

Weeding Landauer's Garden

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Landauer's (1961) “Irreversibility and Heat Generation in the Computing Process” speculated that there is a fundamental link between heat generation in computing machines and the logic of the computation implemented. While this proposal has become the central principle of the new “thermodynamics of computation,” it is a flawed proposal that depends on enduring misapplications of standard results in thermal and statistical physics.

1. An Enticing Proposal

Computing devices generate heat when they compute and hardware designers must provide cooling for the devices to avoid overheating and malfunctioning. One might categorize the problem as a mere technical nuisance that is not amenable or even worthy of independent foundational analysis. Landauer's (1961) paper, “Irreversibility and Heat Generation in the Computing Process” presented us with an enticing alternative. The heating effect, Landauer proposed, is no mere technical nuisance but one that arises from a profound connection between the logic of computation and the fundamental laws of thermodynamics. Whenever a computer computes, the minimum heat that must be generated is fixed by rules independent of the particular physical implementation of the computation. The limits are fixed by the logic. They

are the same no matter the hardware or the particular way the logic is implemented. Landauer's analysis provided the foundation of a new literature that Bennett (1982) later called "the thermodynamics of computation." It appears to be a powerful extension of thermodynamics to computing devices, complete with a new physical law, "Landauer's principle."¹

The principal ideas that this later literature drew from Landauer's paper are:

1. The minimum heat that must be generated in a computation is determined by its logical character, independently of the particular hardware or procedures used.
2. Logically irreversible computations, such as erasure, necessitate heat generation. Logically reversible computations do not.
3. Logically irreversible computations, such as erasure, must be implemented by thermodynamically irreversible processes. Logically reversible computations need not.
4. The erasure of an n-bit memory device reduces the number of states of the memory from 2^n to one. It corresponds to a 2^n fold compression of the device's phase space.
5. Since each of the n bit memory device's 2^n states are equally likely, its erasure moves it from a state of probability $W=1/2^n$ to $W=1$.
6. The decrease in thermodynamic entropy S of a memory device when it is erased can be computed from Boltzmann's celebrated formula,

$$S = k \ln W \tag{1}$$

Thus the entropy of an n bit memory device changes in erasure by

$$\Delta S = k \ln 1 - k \ln (1/2^n) = -nk \ln 2.$$

Since the second law prohibits a decrease of total thermodynamic entropy, this entropy change must be compensated by an increase of entropy in the environment of at least $nk \log 2$. These changes of entropy are connected with heat transfers by the Clausius definition of entropy. It asserts that the increment dS in thermodynamic entropy in a system is

$$dS = dQ_{\text{rev}}/T \tag{2}$$

where dQ_{rev} is the increment of heat passed to the system in a thermodynamically reversible process.

¹ For helpful surveys of this literature and for its connection with the Maxwell demon literature, see Leff and Rex (2003) and Maroney (2009).

7. Applying (2) to the process, the erasure of an n-bit memory device is accompanied by the passing of at least $Q = TS = nkT \ln 2$ of heat to the environment.

This collection of claims has entered into the standard repertoire of modern physics under the rubric of “Landauer’s Principle.” One cannot but be impressed on first seeing these ideas. There is an historic ring to them. In examining the limits of steam engines, Sadi Carnot observed in 1824 that all heat engines must discharge waste heat. This observation became Thomson’s second law of thermodynamics: no heat engine can fully convert heat into work. So began the new science of thermodynamics. Landauer’s 1961 paper promised a similar transformation. Computations must also discharge heat in virtue of their logical specification. So begins, it would appear, the new science of the thermodynamics of computation. Or so we might hope.

As is often the case with new discoveries, Landauer’s original paper was speculative. The results above were made plausible but not demonstrated. It is now over half a century since Landauer’s paper was published. One might think that this is ample time for more careful analysis to provide the secure foundation needed for these claims. My unhappy task in this note is to argue that these secure foundations have eluded us. While much has been written about these claims and they have been employed widely, at best they are supported by flawed argumentation and at worst they are in contradiction with standard thermal and statistical physics. The longstanding failure of attempts to repair these difficulties suggests that they are unlikely ever to be resolved. There is a single origin for many of these problems: a misapplication of Boltzmann’s famous “ $S=k \ln W$ ” through a failure to recognize the dynamical character of the probability W .

