac{\partial \varphi}{\partial x_{\alpha}} \right)^2 \right).$$

Substitution for ρ in (16) from the field equation (15) yields an expression in φ and its derivatives alone and which is an identity:

$$(17) \qquad \sum_{\mu} \frac{\partial}{\partial x_{\mu}} \left(\frac{\partial \varphi}{\partial x_{\nu}} \frac{\partial \varphi}{\partial x_{\mu}} \right) - \frac{\partial}{\partial x_{\nu}} \left(\frac{1}{2} \sum_{\mu} \left(\frac{\partial \varphi}{\partial x_{\mu}} \right)^2 \right) = \frac{\partial \varphi}{\partial x_{\nu}} \sum_{\mu} \frac{\partial^2 \varphi}{\partial x_{\mu^2}}.$$

Einstein's method consists in the realization that one might start with identity (17). Were one to do so, then the field equation (15) could be recovered simply by comparing the form of the identity (17) with the conservation law (16). The field equation (15) would be arrived at immediately since it is just the equation needed to convert (17) to (16). Einstein completed his example by urging that the identity (17) would be very easy to find, resulting almost immediately from an application of the rule for differentiating products to the quantity

$$\left(\frac{\partial \varphi}{\partial x_{\nu}} \frac{\partial \varphi}{\partial x_{\mu}} \right).$$

Einstein then turned to his new theory and showed how this method could be applied to arrive at a set of gravitational field equations. The form of the conservation law that was relevant could be given informally as

Divergence of gravitational field stress-energy tensor	=	Four force density on matter represented by $\Theta_{\mu\nu}$

Einstein presumed a formal statement of this version of the conservation law, which we can develop as follows. The conservation law (12) was written in the 'Entwurf' paper in several forms. On p. 239 it is given as the sum of the divergences of the stress-energy tensor $\Theta_{\mu\nu}$ of non-gravitational matter and of the stress-energy tensor $\theta_{\mu\nu}$ of the gravitational field:

$$(12') \qquad \sum_{\mu\nu} \frac{\partial}{\partial x_\nu} \{\sqrt{-g}\, g_{\sigma\mu}(\Theta_{\mu\nu} + \theta_{\mu\nu})\} = 0.$$

He also wrote the conservation law (12) in the form (p. 232):

$$(12'') \qquad \sum_{\mu\nu} \frac{\partial}{\partial x_\nu} (\sqrt{-g}\, g_{\sigma\mu}\Theta_{\mu\nu}) - \frac{1}{2} \sum_{\mu\nu} \sqrt{-g}\, \frac{\partial g_{\mu\nu}}{\partial x_\sigma} \Theta_{\mu\nu} = 0,$$

where the second term of this equation was identified in his Section 4 of the 'Entwurf' paper as the gravitational four force density on matter. If one uses the field equations (11) to substitute $(1/\kappa)\, \Gamma_{\mu\nu}$ for $\Theta_{\mu\nu}$ in the second term of equation (12″) and then combines equations (12′) and (12″) by eliminating their common first term, one arrives at the equation

$$(18) \qquad \sum_{\mu\nu} \frac{\partial}{\partial x_\nu} (\sqrt{-g}\, g_{\sigma\mu}\theta_{\mu\nu}) = \frac{1}{2\kappa} \sum_{\mu\nu} \sqrt{-g}\, \frac{\partial g_{\mu\nu}}{\partial x_\sigma} \Gamma_{\mu\nu},$$

which is the form of the conservation law sought. Since both the stress-energy tensor $\theta_{\mu\nu}$ and the gravitation tensor $\Gamma_{\mu\nu}$ contain only the metric tensor and its derivatives, the same must hold for the entire equation. Einstein implemented the requirement of the Newtonian limit as requiring the gravitation tensor $\Gamma_{\mu\nu}$ to reduce to the expression $\Box\gamma_{\mu\nu}$ as given in the weak field Equation (14) above, so that he could conclude that the conservation law (18) was reducible to an identity of the form (p. 237)

(19) "Sum of differential quotients

$$= \frac{1}{2} \sum_{\mu\nu} \sqrt{-g}\, \frac{\partial g_{\mu\nu}}{\partial x_\sigma} \left\{ \sum_{\alpha\beta} \frac{\partial}{\partial x_\alpha} \left(\gamma_{\alpha\beta} \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} \right) + \begin{array}{c}\text{further terms which drop away in the}\\ \text{formation of the first approximation}\end{array} \right\}".$$

