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A PARADOX IN NEWTONIAN GRAVITATION THEORY II*

Abstract: Newtonian cosmology, in its original form, is logically inconsistent. I show the inconsistency in a rigorous but simple and qualitative demonstration. "Logic driven" and "content driven" methods of controlling logical anarchy are distinguished.

In traditional philosophy of science, we routinely attribute powers to scientists that are near divine. It is only in desperate circumstances that we may even entertain the possibility that scientists are not logically omniscient and do not immediately see all the logical consequences of their commitments. The inhabitants of the grubby world of real science fall far short of this ideal. In truth they will routinely commit themselves consciously and even enthusiastically to the great anathema of philosophers: a logically inconsistent set of propositions. In standard logics, a logical inconsistency triggers anarchy. From it, one can derive any proposition, so that an inconsistent theory can save any phenomena whatever. Were a Newton to advance an inconsistent gravitation theory, then we know *a priori* that he could derive any planetary orbit he pleases. Whatever the planetary orbits—be they circular, elliptical, square or hexagonal—they can be derived validly within an inconsistent theory. An inconsistent theory can give you any result you want and everything else as well.

Under these bizarre circumstances, the challenge to philosophers of science is to determine whether we can take logically inconsistent scientific theories seriously and, if we can, how we are to do this. As it turns out, there is no shortage of general philosophical schemes which tolerate logical inconsistency without anarchy. What is in short supply are good case studies that can reveal clearly which of these schemes matches the actual practice of science. The problem is that current case studies are typically of two types. Either they are contrived "toy" models, whose logical relations are clear but whose connection to real science is dubious. Or they are instances of real science of such complexity that one must be disheartened by the

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J. Meheus (ed.), Inconsistency in Science, 185—195.

task of mastering the scientific technicalities let alone disentangling its logical structure.[1]

My purpose here is to present an instance of a logically inconsistent theory which is:

- a real and significant piece of science, debated most recently in the primary scientific journal literature of the 1950s;
- indisputably logically inconsistent in the traditional strict sense that both propositions *A* and not-*A* can be derived within the theory; and
- technically so transparent that the inconsistency can be displayed essentially without equations.

This instance is presented with an invitation to apply your own favorite analysis of logical inconsistency in scientific theories in order to see how well your analysis fits.

1. LOGICAL INCONSISTENCY OF NEWTONIAN COSMOLOGY

The logical inconsistency to be displayed here is within Newtonian cosmology. It is a theory whose basic postulates are:

Mechanics. Newton's three laws of motion in Newtonian space and time. Inverse square law of gravitational attraction.

Cosmology. Infinite Euclidean space is filled with a homogeneous, isotropic matter distribution.

The basic result to be developed here is that one can combine standard theorems in Newtonian gravitation theory to conclude that

(1) The net gravitational force on a test mass at any point in space is **F**, where **F** is a force of *any nominated magnitude and direction*.

Thus the theory is logically inconsistent, since we can prove within the theory that the force on a test mass is both some nominated **F** and also not **F**, but some other force.

2. A PICTORIAL REPRESENTATION OF THE NEWTONIAN GRAVITATIONAL FIELD

In order to derive (1) from the postulates of Newtonian gravitation theory, we need essentially only those properties of the Newtonian gravitational field which can be represented in a simple lines of force picture. The essential properties which we shall need are shown in Figure 1 and are:

[1] My own case study of the inconsistency of the old quantum theory of black body radiation (Norton 1987) is a good example, unfortunately. Compare with Smith 1988 and Brown 1990.

- The intensity of the gravitational force on a test mass is given by the density of the lines of force and the direction of the force by the direction of these lines.
- The lines of force can never cross.
- The lines of force may only terminate in a source mass.
- The total number of lines of force terminating in a source mass is proportional to the mass of the source.