2. Dynamic and non-Dynamic Probabilities

What gives Landauer’s proposal its initial plausibility is that it is based on one of the most robust relations in thermal physics, Boltzmann’s formula (1). This relation has a remarkable range of applicability and is the starting point for many important investigations. Indeed it is a generally reliable rule of thumb that, when we encounter a probability W in a thermal system, we can use Boltzmann’s formula to assign a thermodynamic entropy to the system. However even an expansive relation like this has limits to its applicability. The Landauer

proposal breaches those limits and the breach leads to two problems to be delineated in later sections:

- the analysis misuses Boltzmann's formula (1) to ascribe an incorrect thermodynamic entropy to a memory device (Section 3); and
- the analysis neglects the entropy creation required to suppress the fluctuations necessitated by Boltzmann's formula (Sections 4 to 8).

Efforts to provide a more secure foundation for Landauer's proposal must repair this breach. It is a formidable challenge that is, in my view, unmet.

The relevant limit to the applicability of Boltzmann's formula is the type of probability employed. The W of Boltzmann's formula is a dynamic probability whose value is determined by the dynamics of the system. A thermal system migrates dynamically over many accessible states. The dynamic probability of a state is the fraction of time the system will spend in that state in the limit of infinite time. It is fixed by the dynamics.

This probability is contrasted with non-dynamic probabilities. They are fixed by means independent of the dynamics of the thermal system. These probabilities may be fixed as degrees of belief or by any means, other than the dynamical evolution of the thermal state.

This dynamic probability underpins the probabilistic understanding of the second law of thermodynamics. In its migrations, the system is most likely to migrate to and be found in the most probable state. We read directly from Boltzmann's formula that this most probable state has greatest thermodynamic entropy. With that, we recover a familiar probabilistic form of the second law of thermodynamics: systems spontaneously move to states of maximum thermodynamic entropy, where they remain, most probably. If we assume that this dynamic probability is distributed uniformly over the system's phase space, then this same result is expressed as the near certain evolution of the system to states associated with the largest volume of phase space. Correspondingly, thermodynamic entropy is associated by the Boltzmann formula with the logarithm of volumes of phase space. Entropy increases and decreases as the volume of phase space associated with a state increases and decreases.

The entropy of Boltzmann's formula (1) is only associated with heat according to the Clausius definition (2) when the system has arrived at its most probable state, thermal equilibrium. For Clausius' definition is expressed in terms of thermodynamically reversible processes that are only realized in systems brought arbitrarily close to thermal equilibrium. Then

the system's probability distribution over its phase space is Boltzmann or, equivalently, canonical.

These last considerations apply in Boltzmann's approach to statistical physics, which traces the dynamical evolution of the system from a non-equilibrium to an equilibrium state. Such an approach obviously requires dynamic probabilities. One might imagine that they are not required in the alternative Gibbs' approach.² In it, a thermal system at equilibrium is represented by an ensemble of systems, whose number distribution replicates the probability distribution of the Boltzmann approach. However these ensemble counts must match the dynamic probabilities. Only then can the expression for entropy of a canonically distributed system in the Gibbsian approach³

$$S = -k \int_{\Gamma} \rho(x) \ln \rho(x) dx = \frac{\bar{E}}{T} + k \ln \int_{\Gamma} \exp(-E(x)) dx$$

agree with the Clausius, heat based definition (2) of entropy for canonically distributed systems. For the agreement is demonstrated by tracking the heat transferred during a thermodynamically reversible process. Gibb's canonical entropy can only match the heat based entropy of Clausius' formula if we assume that, during the process, each system in the ensemble explores its phase space dynamically with the relative occupation times matching the canonical distribution.