Einstein could then announce that the conditions laid out led to a *uniquely determined* identity which he then presented; the derivation of the identity was provided, as Einstein's footnote indicated, in Section 4.3 of Grossmann's part of the paper. (But there was no support for the uniqueness claim.) It was now a simple matter to read off both the gravitation tensor and gravitational field stress-energy tensor from the identity, completing the construction of the theory.[51] His gravitation tensor was:

$$\text{(20)} \qquad \Gamma_{\mu\nu} = \sum_{\alpha\beta} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x_\alpha}\left(\gamma_{\alpha\beta}\sqrt{-g}\,\frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right) - \sum_{\alpha\beta\tau\rho} \gamma_{\alpha\beta} g_{\tau\rho} \frac{\partial \gamma_{\mu\tau}}{\partial x_\alpha}\frac{\partial \gamma_{\nu\rho}}{\partial x_\beta}$$
$$+ \frac{1}{2}\sum_{\alpha\beta\tau\rho} \gamma_{\alpha\mu} g_{\beta\nu} \frac{\partial g_{\tau\rho}}{\partial x_\alpha}\frac{\partial \gamma_{\tau\rho}}{\partial x_\beta} - \frac{1}{4}\sum_{\alpha\beta\tau\rho} \gamma_{\mu\nu}\gamma_{\alpha\beta} \frac{\partial g_{\tau\rho}}{\partial x_\alpha}\frac{\partial \gamma_{\tau\rho}}{\partial x_\beta}\,.$$

Notice that Einstein's method of arriving at the gravitation tensor and thus the field equations is essentially the same as that described as the standard method for the generally covariant field equations of general relativity at the start of this section. In both cases, one takes the conservation law, substitutes the gravitation tensor for the stress-energy tensor, thereby reducing the law to an identity in the metric tensor and its derivatives. The choice of gravitation tensor is then determined by the choice of identity, which is extremely restricted. In summary form, this comprises an eliminative induction closely comparable in form to the standard derivation of the final field equations discussed above:

Universe of Theories:	Field equations of form (11).
Eliminative Principle:	Requirement of Newtonian limit: $\Gamma_{\mu\nu}$ is composed of first and second derivatives of $g_{\mu\nu}$ and is linear in the latter; the only second derivative term is
Eliminative Principle:	Conservation of energy-momentum (applied via identity (19)).
Conclusion	$\Gamma_{\mu\nu}$ is the 'Entwurf' gravitation tensor (20).

The first and major difference between the two inductions is the omission of the requirement of general covariance in the second. This is compensated for by a strengthening of the implications of the requirement of the Newtonian limit.

This example shows very clearly once again how the justification of a theory can develop hand in hand with its discovery by the eliminative method. For the justification of the field equations given in the 'Entwurf' paper amounts essentially to the recapitulation of the method used to arrive at the equations.

With the completion of the 'Entwurf' theory, Einstein descended into a dark abyss where he would wander for nearly three years as he grappled with the ramifications of the lack of general covariance of this theory. One of its darkest moments came when Einstein decided that his failure to find generally covariant field equations, satisfying his heuristic constraints, was unimportant; for, even if he could find them, they would be physically uninteresting. The principal argument in favor of this conclusion was the "hole argument", which concluded that a field theory such as the 'Entwurf' theory with generally covariant field equations, would violate the principle of causality, or, as we would now put it, would be indeterministic.[52] It was not until November of 1915 that Einstein announced in a communication to the Prussian Academy that he had abandoned the 'Entwurf' field equations and had returned to the search for generally covariant field equations. This search was based on the construction of gravitation tensors from the Riemann curvature tensor and its contractions, the route that Grossmann had earlier designated as the mathematically obvious path. Its direction was governed as before by the application of the same set of eliminative principles that operated in 1912 and 1913. However, Einstein was still not to come directly to his final theory. He sent four communications to the Prussian Academy in that month. In them it is possible to watch Einstein as he slowly unravels the same misconceptions that had originally led him astray in 1912 and 1913, coming to the final result only in the fourth communication of November 25. This series of documents what might well be the most exciting moment of Einstein's scientific career, for in the third communication he could show that his theory with its new field equations was finally able to account for the anomalous motion of Mercury.[53]

7. REFLECTIONS ON THE EXAMPLES OF ELIMINATIVE INDUCTION

The success with which an eliminative induction establishes its conclusion depends on:

(a) our confidence in its premises and most especially our confidence that the universe of theories is sufficiently large; and

(b) the strength of the inference used for elimination.

If inductive (ampliative) inference is used to effect the elimination, then the stronger the inference, the more successfully the conclusion is established. Elimination by deductive inference is, of course, the strongest from.

By these standard, the two most successful eliminative inferences of those examined above are those used to arrive at special relativistic kinematics and at the generally covariant field equations of general relativity. In both cases the universe of theories is sufficiently large to make us very confident that the correct answer to the problem at hand lies within the relevant universe; and the elimination is carried principally by deductive inference.

The least successful eliminative induction is that discussed in Section 4 which concluded the impossibility of an acceptable special relativistic theory of gravitation. The induction is weak judged by criterion (a), for it is difficult to have great confidence in the sufficiency of the size of the universe of theories, when that universe is so vaguely defined. Essentially all we know is that Einstein was looking for a "law of gravity within the framework of the special theory of relativity." More serious problems arise concerning criterion (b). For the bulk of the elimination was carried out by rather fragile inductive arguments. For example, the requirement of a Newtonian limit by no means forces one to retain a scalar potential to represent the gravitational field, although it is a plausible conjecture. There are other choices compatible with the requirement of a Newtonian limit and the other requirements including equality of inertial and gravitational mass.[54] Again special relativity does not force a gravitation theory to be a field theory; there is a lively literature in special relativistic action at a distance theories.

