The Physical Content of General Covariance

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1. Introduction

"My wildest dreams have been fulfilled," wrote a jubilant Einstein in early December 1915 to his dear friend Michele Besso (Speziali, 1972, p. 60) "*General* covariance. Perihelion motion of Mercury wonderfully exact...," he continued, checking off the achievements that brought to a dramatic and successful close nearly three years of struggle with his general theory of relativity, during which time he had mistakenly come to believe that he must forgo general covariance. The general covariance of his theory was to be stressed by Einstein as one of its most distinctive properties and, in particular, the one that gave mathematical expression to the theory's extension of the principle of relativity to all states of motion. He explained this connection to the principle of relativity in lectures he gave at Princeton University:

> We shall be true to the principle of relativity in its broadest sense if we give such a form to the laws that they are valid in every such four-dimensional system of coordinates, that is, if the equations expressing the laws are covariant with respect to arbitrary transformations. (Einstein 1922, p. 60)

Such proclamations are common in Einstein's work.[1] Unfortunately, they have proved to be a problem for later commentators who seek to understand Einstein's views because it is now commonplace for any reasonably coherent space-time theory to have a generally-covariant formulation. One need only formulate it by the standard methods of modern differential geometry. Since these generally-covariant theories include the versions of Newtonian space-time theory that unequivocally violate the usual relativity principles, any interesting connection between general covariance and relativity principles seems extremely dubious. Thus, some modern texts on general relativity lay out the foundations of the theory without recourse to principles of relativity or covariance. See, for example, Hawking and Ellis (1973) and Sachs and Wu (1977), who do not even have entries for these terms in their indices![2]

The purpose of this chapter is not to offer yet another vindication of Einstein's claim that general relativity is a theory that extends the principle of relativity of the special theory. Rather, I explain why a literal reading of Einstein's claims should appear incoherent to modern readers in the first place. Further, I suggest that Einstein's covariance principles are principles with physical content and that they are the analogues of physically significant covariance principles in the modern context. However, I also argue that the relativity-principle-like character of Einstein's covariance principles is a peculiarity of his older and less adequate formulation of general relativity, a conclusion that provides *no* support to the view that general relativity, freed of the peculiarities of specific, known formulations, has effected a generalization of the principle of relativity. My explanation rests on the fact that we now use a far richer set of mathematical machinery in formulating general relativity than Einstein did in the 1910s. One need only scan a text such as Hawking and Ellis' (1973) to find a wealth of mathematical terms, distinctions, and symbols that are just not present in Einstein's work of the 1910s.

Modern readers routinely translate Einstein's claims both consciously and unconsciously into this richer language and are generally very successful. My thesis is that this translation has been routinely carried out incorrectly in one important area, and that this incorrect translation is responsible for much of the apparent incoherence of Einstein's claims about covariance. The root of the problem lies in the following:

> Where we now represent a space-time mathematically by a differentiable manifold with a point set of unspecified elements, Einstein simply used number manifolds, open sets of the R^4.

This representation relation is a *coordination* of an actual or physically possible space-time with a general differentiable manifold or, for Einstein, with a number manifold. Thus Einstein naturally gave this representation the name *coordinate system*, and we have:

> Einstein's coordinate systems are not the coordinate charts of a general differentiable manifold of the modern approach. Rather, they correspond to the modern representation of an actual or physically possible space-time by a general differentiable manifold.

I shall try to show that if we read Einstein in accord with the above proposal, then we shall find that his principle of general covariance is a principle with significant physical content, and that that content is of the character of a generalized relativity principle.

2. The Modern View of Space-Time Theories

We now usually take a space-time theory to be synonymous with its set of models. Thus, general relativity is the set of all triples $\langle M, g_{ab}, T_{ab}\rangle$, where M is a four-dimensional differentiable manifold, g_{ab} is a Lorentz signature metric, and T_{ab} is a symmetric stress-energy tensor, such that the g_{ab} and T_{ab} of each pair satisfy the field equation

$$G_{ab} = kT_{ab} \quad (1)$$

where G_{ab} is the Einstein tensor and k is a constant. In everyday practice, it is usually convenient to treat a single triple $\langle M, g_{ab}, T_{ab}\rangle$ as representing each actual or physically possible space-time and sometimes even to speak of the triple as simply being the space-time. But, in more careful presentations, for example, Hawking and Ellis (1973, p. 56), we are reminded that each physically possible space-time is actually represented by an equivalence class of diffeomorphic triples.

To be more precise, now let us define two requirements that may be satisfied by space-time theories with models of the form $\langle M, O_1, O_2, \ldots\rangle$, where M is a differentiable manifold and $O_1, O_2, \ldots$ are geometric object fields defined on M. The first simply provides for the existence of the members of the equivalence classes of diffeomorphic models previously mentioned. It is automatically satisfied by such theories as general relativity by virtue of the general covariance of its defining field equation Eq. (1).