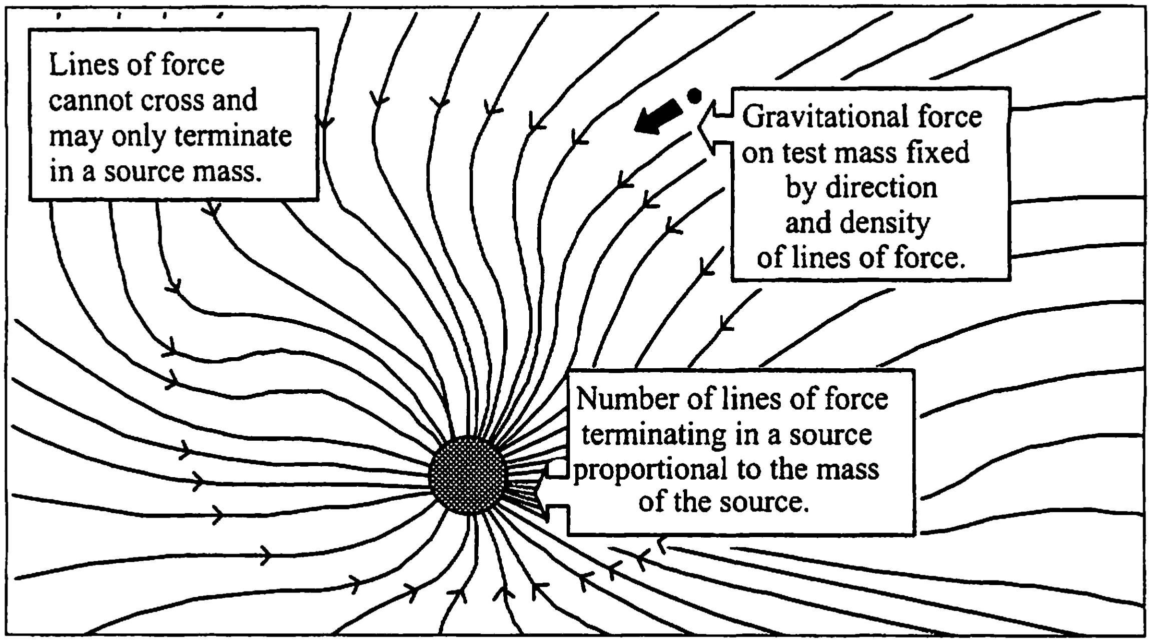

Figure 1. The lines of force model of the Newtonian gravitational field

Notice that these properties already capture a very significant part of Newtonian gravitational theory. For example, they are sufficient to establish that the gravitational force exerted by a source mass on a test mass must diminish with the inverse square of the distance between them in a three dimensional space.[2]

To derive the inconsistency (1) within Newtonian cosmology, we first need two theorems readily demonstrable within the lines of force picture.

Theorem 1

A spherical shell of source masses exerts no net force on a test mass located at any position within the shell.

To see this, imagine otherwise, that is, that there is a net field within the sphere. (See Figure 2.) The lines of force of this field must respect spherical symmetry. This

[2] To see this, consider the spherically symmetric field of the mass. The same total number of lines of force penetrate any sphere centered on the mass. But the area of the sphere increases with the square of its radius. Therefore the intensity of the lines of force on the sphere's surface diminishes with the inverse square of the radius. Since this intensity gives us the magnitude of the gravitational force on a test mass located on the surface of the sphere, this force diminishes with the inverse square of distance from the source mass.

uniquely determines lines of force that lie radially in the shell and cross at the center. Since there is no source mass at the center, this crossing is not allowed. Therefore there can be no field within the shell and no net gravitational force on a test body within it.

Theorem 2
A spherically symmetric source mass distribution has the same external field as a point source of the same mass.

To see this, note that a field is fully specified if we fix its total number of lines of force and require it to be spherically symmetric about some point. (See Figure 3.)