Finally, of intermediate strength are the eliminative inductions of Section 5. Their strength lies in the ability of the principle of equivalence to restrict gravitation theories to those that exploit structures already present in space or spacetime. Their primary weakness lie in the vagueness of delineation of the relevant universes of theories.

At this point, one might wonder whether the inferences described in Section 4 and 5 are eliminative inductions at all, as opposed to attempts at theory construction divorced from any ramifications for rival theories. Here one must not confuse the *weakness* of an eliminative induction with its being no eliminative induction at all. Elimination was the essence of the special relativistic gravitation theory whose construction we saw Einstein sketching in Section 4. Each of its components was selected as the one most likely to be found in a successful theory so that the failure of the resulting, most promising theory should serve to cast doubt upon the possibility of success of any other. Similarly, when Einstein constructed the theory of static gravitational fields and then the 'Entwurf' on the basis of the sequence of requirements described in Section 5, he clearly understood that these requirements be applied eliminatively. Thus he ruled out Mie's theory of gravitation since it failed to satisfy the equality of inertial and gravitational mass (let alone the principle of equivalence).[55]

8. THE CANON OF MATHEMATICAL SIMPLICITY

The Einstein of 1912 and 1913 seemed all too ready to turn away from the obvious mathematical route to the gravitational field equations and thus was destined to spend nearly three years groping for a result that is now blithely spat out in one or two lines in modern text books. It is hard to imagine that this was the same Einstein who later wrote in his 'On the Methods of Theoretical Physics' (p. 274):

> I answer without any hesitation that there is, in my opinion, a right way [to find the axiomatic basis of theoretical physics], and that we are capable of finding it. Our experience hitherto justifies us in believing that nature is the realization of the simplest conceivable mathematical ideas. I am convinced that we can discover by means of purely mathematical constructions the concepts and laws connecting them with each other, which furnish the key to the understanding of natural phenomena . . .

The change is dramatic. The revealing words in the passage are that "our *experience* . . . justifies . . ." and what I would like to argue now is that one of the most important parts of this experience was Einstein's own experience with the field equations of general relativity.

Einstein, by his own account, had neglected mathematics as a student.[56] He preferred physics since he was able to "scent out that which might lead to fundamentals and to turn aside from everything else." His intuition in mathematics was not comparably strong and, he confessed further, that "it was not clear to me as a young student that access to to a more profound knowledge of the basic principles of physics depends on the most intricate mathematical methods." This last scepticism followed Einstein to 1912, the time at which he began work on the 'Entwurf' theory. In 1911, Laue had published his *Das Relativitätsprinzip*,[57] which included a development of the four dimensional vector methods of Minkowski and brought a level of mathematical sophistication to an introductory text book in relativity theory that outstripped the level of Einstein's own publications. So Einstein quipped that he could "hardly understand Laue's book."[58] At that time, Einstein was quick to use this scepticism to attack Abraham's rival theory of gravitation in his private communications behind the major lines of the public battle with Abraham. Einstein wrote to Zangger in January 1912 of the incorrectness of Abraham's theory, lamenting that Abraham's errors result from operating formally without thinking physically.[59] A few months later, he complained to Besso in a letter of March 26, 1913,[60] that Abraham's theory was based purely on considerations of mathematical beauty and completely untenable.

By October of 1912, Einstein had moved from Prague to Zurich and had been introduced to the absolute differential calculus of Ricci and Levi-Civita. He began to see that he had been too hasty in his assessment of the role of mathematics in physics and he wrote to Sommerfeld on October 29:[61]

> I occupy myself now exclusively with the problem of gravitation and now believe, with the help of a local, friendly mathematician, that I will be master of all difficulties. But one thing is certain, that I have never before had to toil anywhere near as much, and that I have been infused with great respect for mathematics, which I had up until now in my naivety looked upon as a pure luxury in its more subtle parts. Compared to this problem, the original theory of relativity is child's play.

However Einstein did not master all the difficulties as rapidly as he would have liked. By March of 1914, he believed that the natural mathematical pathway laid out for him by the absolute differential calculus was just a dead end and that direct physical reasoning was the correct way. In that month, he informed Besso of his satisfaction with his non-generally covariant 'Entwurf' theory and expressed some disenchantment with the lure of the mathematical route:[62]

> Now I am completely satisfied and no longer doubt the correctness of the whole system, whether the observation of the solar eclipse works out or not. The sense [*vernunft*] of the

matter is too evident . . . The general theory of invariants functioned only as a hindrance. The direct path proved itself to be the only passable one.