These results are fundamental to the integration of thermal and statistical physics. Ehrenfest and Ehrenfest (1990, pp. 60-61) summarize the results that assert the agreement with the Clausius definition of entropy for both Boltzmannian (Eq. 75') and Gibbsian (Eq. 74) entropies. Norton (2005, Sections 2, 3 and Appendix A) provides an analysis of Gibbs' canonical entropy and the ensemble approach that makes clear its dependence on dynamic probabilities.

3. Misattribution of Thermodynamic Entropy to Memory Devices

The basis of Landauer's analysis is that erasure reduces the number of states in a memory device; that this is a compression of the device's phase space; and that this leads to a reduction in the thermodynamic entropy of the device. The error here is obvious. Prior to erasure, the memory

² For a comparison of the approaches see Ehrenfest and Ehrenfest (1990, Ch. III).

³ The system is canonically distributed with probability density $\rho(x) = \exp(-E(x)/kT)/Z$ over a phase space Γ with generalized coordinates x and energy at each phase point $E(x)$.

device is in just one of its 2^n states; and after erasure it is in another. The logical specification of the process of erasure does not require any compression of the phase space. It merely requires a relocation of which part is occupied.

The need to compress phase space arises because of the particular erasure protocol employed by Landauer and virtually all authors who followed him. They insist on a dissipative erasure procedure in which the memory device is thermalized. The energetic barriers that prevent each bit from flipping are dropped, so that the bits can flip to and fro in thermal agitation and the memory device can migrate freely over all its 2^n states. This is a thermodynamically irreversible process that is the analog of allowing an n molecule ideal gas to expand twofold without doing work. This thermalization process is responsible for the creation of the thermodynamic entropy that appears in subsequent calculations.

The same error is committed in terms of the probabilities W in Boltzmann's formula in propositions 5 and 6 above, where the erasure is portrayed as moving the system from a state of probability $W=1/2^n$ to $W=1$. The error is that this probability ratio is not a ratio of dynamic probabilities. The un erased memory device is *not* migrating over all possible 2^n states. The probability $W=1/2^n$ assigned to the un erased memory device does not correspond with relative occupation times. If the device did so migrate, the un erased device would be useless for storing data. The probability $W=1/2^n$ is derived from another source. Perhaps we believe that each of the possible states is equally likely; or we expect that in typical computations we will run into each configuration equally often. Both of these are non-dynamic probabilities that cannot be substituted into the Boltzmann formula (1) if a thermodynamic entropy is to be recovered.

Dynamic probabilities do appear as an intermediate in the particular erasure procedure employed. They appear when the memory device is thermalized so that it can migrate freely over all its 2^n states. As before, this thermodynamically irreversible thermalization step is responsible for the creation of $nk \log 2$ of thermodynamic entropy. It does not derive from the logic of erasure, but from a step in the particular erasure procedure employed.

This treatment of the memory state as if it were the thermalized state is pervasive. In response to earlier analyses, I had been assured that newer, better demonstrations of Landauer's principle avoid the conflation. However, as Norton (2011, Appendix) reports, examination of these purported improvement show the conflation remains the basis of virtually all the

demonstrations. However it is harder to see since the conflation is buried more deeply in an ever-growing thicket of formalism.⁴

4. Fluctuations in Small n Systems are Unavoidable

That the probabilities in Boltzmann's formula (1) are dynamic necessitates thermal fluctuations. Any analysis based on the formula cannot ignore them. To see them, take an ideal gas consisting of n molecules momentarily confined to half a vessel. The gas will most probably expand to fill the vessel. Since each molecule moves independently, the ratio of probabilities of the final expanded state to the initial state is $W = 2^n$. Boltzmann's formula then assigns an entropy change of $S = k \ln 2^n = nk \ln 2$ to this twofold expansion in volume. It matches the expression from the ordinary thermodynamics of ideal gases.⁵ The dynamical character of the probabilities permits a reversal of this expansion. With the small probability of $1/2^n$, all the molecules may momentarily be located in the original half of the vessel. This spontaneous recompression of the gas would be an improbable thermal fluctuation.