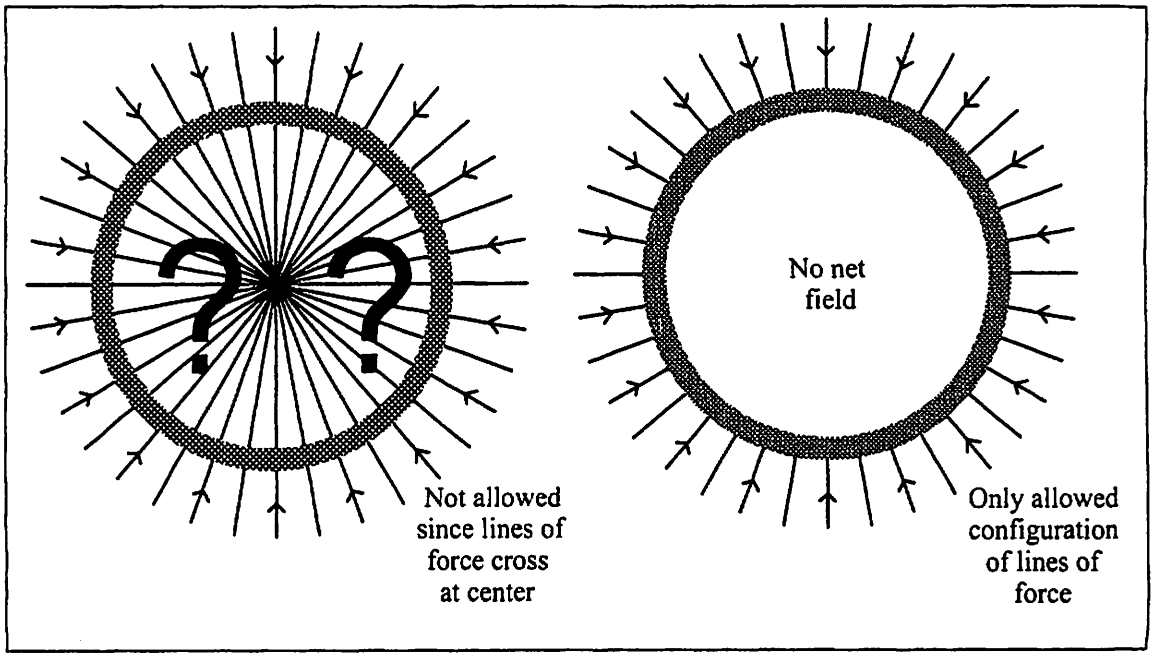

Figure 2. No net gravitational field within a spherically symmetric shell

In this case, both external fields have the same number of lines of force, since their sources have the same mass. Again both fields must be arranged spherically symmetrically about the center of their source masses. Therefore both external fields are the same.

3. DERIVATION OF THE CONTRADICTION

Since Newtonian gravitation theory is a linear theory, we can compute the net gravitational force on a test mass as the sum of the forces exerted by all the individual source masses. To find the net gravitational force on a test mass in Newtonian cosmology, we may consider the infinite source mass distribution divided into finite parts. Each part exerts some (possibly vanishing) force on the test mass and the net force is simply the sum of these forces. It turns out that dividing upthe sources masses in different ways, in this case, can lead to a different final net force. In particular, for *any* nominated force **F** on a test mass, it turns out that we can always find a way of dividing the source masses of Newtonian cosmology so that the resultant net force is **F**.