By November of 1915 when Einstein had decided to return to the search for generally covariant field equations, he had certainly come to regret this assessment of the theory of invariants, for it was his greatest resource in that search. He later recalled how he "ruefully returned to the Riemann curvature."[63] What may have deepened those feelings was the knowledge that Hilbert in Göttingen was also working on the problem of the field equations. Hilbert's assault on the problem was an essentially mathematical one. He constructed generally covariant field equations for gravitation and electromagnetism from an action principle arriving, by essentially pure formal manipulation, at the same equations as Einstein. Einstein communicated his final field equations to the Prussian Academy on November 25 of 1915; Hilbert communicated his equations to the Göttingen Academy on November 20, 1915, five days before Einstein.[64] Coming so close to having the capstone of the general theory of relativity stolen from him was surely a memorable lesson. It may well have been in his mind along with the other events of the three years leading up to November 1915, when he recalled in his *Autobiographical Notes*:[65]

I have learned something else from the theory of gravitation: no collection of empirical facts however comprehensive can ever lead to the setting up of such complicated equations [as non-linear field equations of the unified field]. A theory can be tested by experience, but there is no way from experience to the construction of a theory. Equations of such complexity as are the equations of the gravitational field can be found only through the discovery of a logically simple mathematical condition that determines the equations completely or almost completely. Once one has obtained those sufficiently strong formal conditions, one requires only little knowledge of facts for the construction of the theory; in the case of gravitation it is the four-dimensionality and the symmetric tensor as expression for the structure of space that, together with the invariance with respect to the continuous transformation group, determine the equations all but completely.

9. CONCLUSION

A principal burden of this paper has been to demonstrate that, in broad outline, the process of Einstein's discovery of general relativity followed what I described as the "eliminative model of scientific discovery" in the introduction. It follows from this result that Einstein's process of discovery was, at this broad level, a process of reasoned investigation, not inscrutable creativity, and that it is amenable to logical analysis just as much as is any other part of scientific activity. In this century, there seems to be a strong temptation to represent the generation of scientific discoveries, especially those of the caliber of general relativity, as somehow miraculously transcending reason and analysis. Perhaps the fear is that we would respect Einstein less if we realized that his toolbag was filled with the same instruments as are used in the common reasoning of science. Such a fear is surely unwarranted. We ought to respect an Einstein all the more when we find that he wrought his miracles with tools and materials available to everyone, day to day.

Needless to say, the eliminative model has oversimplified many of the subtleties of Einstein's process of discovery and there are quite possibly components of the process of discovery that do transcend rational analysis. Indeed one could hardly expect the process of discovery of one of the major achievements of science to be fully reducible to that simple a recipe. However I find it striking that so simple a model can do as much as it has in this paper. It suggests that the process of discovery, even in the case of general relativity, admits of quite simple logical schemas. In any case, since commentaries on scientific discoveries so easily reduce to vacuous praise, I should prefer to err on the side of oversimplification rather than mystification.

The eliminative model does leave room in many places for the insertion of arational procedures. One of the most tempting is associated with the initial construction of the universe of theories or with its expansion, when an earlier eliminative search yields unacceptable results. This move would seem to involve the conception of hitherto never conceived possibilities, perhaps even Einstein's "free inventions of the human mind." But even here it is not clear just how much arational activity is involved. In the examples we have seen in this paper, the universes of theories dealt with often have been very vaguely specified. While it is true that each universe of theories is populated with numerous hitherto never conceived possibilities, the bulk of them remain just that – never conceived. Moreover, as we have seen, the eliminative induction actually enables construction of the principal content of those favored few theories in the universe upon which attention is lavished.[66]

The assumption that Einstein's discovery process is essentially a rational exploration explains a phenomenon familiar to Einstein's readers. An explanatory account or justification of one of his theories very commonly involves a recapitulation of the historical path that led him to the theory. I have indicated several instances of this in the course of the paper, although Einstein does not usually himself point out when this happens.[67] If the discovery process is predominantly a process of rational exploration and elimination, then we should expect its recapitulation to provide abundant material for construction of an account of the justification of the theory.

Finally we might well wonder just how plausible it is for the process of discovery of a theory such a general relativity to be dominated by arational maneuvers. What faces any such process is an enormous number of candidate theories, the bulk of them essentially unarticulated. What kind of a process could select and articulate from this overwhelming flood a theory as able as general relativity to stand up to extensive later rational testing – both as to its internal logical structure and its foundation in experience? Could it be that a set of canons of rationality that cannot embrace such a process is in need of revision? Or are we prepared to entertain the possibility of mysterious processes realized in the human mind that achieve eminently rational ends by predominantly arational means?

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ADDENDUM

In the paper that follows, Professor Stachel expresses some fundamental doubts about my paper. I should like to affirm that I stand by my account as it appears in this volume. In response to some of his specific points, I note:

To say that Einstein proceeded from stage A to stage B of the development of his theory using method X (be it eliminative induction, transcendent revelation, the reading of entrails *etc.*) is perfectly in accord with Stachel's general remarks on the nature of explanation in history of science. Nowhere do I say that it could not be otherwise or that stage A *had* to develop to stage B in the same way as kinetic gas theory tells us that a gas in state A has to develop into state B.