For macroscopic systems, these fluctuations are imperceptible. In systems with smaller numbers of components, where "n" is small, the probabilities for fluctuations are substantial and will reverse processes that would otherwise complete without difficulty in large n systems. The twofold expansion of a three molecule ideal gas, for example, is reversed by a fluctuation with probability $(1/2)^3 = 1/8$. The reversal will occur routinely.

This result for small n systems is quite general. It is recovered most easily from Einstein's fluctuation formula (1904, p. 360). Consider a system that has come to thermal equilibrium with a large heat reservoir at temperature T. The system's energy E is canonically distributed. Using $\langle \dots \rangle$ to designate expectation values, the variance of the energy is related to the mean energy $\langle E \rangle$ by

$$\langle \epsilon^2 \rangle = kT^2 d\langle E \rangle / dT$$

⁴ A welcome exception is the demonstration provided by Ladyman et al. (2007, 2008, 2013). The problems of these demonstrations have been laid out in Norton (2011, 2013).

⁵ This example is Einstein's (1905) from his light quantum paper in which he memorably labels formula (1) "Boltzmann's Principle."

where $\epsilon = E - \langle E \rangle$ is the deviation of the energy from its mean value. Most systems commonly considered in thermal physics have Hamiltonians that are quadratic in their canonical phase space variables. The equipartition theorem applies to such systems. Each has an additive contribution of $kT/2$ to its mean energy for each degree of its m degrees of freedom. The mean energy is:

$$\langle E \rangle = mkT/2$$

It follows from Einstein's fluctuation formula that the spread in the energy, as measured by the root mean square ("rms") deviation is

$$\langle \epsilon^2 \rangle^{1/2} = (m/2)^{1/2} kT$$

The key dependency is that the spread in energy grows slowly with the square root of the number of degrees of freedom m , whereas the mean energy $\langle E \rangle$ grows faster, linearly in m . That means that fluctuations become negligible for macroscopic systems with large values of m . For such systems, the mean energy is of the order of 10^{24} in units of kT . Energy fluctuations are merely of the order of 10^{12} in energy units of kT .

Matters are quite different for systems with small numbers of components. Then the energy fluctuations will be substantial in relation to the system's mean energy. A monatomic ideal gas is often used in a simple model of a one bit memory device in the thermodynamics of computation. This gas has three degrees of freedom, so the spread in its energy is given by

$$\langle \epsilon^2 \rangle^{1/2} = (3/2)^{1/2} kT = 1.22 kT$$

That means that gas energy fluctuates over an rms range of $0.28kT$ to $2.72kT$. These are substantial fluctuations that will present an obstacle to completion of processes. Imagine, for example, that we want to double the energy of the gas by heating it to twice the initial temperature. A fluctuation that reverts the energy to its original energy lies well within this rms range. The energy increase will be spontaneously undone and redone, repeatedly, by fluctuations.

5. No Go Result: The Simple Idea⁶

The last section recounted two instances of how thermal fluctuations disrupt the completion of process in systems with small numbers of components. This disruption is part of a general “no go” result that applies to all such systems. We should like process in such systems to:

- (i) be minimally dissipative, that is, create the minimum of thermodynamic entropy; and
- (ii) complete with certainty or at least high probability.

The no go result is that satisfaction of one of these conditions precludes satisfaction of the other. If we seek minimum thermodynamic entropy creation, we must forgo any workable probability of completion. If we seek substantial probabilities of completion, we must create quantities of thermodynamic entropy that are large on molecular scales.

Since the n-bit memory devices of Landauer’s proposal are small n system, this tension applies to all the processes presumed in developing the proposal. No process at this scale can be brought to completion with high probability unless there are dissipative, entropy creating processes somewhere in the system. Yet their necessary presence is routinely neglected and the associated entropy creation ignored. In the standard erasure protocol, for example, it is assumed that the state space of a thermalized n-bit memory device can be compressed reversibly without creation of thermodynamic entropy.