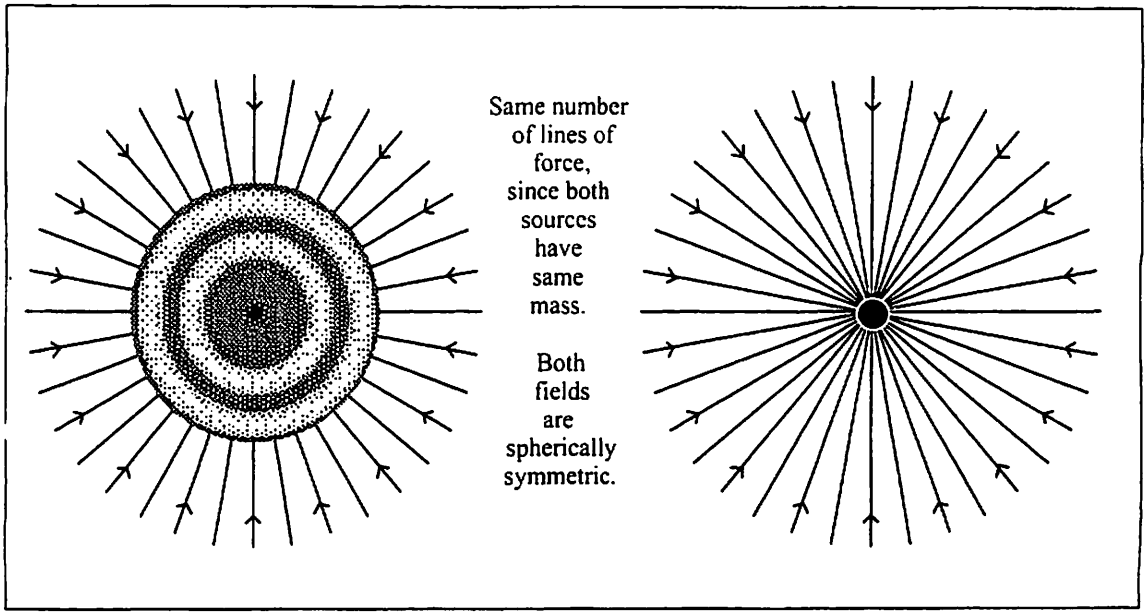

Figure 3. External field of a spherically symmetric source distribution same as field of a point source with same mass

How this may be done is summarized in Figure 4. First we consider a test mass within a Newtonian cosmology and nominate a force **F** of arbitrary magnitude and direction. We then divide the source mass distribution into a sphere surrounded by concentric shells. The sphere is chosen so that the test mass sits on its surface. The sphere's position and size are determined by the requirement that the net force exerted by the sphere on the test mass be **F**. Theorem 2 ensures that we can always find a sphere of suitable size and position to satisfy this requirement.[3] The test mass lies within the concentric shells of the remaining source masses. Therefore, from Theorem 1, each of these shells exerts no net gravitational force on the test mass. Summing, the total force exerted by all source masses—the sphere and the concentric shells—equals the arbitrarily chosen **F** and we have recovered (1) stated above.

[3] The point is intuitively evident, but I give the details for zealots. From the theorem, the force due to the sphere is the same as the force due to a corresponding point source of equal mass located at the sphere's center. Thus, by placing the center of the sphere in some arbitrarily nominated direction from the test mass, we can fix the direction of **F** arbitrarily. Similarly we can set the magnitude of **F** arbitrarily by choosing an appropriate radius for the sphere. It turns out that the force exerted by the sphere grows in direct proportion to its radius, so that all magnitudes are available. To see this linear dependence, note that the force exerted by the corresponding point source of Theorem 2 grows in direct proportion to its mass and decreases as the inverse square of the radius of the sphere. However the mass of the sphere grows in direct proportion to its volume, that is, as the radius cubed. Combining the two dependencies, we recover the direct proportion of the force and the radius.