My remarks on simplicity are intended to apply to mathematical simplicity specifically. While the sense may be "narrow," to use Stachel's word (§3), it is far from trivial given its prominence in Einstein's later thought.

Einstein's "fruitful error" (Stachel, §5), was the acceptance of the conclusion of an *inductive* argument. Since the strength of the induction was weak, the acceptance involved considerable inductive risk. As it turned out, Einstein lost the gamble. The conclusion was false. Einstein's procedure, however, was rational.

As an eager student of Stachel's work, I was fully aware of the account of the history of special relativity that Stachel lays out in his Section 5 when I wrote the original version of my paper. I do not believe that my story contradicts Stachel's account. I never intended the short section on special relativity to give a comprehensive account of the emergence of special relativity or to claim that Einstein discovered his two principles simultaneously. I carefully excised any remarks in the original version that could even vaguely suggest otherwise.

To answer Stachel's question in Section 5: If a method of solving a set of simultaneous equations is known to yield the *only* admissible solution, then the use of the method embodies a (non-ampliative) eliminative induction. You will find this fact trivial if you think that a guarantee of uniqueness of the solution is trivial. I do not find it trivial.

Stachel's pessimism (§9) over our ability to make well grounded normative judgments about methods of scientific discovery is unwarranted. The goal of scientific theorizing is theories that are confirmed or justified by experience. If such a theory is in their universe of candidates, scientists who proceed eliminatively, as did Einstein, have at least some reasonable prospect of finding it and of knowing when they have found it, because their procedure automatically generates a partial justification. I have no similar confidence in the prospects of scientists who start at the same point but proceed by the reading of tarot cards as a method of discovery in science. Unless Stachel's view of tarot cards is very different from mine, I cannot see that he could disagree.

NOTES AND REFERENCES

[1] A.' Einstein to P. Hertz, 22 August 1915, EA 12 203, as quoted in D. Howard and J. D. Norton, 'Out of the Labyrinth? Einstein, Hertz, and the Goettingen Answer to the Hole Argument,' in J. Earman, M. Janssen and J. D. Norton (eds.), *The Attraction of Gravitation: New Studies in the History of General Relativity*, Boston: Birkhaueser, 1993, p. 40. The notation EA 12 203 denotes the document with control number 12 203 in the duplicate Einstein archive, Mudd Manuscript Library, Princeton, NJ.

[2] Einstein's discovery of general relativity had been described in many places. See for example, A. Pais, *Subtle is the Lord . . . : The Science and Life of Albert Einstein* (Oxford: Clarendon, 1982) Part IV; J. Stachel, 'The Genesis of General Relativity', in H. Nelkowski *et al., Einstein Symposion Berlin auf Anlass der 100, Wiederkehr seines Geburtstag*, Lecture Notes in Physics, vol. 100, Berlin: Springer, 1972; R. Torretti, *Relativity and Geometry*, Oxford: Pergamon, 1983, Ch. 5.

[3] J. Earman, "A Plea for Eliminative Induction", unpublished manuscript.

[4] J. S. Mill, *A System of Logic: Ratiocinative and Inductive*, London: Longmans, Green and Co., 1916, Bk. III, Ch. VIII, especially §3, p. 256.

[5] See for example G. H. von Wright, *The Logical Problem of Induction*, Oxford: Basil Blackwell, 1957, Ch. IV; *A Treatise on Induction and Probability*, Peterson, N.J.: Littlefield, Adams and Co., 1960, Ch. 4;

[6] My generalization of the definition given in W. E. Johnson, *Logic*, Part II, New York: Dover, 1964, p. 210.

[7] For example, J. Dorling, "Demonstrative Induction: Its Significant Role in the History of Science", *Philosophy of Science*, **40** (1973), 360–372; 'Einstein's Methodology of Discovery was Newtonian Deduction from the Phenomena", this volume.

[8] This situation is familiar. It arises, for example, for someone seeking to evaluate *c*, the speed of light, experimentally, Prior to the experiment, the universe of choices contains a wide range of possible values, defined by the hypothesis that c has some real, positive value. The vast majority of them are unarticulated – this simply means that no one *explicitly* entertained the idea that c = 54,321 km/hr, for example. Assuming that the experiment leads to values very different from 54,321 km/hr, this choice along with the bulk of the others will remain unarticulated. Nevertheless, that particular value was a possible outcome, even if it was ruled out along with the bulk of the other values without ever being articulated.

[9] Einstein's term for the source of *a priori* views on space and time in A. Einstein, *Meaning of Relativity*, London: Chapman & Hall, 1976, p. 2.

[10] K. Popper, *The Logic of Scientific Discovery*, London: Hutchinson, 1975, pp. 31–32. See also H. Reichenbach, *The Rise of Scientific Philosophy*, Berkeley: University of California Press, 1951, p. 231, who distinguished the "context of discovery" from the "context of justification" and wrote of the former: "The act of discovery escapes logical analysis; there are no logical rules in terms of which a 'discovery machine' could be constructed that would take over the creative function of the genius."