The basic idea of the no go result is recoverable without computation from Boltzmann’s formula (1). Consider some process that we set up to proceed from an initial state “init” to a final state “fin,” where the process moves forward in virtue of the dynamics of the system. In order to minimize entropy creation, we must keep all processes as close to thermodynamic reversibility as possible. Such processes are constant thermodynamic entropy processes. In the limit case sought, we seek a process whose initial entropy S_{init} and final entropy S_{fin} are equal:

$$S_{\text{init}} = S_{\text{fin}}$$

⁶ This “no go” result that has been developed in Norton (2011, Section 7; 2013a, Part II; 2013b; 2017). These papers contain computations of fluctuations in specific processes, including the expansion of ideal gases of few and many molecules and the measuring of the state of an electric dipole.

It follows immediately from Boltzmann's formula (1) that the dynamic probabilities of the two states are the same.

$$W_{\text{init}} = W_{\text{fin}}$$

Since these probabilities are dynamic, they describe fluctuations. We learn that fluctuations so confound a constant entropy process that we are as likely to find it in its initial state as in its final state.

For any small component process, computational or otherwise, we can only assure its completion probabilistically by raising the entropy of the final state in relation to the initial state. The resulting entropy costs are substantial. To secure a modest ratio of success of merely $(W_{\text{fin}}/W_{\text{init}}) = 20$, we must have a process that creates at least $3k$ of entropy:

$$\Delta S = k \ln (W_{\text{fin}}/W_{\text{init}}) = k \ln 20 = 3k$$

The quantities of entropy required by this formula to suppress fluctuations are large in comparison to those tracked by Landauer's principle. They cannot be ignored as a nuisance to be dealt with in other ways. They arise from the same relation $S = k \ln W$ that is essential to Landauer's proposal.

These quantities of entropy are independent of the logical specification of whichever computation is implemented by the process. They are determined merely by the probability of successful completion specified. If, as is commonly the case, a computation requires multiple steps to be completed successively, then there will be a corresponding quantity of entropy associated with the completion of each step. This means that an attainable lower limit to dissipation in molecular scale processes is neither given by Landauer's expression nor is it independent of the details of the implementation of the computation. Any estimate of an attainable lower limit has to include the thermodynamic entropy that must be created to assure completion with the specified probability of each step of the implementation used.

6. No Go Result: A More Developed Version

The version of the no go result described in the last section was simplified by its neglect of the states intermediate between the initial and final states of the process. Including them reveals more sources of thermodynamic entropy creation when we try to implement thermodynamically reversible processes on molecular scales.

Consider a process whose degree of completion is tracked by a continuous variable λ . The process could be an expansion or contraction of the accessible volume of a system's configuration space; or a measurement process in which the state of some measuring device is brought to match that of a target system; or a transfer of data from one memory device to another; or the setting of the content of one memory device as some specified function of another; or any other process required of some computational system that has a definite initial and final state. In seeking the minimum dissipation, we seek a thermodynamically reversible process, that is, one in which the thermodynamic entropy S of the system and its environment remain as close as we can achieve to constant throughout:

$$dS(\lambda)/d\lambda = 0 \quad \text{and thus} \quad S(\lambda_{\text{init}}) = S(\lambda_1) = S(\lambda_2) = S(\lambda_{\text{fin}})$$

Applying Boltzmann's formula (1) to this case, we recover a uniform probability density w over the path parameter λ :

$$w(\lambda_{\text{init}}) = w(\lambda_1) = w(\lambda_2) = w(\lambda_{\text{fin}})$$

Since these probabilities are dynamic, it follows that the system is equally likely to be found in any of the stages, including the initial and final states and any arbitrarily chosen intermediate stages 1 and 2. The system fluctuates back and forth with limiting occupation times matching these probabilities. Attempting to implement a thermodynamically reversible process leads to one so confounded by fluctuations that it could be equally in any of its stages.