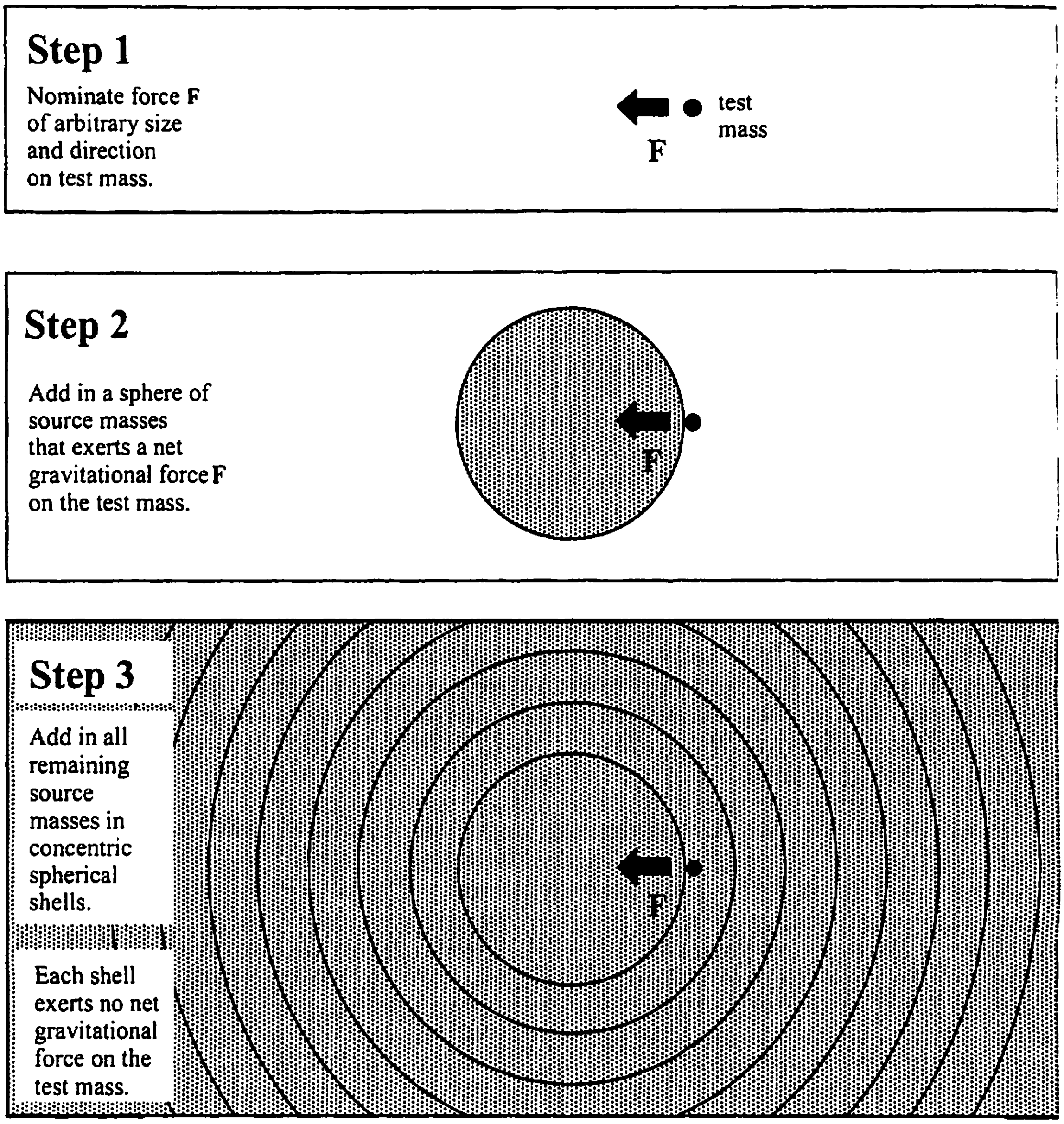

Figure 4. Proof that net gravitational force on a test mass is any arbitrarily nominated force within Newtonian cosmology

4. REACTIONS TO THE INCONSISTENCY[4]

Although the inconsistency of Newtonian cosmology is structurally similar to Olber's celebrated paradox of the darkness of the night sky, the inconsistency was not pointed out clearly and forcefully until the late nineteenth century in the work of Seeliger. Einstein doubtlessly contributed to its dissemination when he invoked it as

[4] For this section, I am grateful to Philip Sharman for bibliographic assistance.

a foil to assist his development of relativistic cosmology in the mid 1910s.[5] Research into Newtonian cosmology did not cease with the advent of relativistic cosmology. In the early 1930s, Milne and McCrea discovered that, in certain aspects, Newtonian cosmology gave a dynamics identical to that of the relativistic cosmologies. This result engendered a tradition of research in neo-Newtonian cosmology in which the inconsistency had eventually to be addressed.

Within these many analyses of Newtonian cosmology, there seem to be three types of responses by physical theorists to the inconsistency of Newtonian cosmology:

- They are unaware of the inconsistency and derive their results without impediment. This was Newton's response when Bentley outlined the problem to him in their celebrated correspondence; Newton simply denied there was a problem![6] Milne and McCrea's early papers make no reference to the problem even though they use the very construction of Figure 4 to arrive at a *non*-uniform force distribution. (See Milne 1934 and McCrea and Milne 1934.)
- They are aware of the inconsistency but ignore the possibility of deriving results that contradict those that seem appropriate. This seems to be the case with Narlikar (1977, 109-110), and certainly with Arrhenius (1909, 226) whose diagnosis is that the paradox "only proves that one cannot carry out the calculation by this method".[7]
- They find the inconsistency intolerable and seek to modify one or other of the assumptions of the theory in order to restore its consistency. (See Seeliger 1895, 1896; Einstein 1917, §1; Layzer 1954.) As my survey (Norton 1999) shows, of those who explicitly address the problem, this is by far the most common response. At one time or another, virtually every supposition of Newtonian cosmology has been a candidate for modification in the efforts to eliminate the inconsistency. These candidates include Newton's law of gravitation, the uniformity of the matter distribution, the geometry of space and the kinematics of Newton's space and time itself.