[11] For illumination of the influence of Schlick on Einstein in these questions, see D. Howard, "Realism and Conventionalism in Philosophy of Science: the Einstein-Schlick Correspondence", *Philosophia Naturalis*, **21** (1984), 616–629.

[12] A. Einstein, "Principles of Research", in *Ideas and Opinions*, New York: Bonanza, n.d., p. 226. Such remarks are made throughout his working life. Compare with ". . . there is no method capable of being learned and systematically applied so that it leads to the goal [of establishing the first principles of a theory]." in "Principles of Theoretical Physics" (1914) in *Ideas and Opinions*, p. 221. In a letter of May 7, 1952 to Maurice Solovine he wrote, concerning a sketch of the structure of a scientific theory, ". . . there exists no logical path leading from the E [direct experiences] to A [axioms], only an intuitive (psychological) connection, which is always merely 'until further notice'." (Letter reproduced and translated in pp. 270–272 of A. P. French (ed.), *Einstein: A Centenary Volume*, Cambridge, MA: Harvard University Press, 1979.)

13 In *Ideas and Opinions*, p. 272.

14 In *Ideas and Opinions*, pp. 294–295. Presumably this faith is related to the "faith in the simplicity, i.e., intelligibility of nature" which he expresses as the view that "nature is so constituted that it is possible logically to lay down such strongly determined laws that within these laws only rationally completely determined constants occur (not constants, therefore, whose numerical value could be changed without destroying the theory.)" *Autobiographical Notes*, La Salle and Chicago, Illinois: Open Court, 1979, p. 59.

15 In addition to the usual sources in the history of special relativity, see J. Stachel, "Einstein Michelson: The Context of Discovery and the Context of Justification", *Astron, Nachr.*, **303** (1982), 47–53 especially for evidence that Einstein had toyed with an emission theory of light. See also J. Earman, C. Glymour and R. Rynasiewicz, "On Writing the History of Special Relativity", *Philosophy of Science Association, Proceedings, 1982*, Volume 2, pp. 403–416; "Reconsidering the Origins of Special Relativity", never (?) to be published manuscript.

16 This thought experiment launches Einstein's 'On the Electrodynamics of Moving Bodies', pp. 37–55 in *Principle of Relativity* (Dover, 1952). See my 'Thought Experiments in Einstein's Work", in T. Horowitz and G. Massey (eds.), *Thought Experiments in Science and Philosophy*, Savage, MD: Rowman and Littlefield, 1991, pp. 129–148 (University of America Press, forthcoming).

17 A. Einstein, "What is the Theory of Relativity", in *Ideas and Opinions*, p. 228.

18 In a letter of January 17, 1952 to Max von Laue (EA 16 167, 168) Einstein recalls that he could not base special relativity on Maxwell's theory because of its failure to yield acceptable results for the fluctuations in black body radiation pressure and the need for an atomic structure for radiation incompatible with that theory. Thus he based special relativity on the constancy of the velocity of light.

19 A. Einstein, "Zum Relativitäts-Problem", *Scientia*, **15** (1914), pp. 340–341.

20 The spacetime perspective coupled with modern differential geometry has exposed just how rough this assertion is and how many hidden assumptions it contains. See M. Friedman, *Foundations of Space-Time Theories*, Princeton: Princeton University Press, 1983, pp. 138–142.

21 Einstein seems to have expected an experiential or experimental foundation for the principles of his theories of principle. In 'Physics and Reality' (p. 307) he insists that "there is no inductive method which could lead to the fundamental concepts of physics." However he does then concede that "the most satisfactory situation is evidently to be found in cases where the new fundamental hypotheses are suggested by the world of experience itself." He then lists as examples, the non-existence of a perpetual motion machine, Galileo's principle of inertia and the "fundamental hypotheses of the theory of relativity."

22 A. Einstein, "Notes on the Origin of the General Theory of Relativity", in *Ideas and Opinions*, p. 286. We must rely on later recollections, since Einstein did not report on this reasoning in the review article that he published in *Jahrbuch der Radioaktivität und Elektronik*, **4** (1907), 411–462; **5** (1908), 98–99.

23 This problem turns out not to be straightforward. It finds its fullest development in the gravitation theory of Nordström, advanced and developed in the period 1912–1914. As a first pass at the problem, note that the four dimensional analog of the Newtonian force law is $F_i = m \nabla_i \varphi$, where F_i is the four-force on the mass and ∇_i the derivative operator of a Minkowski spacetime. Since the four-force F_i satisfies $F_i = mA_i$, the four dimensional force law will not in general allow the necessary orthogonality of the body's four acceleration A_i to the body's four velocity U_i, unless $U^i \nabla_i \varphi = 0$, which amounts to the severe restriction that the the scalar potential φ be constant along the world line of the mass.