We can now see how the accessibility of the intermediate stages leads to further entropy creation than suggested by the simpler analysis of Section 5. Assume for convenience that the stages are divided into n steps: $\lambda = 0$ to 1, $\lambda = 1$ to 2, ..., $\lambda = n-1$ to n . We initialize the system in a state corresponding to a range of values, $\lambda = 0$ to 1. It is released and allowed evolve dynamically over the full range of stages. The final state "eq" is not the intended final state $\lambda = n-1$ to n . Rather, because all intermediate stages are accessible, it is a state uniformly distributed over all stages $\lambda = 0$ to n . The ratio of probabilities is $W_{\text{eq}}/W_{\text{init}} = n/1$. Hence the thermodynamic entropy creation is

$$\Delta S = k \ln (W_{\text{eq}}/W_{\text{init}}) = k \ln n$$

This entropy is created without any assurance that stages of larger λ have greater probability. The probability of the intended final state, $\lambda = n-1$ to n , is just

$$W_{\text{fin}} = 1/n$$

If we are to improve the probability of successful completion of the process, it must be designed so that later stages have higher thermodynamic entropy:

$$dS(\lambda)/d\lambda > 0$$

The greater the entropy change over the stages, the higher the probability of later stages.

7. An Illustration

A simple example illustrates how such an entropy gradient enhances the probability of completion. Assume that we have a system in thermal contact with a large heat reservoir at temperature T and a system Hamiltonian given by

$$H(\lambda) = f(\pi) - \varepsilon\lambda$$

The stages of the process are tracked by the parameter λ , which is also assumed to be canonical. The quantity $\varepsilon \geq 0$ introduces an energy gradient that inclines the system to evolve towards larger values of λ . The remaining canonical coordinates of the system are represented jointly by π and their contribution to the Hamiltonian by the term $f(\pi)$, which is independent of the stage of completion of the process. They need not be represented more completely, since they will drop out of the calculation.

Two instantiations of systems with this Hamiltonian are given in Norton (2013a, §10 and §11). The first is a small bead that slides frictionlessly on a straight wire. The inclination of the wire to the horizontal yields a gravitationally induced energy gradient that drives the bead from one end of the wire to the other. The second is a charge that we seek to move in channel. The energy gradient that moves the charge is provided by a constant electric field.

The probability that the system lies between two stages λ_1 and λ_2 is proportional to the partition integral:

$$Z(\lambda_1, \lambda_2) = \int_{\lambda_1, \pi}^{\lambda_2} \exp\left(-\frac{H(\lambda)}{kT}\right) d\lambda d\pi = \text{const.} \int_{\lambda_1}^{\lambda_2} \exp\left(\frac{\varepsilon\lambda}{kT}\right) d\lambda = \text{const.} \frac{kT}{\varepsilon} \left[\exp\left(\frac{\varepsilon\lambda_2}{kT}\right) - \exp\left(\frac{\varepsilon\lambda_1}{kT}\right) \right]$$

This probability is no longer uniform and, for larger ε , favors larger λ . If the intended final state is $\lambda = n-1$ to n , then we compute its probability as:

$$W_{fn} = \frac{Z(n-1, n)}{Z(0, 1)} = \frac{\exp(\varepsilon n / kT) - \exp(\varepsilon(n-1) / kT)}{\exp(\varepsilon n / kT) - 1} = \frac{1 - \exp(-\varepsilon / kT)}{1 - \exp(-\varepsilon n / kT)}$$

The thermodynamic entropy created by the process that released the system from its initial state $\lambda=0$ to 1 is:

$$\Delta S = k \ln \left[\frac{Z(0,n)}{Z(0,1)} \right] = k \ln \left[\frac{\exp(\epsilon n / kT) - 1}{\exp(\epsilon / kT) - 1} \right]$$

These last two formulae have two revealing limiting cases.

If take the limit as ϵ goes to zero, we have the case of no driving force, a zero energy gradient, and we recover:

$$W_{\text{fin}} = 1/n \quad \text{and} \quad \Delta S = k \ln n$$

As expected, we have the least entropy creation of Section 6 but with an unsatisfactory probability of success.