In all three cases, logical anarchy is avoided. In the first two cases, however, it is not at all clear how it is avoided. At first glance, it would seem that the physical theorists avoid logical anarchy by the simple expedient of ignoring it! Philosophical work in logical inconsistency presupposes that something more subtle may really be guiding the avoidance of logical anarchy and that it may be controlled by quite principled methods. Most of these analyses implement what I shall call "*logic driven control of anarchy*". Logical anarchy is avoided by denying or restricting use of certain standard inference schemes within a non-standard or paraconsistent logic.

A difficulty of this approach is that it is hard to recover it from the actual practice of physical theorists who do work with logically inconsistent theories. Typically, it is hard to discern any principled approach to the control of logical anarchy by such theorists. One certainly does not find explicit recourse to a modification of something as fundamental and universal as basic schemes of logical

[5] For a survey of work on the problem up to the end of the 1920s, see Norton 1999.

[6] See Norton 1999 (Section 7.1) for details.

[7] For further discussion see Norton 1999 (Section 7.2).

inference. Rather—in so far as any strategy is discernible—it seems to be based on a reflection on the specific content of the physical theory at hand. I will dub this approach "*content driven control of anarchy*".[8]

The example of Newtonian cosmology illustrates how this approach operates. If a theory has inconsistent postulates, then one can derive a range of contradictory conclusions from them. We expect the approach to tell us which of these conclusions to take seriously and which to ignore as spurious. In Newtonian cosmology, we can deduce that the force on a test mass is of any nominated magnitude and direction. Which force are we to take seriously?

The simplest answer arose in the context of the Newtonian cosmologies considered by Seeliger and his contemporaries around 1900. The source mass distribution was presumed static as well as homogeneous and isotropic. There is only one force distribution that respects this homogeneity and isotropy, an everywhere vanishing force distribution. and so this is the only one we should entertain.[9]

In the neo-Newtonian cosmologies of Milne and McCrea, things are more complicated. The cosmic masses are gravitationally accelerated so that the gravitational force distribution cannot be homogeneous and isotropic. Several considerations of content direct the choice that is made. In these cosmologies it is presumed that the force distribution throughout space can be combined to yield a gravitational potential. This reduces the force fields to a family of canonical fields all of which display the particular dynamics desired. [10] These canonical force fields are $\mathbf{F} = -(4/3)\pi G\rho(\mathbf{r}-\mathbf{r}_0)$, where $\mathbf{F}$ is the force on a unit test mass at vector position $\mathbf{r}$ in a force distribution due to source mass density ρ with G the gravitational constant. The position $\mathbf{r}_0$ is an arbitrarily chosen force free point. That this is the one to take seriously is also suggested by another result: a spherical mass distribution of arbitrarily large but finite size in an otherwise empty infinite space uniquely displays one of these solutions without any complications of the infinite case. Finally these particular force distributions are made very attractive by the agreement between their dynamics and that of general relativity in analogous cases.[11] We expect the two theories to agree as we pass from general relativity to Newtonian theory in some suitable limiting procedure. This agreement can arise most simply if they already agree on the dynamics.

[8] See Smith 1988 for an account of a content driven approach to the control of anarchy.

[9] Arrhenius (1909; quoted in Norton 1999, Section 7.1) mounts exactly this argument.

[10] The indeterminateness of gravitational force on any given test body still remains. By appropriate selection of $\mathbf{r}_0$, the force of a test body can still be set at any designated force. This choice however now fixes the gravitational force distribution throughout space and thus the forces on all other bodies. Malament (1995) has described this condition most clearly. This condition raises another issue of interest elsewhere. It is usually assumed that the reformulation of Newton's theory in terms of a potential field does not alter the physical content of the theory, or at least not its observable content. We see in this cosmological case that it does. The reformulation does not admit many force distributions and thus many observable motions that the original version did admit.

[11] This remarkable agreement resides in the following. Let R be the distance separating two different, arbitrarily chosen masses in the Newtonian cosmology. Then the time dependence of R agrees exactly with the time dependence of the scale factor (usually written as R also) in the Robertson-Walker cosmologies of general relativity.