> (*Active*)[3] General Covariance: If $\langle M, O_1, O_2, \ldots\rangle$ is any model of the space-time theory and h any diffeomorphism from M to hM, then the carried along tuple $\langle hM, h^*O_1, h^*O_2, \ldots\rangle$ is also a model of the theory.
>
> (*Active*) Leibniz Equivalence: If $\langle M, O_1, O_2, \ldots\rangle$ and $\langle hM, h^*O_1, h^*O_2, \ldots\rangle$ are diffeomorphic models of a space-time theory, then they represent the same physically possible space-time.

The following is crucial to my story:

> The requirements of (active) general covariance and Leibniz equivalence are not forced on us by mathematical necessity; they are physical principles that we can choose to accept or deny.

The justification for this claim depends on the fact that two diffeomorphic models are distinct mathematical objects, unless the diffeomorphism concerned is the identity map or a symmetry of the model. Prior to further assumptions, we must assume that each property of a theory's model represents some physical property of a physically possible space-time. General covariance, asserted without Leibniz equivalence, allows us to take a model of a theory and from it generate arbitrarily many new and distinct mathematical models, each of which asserts the physical possibility of a distinct space-time. Leibniz equivalence asserts that two diffeomorphic models represent the same physically possible space-time. This can only be the case if the properties that distinguish the two models have no physical significance, that is, represent nothing in the physically possible space-time. Finally, general covariance and Leibniz equivalence, asserted jointly, amount to the assertion that the physically significant properties of a theory's models are just those that are invariant under arbitrary diffeomorphism. They assert that a model and all possible diffeomorphic copies of it represent the same physically possible space-time. Thus, the physically significant parts of each model can only be those upon which they all agree, that is, their invariants under arbitrary diffeomorphism. In sum, since these principles take a stand on which space-times are physically possible and what their physical properties are, the principles are physical principles, not mathematical stipulations.

The properties that distinguish diffeomorphic models are hardly of great importance because of the sustained attempt to purge our formulations of space-time theories of properties without physical significance. But, they are there. The simplest case is the one in which the manifolds M and M' are the same and the diffeomorphism h is not an identity or symmetry of the geometric object fields. Then, we can always find a point p of M at which O_1 and h^*O_1 differ. If the point sets of M and M' are disjoint, the two models are still mathematically distinct entities distinguished by the property of set membership. If p is a member of the point set of M, then it cannot also be a member of the point set of M'. This difference—if noticed at all—is usually not taken to be physically significant. But, no incoherence would follow from deciding otherwise.[4]

There are two grounds for accepting Leibniz equivalence, and since Leibniz equivalence is a physical principle, these grounds depend on physical considerations and cannot be proofs:

1. Under canonical interpretation, the properties that distinguish two diffeomorphic models do not correspond to any observable physical properties. So, we admit Leibniz equivalence in order to minimize instances of distinct physical states of affairs that cannot be distinguished

> by any possible observation.

Of course, we would not be guilty of incoherence if we denied Leibniz equivalence and allowed the possibility of observationally indistinguishable but distinct space-times.

> 2. The generation of diffeomorphic models is a gauge freedom of theories with generally-covariant field equations. If we deny Leibniz equivalence, we force indeterminism in many space-time theories. For example, even general-relativistic space-times that admit Cauchy surfaces become indeterministic.

The forcing of indeterminism in cases such as these seems unwarranted. In particular, there are no observables corresponding to the properties that remain undetermined, that is, those properties that distinguish the diffeomorphic models. Notice again, however, that there is no incoherence in denying Leibniz equivalence and thereby forcing indeterminism.

These two considerations are essentially the "point-coincidence" and "hole" arguments presented in greater detail in Earman and Norton (1987) and Norton (1987). Both arguments find early expression in the work of Einstein in the 1910s, and I shall return to the first argument in Section 8.

3. Number Manifolds

The ability of space-time theories to provide mathematical representations of physically possible space-times depends on the availability of a continuous, finite-dimensional mathematical structure. The modern study of such structures was initiated by Riemann (1854) in his classic inaugural lecture in which he introduced the concept of the "n-fold extended manifold." Riemann's analysis of his n-fold extended manifolds was brief and imprecise. We can now turn to the theory of point set topology for a very detailed account of precisely what it is to have manifold structure. But, this resource was not available to those geometers who sought to develop Riemann's idea in the later part of the 19th century. Fortunately, this proved not to be a serious obstacle since these geometers had at their disposal one example of a class of differentiable manifolds that had properties that were well understood and could be used whenever their theories called for a differentiable manifold. That class of differentiable manifold was the number manifold R^n. Thus, Felix Klein made Riemann's concept precise by explaining:

> At the foundation of his research, Riemann laid n variables $x_1, x_2, \ldots, x_n$, each of which can take all real values. Riemann denoted the totality of their value systems as a manifold of n dimensions; by a fixed value system $x'_1, x'_2, \ldots, x'_n$, he meant a point in this manifold. (Klein 1928, p. 289)