24 A. Einstein, "Relativität und Gravitation: Erwiderung auf eine Bemerkung von M. Abraham", *Annalen der Physik*, **38** (1912), pp. 1059–1064 on pp. 1062–1063.

25 Einstein's conclusion was hasty. He conceded a year later that Nordström's special relativistic, scalar theory of gravitation did satisfy the requirement of the equality of inertial and gravitational masses of closed systems. (A. Einstein, "Zum gegenwärtigen Stande des

Gravitationsproblems", *Physikalische Zeitschrift* **14** (1913), pp. 1249–1262 on p. 1253.) The only defect of the theory. Einstein remarked (p. 1254), was that according to it the inertia of a body was influenced but not caused by all other bodies.

26 A. Einstein, "Über das Relativitätsprinzip und die aus demselben gezogenen Folgerungen", in *Jahrbuch der Radioaktivität und Elektronik*, **4** (1907), 411–462; **5** (1908), 98–99.

27 A. Einstein, "Über den Einfluss der Schwerkraft auf die Ausbreitung des Lichtes", *Annalen der Physik*, **35** (1911), 898–908; "Lichtgeschwindigkeit und Statik des Gravitationsfeldes", *Annalen der Physik*, **38** (1912), 355–369; "Zur Theorie des statischen Gravitationsfeldes", *Annalen der Physik*, **38** (1912), 443–458.

28 (New York: Dover, n.d.). The article is translated as "On the Influence of Gravitation on the Propagation of Light", pp. 90–108.

29 In 1907 and 1911, he gave the principle no name. In 1912 it was called the "hypothesis of equivalence" which gave way to the "principle of equivalence".

30 Of course that quantity would be energy so that this last constraint really amounts to requiring compatibility with force × distance = energy.

31 "Relativität und Gravitation", p. 1063. See also the letter from Einstein to P. Ehrenfest received July 7, 1912, EA 9 333.

32 This problem became even more acute in Einstein's gravitation theory of 1913–1915 in which the principle even failed in the homogeneous case *after* the restriction to infinitesimally small regions. See Section 4.3 of my "What was Einstein's Principle of Equivalence", *Studies in the History and Philosophy of Science*, **16** (1985), 203–246.

33 Letter from Einstein to P. Ehrenfest received July 7, 1912, EA 9 333.

34 "Notes on the Origin . . .", p. 286.

35 A. Einstein, "Gibt es eine Gravitationswirkung, die der elektromagnetischen Induktionswirkung analog ist?", *Vierteljahrsschrift für gerichtliche Medizin*, **44** (1912), 37–40.

36 "Relativität und Gravitation", submitted July 4, 1912, pp. 1063–1064.

37 A. Einstein and M. Grossman, "Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation", *Zeitschrift für Mathematik und Physik*, **62** (1913), 225–261.

38 As quoted on p. 12 in J, Stachel, "Einstein and the Rigidly Rotating Disk", pp. 1–15, in A. Held (ed.), *General Relativity and Gravitation: One Hundred Years After the Birth of Albert Einstein, Vol. 1*, New York: Plenum, 1980.

39 "Einstein and the Rigidly Rotating Disk".

40 And also in his *Autobiographical Notes*, pp. 63–65.

41 This generalized vector analysis has come to be known as "tensor calculus". The term "tensor" comes from vector analysis, where it labelled a quantity we would now call a second rank symmetric tensor. It was Einstein and Grossmann who generalized the use of the term and are responsible for its current prominence. Ricci and Levi-Civita do not use the term. For a review of Einstein and Grossmann's combining of the two mathematical traditions see my 'The Physical Content of General Covariance', in J. Eisenstaedt and A. J. Kox (eds.), *Studies in the History of General Relativity: Einstein Studies* **3**, Boston: Birkhäuser, 1992, pp. 281–315.

42 "Zur Theorie. . .", p. 458.

43 A. Einstein, "Physikalische Grundlagen einer Gravitationstheorie", *Naturforschende Gesellschaft. Vierteljahrsschrift (Zürich)*, **58** (1913), 284–290 on p. 285.

44 This latter point is made very clearly in A. Einstein, "Über Friedrich Kottlers Abhandlung 'Über Einstein Äquivalenzhypothese und die Gravitation'", *Annalen der Physik*, **51** (1916), 639–642. The relevant passage is translated in my 'What was Einstein's Principle of Equivalence', p. 207.

45 See, for example, A. Einstein, "The Foundation of the General Theory of Relativity", pp. 111–164 in *Principle of Relativity*, §§2, 3. See also my "The Physical Content of General Covariance".

46 EA 16 148 and EA 16 154. The translations are from my "What was Einstein's Principle of Equivalence?", p. 243 and p. 234.

47 EA 6 074 and EA 6 075. See my "What was Einstein's Principle of Equivalence?", pp. 230–231 for details.

48 Notice that result is introduced as an independent assumption. Einstein did *not* argue that the principle of equivalence could be applied to infinitesimal regions of spacetime in which special relativity would hold, so that we could infer the equation of motion from special relativity. In fact, Einstein never endorsed this infinitesimal version of the principle. He corrected Schlick in 1917, when the latter tried to use the argument just outlined, and made an objection to the infinitesimal principle that renders it essentially impotent. For further details, see my "What was Einstein's Principle of Equivalence?".