5. CONCLUSION AND A PROPOSAL

The difficulty with the content driven control of anarchy sketched above is that it appears to be a completely *ad hoc* maneuver. What justifies ignoring all but one preferred member of a set of conclusions derived validly from postulates? One program would be to seek this justification in the logic driven control of anarchy. Perhaps if we impose the restrictions of one or other non-standard logic upon Newtonian cosmology, then we will recover the apparently *ad hoc* rules of the content driven approach. This is an interesting possibility worth pursuing, but it is not the only one.

We can also justify the strategy of content driven control without tinkering with something as fundamental and universal as logic. If we have an empirically successful theory that turns out to be logically inconsistent, then it is not an unreasonable assumption that the theory is a close approximation of a logically consistent theory which would enjoy similar empirical success. The best way to deal with the inconsistency would be to recover this corrected, consistent theory and dispense with the inconsistent theory. However, in cases in which the corrected theory cannot be identified, there is another option. If we cannot recover the entire corrected theory, then we can at least recover some of its conclusions or good approximations to them, by means of meta-level arguments applied to the inconsistent theory.

The clearest example is the case of homogeneous, isotropic cosmologies with static mass distribution. In *any* such cosmology—Newtonian or otherwise, symmetry considerations will require the vanishing of the net gravitational force on a test mass. Thus, when we use these symmetry considerations to exclude all but vanishing forces on a test mass in a static, Newtonian cosmology, we are in effect saying:

> We know that this cosmology is inconsistent. However, we expect that a small modification would eliminate the inconsistency and in the resulting, corrected theory the only force derivable would be the one satisfying the symmetry requirement, that is, the vanishing force.

In many cases, we might even guess what this corrected theory might be. Seeliger (1895, 1896) noticed that merely adding an exponential attenuation factor to the inverse square law of gravity would suffice. At very large distances only, the force of gravity would fall off faster with distance than the inverse square. Because of the enormous empirical success of Newton's theory, such attenuation factors must have nearly negligible effects within our solar system, so that unambiguous empirical determination of the factor is extremely difficult, as Seeliger found.

The case of neo-Newtonian cosmologies is similar and harbors a surprise. The selection of the canonical fields can be justified by displaying a corrected theory in which these canonical fields arise without inconsistency. Because of the agreement over dynamics, one might imagine that the corrected theory would simply be general relativity. But, in response to an earlier version of this paper, David Malament (1995) showed me and the readers of *Philosophy of Science* that the correction can be effected in the simplest and most desirable way imaginable. That is, we should like the correction merely to consist in the elimination of a superfluous assumption that plays no essential role in the theory in the sense that its elimination does not alter the observable consequences of the theory. The eliminable assumption proves to be the assumption that there are preferred inertial states of motion in space. It amounts to the adoption of a kind of relativity of acceleration. The corrected theory no longer portrays the motion of cosmic masses as deflections by gravitational forces from the preferred inertial motions. Instead the free fall motions of the cosmic masses are taken as primitive. Using techniques introduced by Cartan and Friedrichs in the 1920s, one constructs a gravitation theory that is observationally identical with the original theory. Its novelty is that the free fall motions are represented by a curvature of the affine structure of spacetime in a way that is strongly analogous to the corresponding result in general relativity. The removal of the inconsistency of Newtonian cosmology proves to be a natural and compelling path to the notion that gravitation is to be associated with a curvature of the geometrical structures of spacetime.[12]

In sum, my proposal is that the content driven control of anarchy can be justified as meta-level arguments designed to arrive at results of an unknown, consistent correction to the inconsistent theory. The preferred conclusions that are picked out are not interesting as inferences within an inconsistent theory, since everything can be inferred there. Rather they interest us solely in so far as they match or approximate results of the corrected, consistent theory.[13]

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[12] Or so I argue in my response (Norton 1995) to Malament (1995).

[13] This proposal also works in the case of the old quantum theory of black body radiation, as analyzed in Norton 1987, where I attempt to identify the corrected, consistent theory as a consistent subset of the commitments of the old quantum theory. The decision of a quantum theorist over whether to use some result in a given calculation amounts to a deciding whether that result belongs to the relevant subtheory.

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