If we take the case of large ϵ in which the process is driven forward by a steep energy gradient, we recover

$$W_{\text{fin}} \approx 1 - \exp(-\epsilon/kT) \quad \text{and} \quad \Delta S \approx \epsilon(n-1)/T$$

In this last case, we have a probability of successful completion that can be brought as close to one as we wish by making ϵ sufficiently large. However we have large quantities of entropy created in proportion to $(n-1)$. The entropy created allows a simple interpretation in terms of the Clausius definition of thermodynamic entropy (2). It is the entropy created by a reversible transfer of heat to the heat reservoir in the amount $\epsilon(n-1)$. That heat corresponds to the energy lost by the system in moving down the energy gradient from a stage with $\lambda=1$ to one with $\lambda=n$.

8. Least Dissipative Processes Derived Using Fluctuations

The analysis of the last two sections shows that the accessibility of intermediate stages in a process creates more thermodynamic entropy than called for by the Boltzmann formula $\Delta S = k \ln (W_{\text{fin}}/W_{\text{init}})$. It turns out that, if we eschew the attempt to keep processes everywhere thermodynamically reversible, it is possible to use fluctuations to arrive at this lower amount of entropy creation. Fluctuations become part of the solution and are not just the source of the problem.

This lower dissipation is achieved by assuring that all intermediate states “inter” have energies far higher than the energies of the initial and final states.

$$E_{\text{inter}} \gg E_{\text{init}} \quad E_{\text{inter}} \gg E_{\text{fin}}$$

If the system is in thermal equilibrium with a heat reservoir at temperature T , the probability of a state is canonically distributed. Thus the probability of these intermediate states is proportional to $\exp(-E_{\text{inter}}/kT)$. This factor will be significantly less than the corresponding factors for the probabilities of either the initial or final states, $\exp(-E_{\text{init}}/kT)$ and $\exp(-E_{\text{fin}}/kT)$.

The process advances only when an improbable fluctuation leads the system to jump from the initial state to an intermediate state of much higher energy, from which it can then revert to either the initial or the final state. If we assume that the factor $\exp(-E_{\text{inter}}/kT)$ is negligible, the probability that the process over the longer term delivers the system in the final state is given as

$$W_{\text{fin}} = \frac{\exp(-E_{\text{fin}} / kT)}{\exp(-E_{\text{fin}} / kT) + \exp(-E_{\text{init}} / kT)} = \frac{1}{1 + \exp((E_{\text{fin}} - E_{\text{init}}) / kT)}$$

Thermodynamic entropy is created in the transition from the initial state “init” to a state that is the probabilistic weighted combination of the initial state “init” and the final state “fin.” It is given by

$$\Delta S = k \ln \frac{\exp(-E_{\text{init}} / kT) + \exp(-E_{\text{fin}} / kT)}{\exp(-E_{\text{init}} / kT)} = k \ln [1 + \exp(-(E_{\text{fin}} - E_{\text{init}}) / kT)]$$

The force driving the process is the energy difference $E_{\text{fin}} - E_{\text{init}}$. As before, we can take two limiting cases.

When that energy driving force goes to zero, $E_{\text{fin}} - E_{\text{init}} = 0$, then we have

$$W_{\text{fin}} = 1/2 \quad \text{and} \quad \Delta S = k \ln 2$$

It is an improvement on the corresponding results of Section 7. The probability of successful completion is increased from $1/n$ to the still modest $1/2$ and the entropy created reduced from $k \ln n$ to $k \ln 2$.

When the energy driving force is large, $E_{\text{fin}} - E_{\text{init}} \ll 0$, then we have

$$W_{\text{fin}} \approx 1 - \exp((E_{\text{fin}} - E_{\text{init}})/kT) \quad \text{and} \quad \Delta S \approx -(E_{\text{fin}} - E_{\text{init}})/T$$

Once again, the probability of successful completion, W_{fin} , can be brought as close to one as we wish by making $E_{\text{fin}} - E_{\text{init}}$ sufficiently negative. However the entropy created rises correspondingly. The thermodynamic entropy created allows a simple interpretation in terms of the Clausius definition (2). It is the thermodynamic entropy that would be gained by the thermal

reservoir if the energy lost by the system $-(E_{\text{fin}} - E_{\text{init}})$ were to be imparted to the reservoir as heat in a thermodynamically reversible process.