49 Note that, unless otherwise indicated, I use Einstein and Grossmann's notation of 1913. Thus contra- and covariant components of a quantity are not indicated by the raising and lowering of indices. Latin letters represent covariant quantities; the corresponding Greek letters their corresponding contravariant forms. Thus the covariant metric tensor is "$g_{\mu\nu}$" and its contravariant form is "$\gamma_{\mu\nu}$". Summation over repeated indices is *not* implied.

50 "How Einstein Found his Field Equations", *Historical Studies in the Physical Sciences*, **14** (1984), 253–316.

51 For further discussion of this derivation as well as the ensuing three year digression leading up to the final theory, see my "How Einstein Found . . .".

52 For a discussion of the the emergence of the hole argument and its place in Einstein's thought, see my "Einstein, the Hole Argument and the Reality of Space", in J. Forge (ed.), *Measurement, Realism and Objectivity*, Dordrecht: Reidel, 1987, pp. 153–188.

53 For details of the hectic proposals of this final month, see my "How Einstein Found . . .".

54 For example, one can construct one by forming a three space version of the spacetime theory in which gravitation is represented by a second rank tensor.

55 A. Einstein, "Zum gegenwärtigen Stande des Gravitationsproblems", *Physikalische Zeitschrift*, **14** (1913), p. 1263.

56 *Autobiographical Notes*, p. 15. See also R. McCormmach, "Editor's Foreword" to *Historical Studies in the Physical Sciences*, **7** (1976), xi-xxxv, for a survey of Einstein's mathematical development and its relation to then current debates over mathematics.

57 Braunschweig: Vieweg, 1911.

58 As reported in McCormmach, p. xxvii.

59 June 27, 1912, EA 39 644.

60 EA 7 066. This letter was *not* published in the collection *Albert Einstein Michele Besso: Correspondance 1903–1955*, P. Speziali (ed.), Paris: Hermann, 1972.

61 In A. Hermann (ed.), *Albert Einstein/ Arnold Sommerfeld: Briefwechsel*, Basel: Schwabe, 1968, p. 26.

62 In Speziali, p. 53. Notice that at this time Einstein had not completely turned away from the theory of invariants in his work on spacetime theory. In a joint paper submitted in February 19, 1914, he had shown that Nordström's theory of gravitation was, in effect, the theory of a conformally flat semi-Riemannian spacetime and that its field equation was actually $R = \kappa T$ where R is the Riemann curvature scalar and T the trace of the stress-energy tensor. A. Einstein and A. D. Fokker, "Die Nordströmsche Gravitationstheorie vom Standpunkt des absoluten Differentialkalküls", *Annalen der Physik*, **44** (1914), 312–328.

63 "Notes on the Origin . . .", p. 289.

64 This fact and the fact that Einstein suspended essentially all correspondence in that month in favor of a lively exchange of mail with Hilbert has raised the question of priority of discovery and even plagiarism. See my "How Einstein Found . . ." for my analysis of the relation of Hilbert's work to Einstein's in this month. I urge that Einstein's path was essentially independent in its nature from Hilbert's, so that we have a genuine case of independent discovery.

65 P. 85. We should note that this 1912–1915 episode was certainly the most prominent case in Einstein's experience in which mathematical simplicity was vindicated. But there were others. For example, within two years of its introduction, Einstein described the addition of

the cosmological term to the field equations of general relativity as "gravely detrimental to the formal beauty of the theory." A. Einstein, "Do Gravitational Fields Play an Essential Part in in the Structure of the Elementary Particles of Matter?", in *Principle of Relativity*, pp. 191–198 on p. 193. It is well known that Einstein renouced the additional term with the discovery of the expansion of the universe. See for example, A. Einstein, Review of R. C. Tolman, *Relativity. Thermodynamics and Cosmology*, in *Science*, **80** (1935), 358. Einstein later described to Gamow the introduction of the term as ". . . one of the biggest blunders he had made in his entire life." G. Gamow, "The Evolutionary Universe", in *The Universe*, London: Bell & Sons, 1958, p. 67.

[66] A mundane analogy: in computer implemented tree searches, such as are carried out by chess playing programs, one speaks as though the entire tree structure is present to be inspected. In fact very little of it need be present in the computer's memory at any time. The program need only construct those parts of the tree which are actually being visited so that in a limited sense the search actually creates the tree.

[67] There are exceptions, such as when he announced in his 1917 paper on the cosmological problem: "In the present paragraph I shall conduct the reader over the road that I have myself travelled, rather a rough and winding road, because otherwise I cannot hope that he will take much interest in the result at the end of the journey." "Cosmological Considerations on the General Theory of Relativity", in *Principle of Relativity*, pp. 177–188 on pp. 179–180.