A comparison with the formulae of Section 7 for the case of a large driving force shows a reduction in the entropy created for the same probability of completion. Fix some desired value for W_{fin} . To provide it, the process of Section 7 with accessible intermediate states requires creation of $n-1$ times as much entropy as the process of Section 8 that does not allow these intermediate states to be accessible.

While rendering the intermediate states energetically inaccessible may be appealing for the ensuing reduction in thermodynamic entropy creation, it has its own problem. The probabilities computed here are dynamic and correspond to the relative occupation times of the system. Since the process must still pass through the intermediate states and they have very low probability, the process will take a long time to complete, while we wait for a highly improbable random fluctuation.

9. Confusions over Thermodynamic Reversibility

The issues of the preceding sections are, in my view, the principal difficulties faced by Landauer's proposal. Seeing them clearly, however, has proven difficult because they are obscured by further confusions. An important one concerns thermodynamically reversible processes.

Giving a precise characterization of thermodynamic reversibility is delicate and even more so that one might imagine, as I found in Norton (2016). Loosely speaking, a thermodynamically reversible process is one that proceeds with the most minute deviations from equilibrium. Within those minute deviations, it can proceed in either direction.

The most troublesome confusion in the Landauer principle literature is the erroneous claim that the (irreversible) thermalization process is thermodynamically reversible.⁷ The basic argument is that, prior to thermalization, there is a probability of $1/2^n$ that the memory device is in each of the possible 2^n states. After thermalization, the same probabilities obtain. Therefore (?), by the Boltzmann formula, the entropy of the memory device is unchanged during thermalization and the process is thermodynamically reversible. Of course the fallacy is that the

⁷ This is reported as a standard result in Leff and Rex (2003, p. 21).

probability prior to thermalization is not a dynamic probability, so that the Boltzmann formula cannot be applied.

This confusion then threatens to undo the claim that a logically irreversible process like erasure must be implemented by a thermodynamically irreversible process. For if that confusion is accepted, the erasure process can proceed entirely with thermodynamically reversible steps. To erase a memory device we first thermalize it, which is itself already a logically irreversible process. We then compress its state space to a single state. The compression can be carried out in a thermodynamically reversible manner, analogous to the thermodynamically reversible compression of a gas. The compression does require that thermodynamic entropy be passed to the environment and, following the Clausius definition (2), that the environment is heated. We recover the heating effect that grounded Landauer's original paper but now in a process that we are to suppose is thermodynamically reversible.

10. Conclusion

We now see that each of the claims of Landauer's original proposal, as enumerated in Section 1 above, turns out to be unfounded or refuted.

- Contrary to Propositions 1 and 2, the need to suppress fluctuations places a lower limit on entropy creation that is unconnected with the logic of the computation. It is set merely by the number of steps chained in the computation and the probability of completion specified for each.
- Proposition 3 is contradicted by the Landauer principle literature itself when it supposes erroneously that the logically irreversible thermalization of a memory device is thermodynamically reversible.
- Proposition 4 fails since erasure only relocates the occupied location of a phase space and does not compress it.
- Proposition 5 mixes non-dynamic and dynamic probabilities, so that their insertion into Boltzmann's formula (1) in Proposition 6 fails to determine a thermodynamic entropy.
- Proposition 7 fails to recover quantities of heat since the entropy changes computed in Proposition 6 are not thermodynamic entropies that are subject to Clausius' definition (2).

While the concerns described above have been laid out in simple terms, I believe that they are decisive and ultimately unanswerable. Much more can and has been said. The debate is elaborate and can be entered through the references to which the reader is now referred.

If I have established well enough that all is not well with the present foundations of the thermodynamics of computation, might one plead for indulgence since Landauer's proposal has been fertile? Should we not give it a chance to prove itself? "Let a thousand flowers bloom." No—I do not see the fertility. All I see is the endless repetition in various forms of the same ill-founded claims. This literature has been given chance enough to produce the promised sound foundation. Flowers cannot bloom in a garden overgrown with weeds. Over fifty years after Landauer planted the seeds, we should realize that what sprang from them are weeds that threaten to overtake our garden. Let us weed the garden and give the real flowers a chance.

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