This identification of a manifold of n-dimensions with the number manifold R^n was already well established in 1873, at which time Klein (1873, p. 315) could report that it was "in agreement with the usual terminology." The tradition survived into the 1920s and appears on the opening page of Levi-Civita's 1925 treatise on the "absolute differential calculus," as the relevant branch of mathematics was then called. (Levi-Civita 1925, p. 9; 1926, p. 1). While the point set topological approach was being developed in the 1910s through such works as Weyl (1913) and Hausdorff (1914), the identification of geometrical manifolds with number manifolds remained attractive since it promoted the application of geometrical methods to the problems of real analysis and vice versa.[5]

This number manifold tradition entered relativity theory in its infancy. Minkowski's famous "world" was none other than R^4. He wrote in his 1908 address[6] (Minkowski 1908, pp. 56–67):

> We will try to visualize the state of things by the graphic method. Let x, y, z be rectangular coordinates for space and let t denote time.... A point of space at a point of time, that is, a system of x, y, z, t, I will call a world point. The manifold of all thinkable x, y, z, t systems of values we will christen the world.

The following summarizes the crucial difference between the older approach of Klein and Minkowski and the modern approach.

> Modern approach: An actual or physically possible space-time is represented mathematically by a general differentiable manifold, that is, one with a point set of unspecified elements.
>
> Older approach: An actual or physically possible space-time is represented mathematically by a special case of the general differentiable manifold, a number manifold, R^n, or its open subsets. The representation, which may only be "patchwise," is a coordination of the space-time with points of R^n, so that the maps $x_1, x_2, \ldots, x_n$ (Klein) or x, y, z, t (Minkowski) that effect this coordination are called "coordinate systems."

Einstein's work in space-time theories lies within the older approach. I refer here in particular to the three major expositions of general relativity that Einstein gave in the 1910s: Einstein and Grossmann (1913), Einstein (1914), and Einstein (1916a). These expositions set the pattern for

Einstein's and many others' later expositions. Each of these three expositions presented a review of the current state of the theory and included a self-contained primer on the mathematical techniques needed to work with the theory. Einstein's assumption was that his physicist readers were unacquainted with these mathematical techniques. Einstein himself had needed the assistance of his friend Marcel Grossmann in 1912 and 1913 to gain access to them. Grossmann drew heavily on the review article by Ricci and Levi-Civita (1901) of the absolute differential calculus, and he provided a primer on the novel mathematics needed for the new theory in his mathematical part of their joint work, Einstein and Grossmann (1913). The following year, Einstein felt that he had sufficiently mastered the mathematical techniques to write the primer himself and in such a way that it would "enable a complete understanding of the theory without the need to read other pure mathematical treatises" (Einstein 1914, p. 1040).

The compliance of Einstein with the older number manifold tradition is not immediately obvious from a reading of the above expositions. Unlike Klein, Minkowski, and others, neither Einstein nor Grossmann defined what their manifolds are. Rather, their unexplicated primitive is the space-time coordinate system, x_1, x_2, x_3, x_4, from which the expositions move immediately to the treatment of coordinate transformations and the transformation laws for vectors and tensors. The standard practice of modern readers is to interpret these coordinate systems as coordinate charts of another mathematical object, a general differentiable manifold, which in turn represents the actual or physically possible space-time. Thus, this modern reading requires the presence of an intermediate level of mathematical structure that is never explicitly addressed in the expositions. If we take Einstein at his word and accept that the expositions are intended to be self-contained, then we must assume that Einstein did not suppress a major level of mathematical structure, and we must read Einstein's coordinate systems as being just like those of Klein and Minkowski. They coordinate the actual or physically possible spacetimes with the mathematical structure used to represent them, R^4 or its open subsets.

In the appendix, I review in greater detail the mathematical traditions upon which Einstein and Grossmann drew in their work on general relativity. This review helps explain their failure to define the nature of their manifolds.

4. The Mathematical Structure of Einstein's Space-Time Theories

The use of a number manifold rather than a general differentiable manifold in the older tradition is just one manifestation of the fact that the older tradition used a much simpler set of mathematical machinery than we use now. To illustrate the point, I will consider the mathematical structures that Einstein used in his formulation of general relativity and to enable close comparison with modern methods, I will read Einstein's formulations in a way that mimics the modern extensional or model theoretic formulations of space-time theories.

4.1 Models

Suppressing the stress-energy tensor T_{ab}, in the modern formulation, general relativity has the models[7]

$$\langle M, g_{ab} \rangle. \tag{2}$$

To recover Einstein's models, we now know that we should replace the manifold M by an open set of R^4, for example, A, so that we have the models

$$\langle A, \text{metrical object} \rangle. \tag{3}$$

The second position in Eq. (3) requires a mathematical object representing a metrical structure. Einstein routinely introduced a non-Minkowskian metrical structure into his expositions by considering two infinitesimally close space-time points x_i and $x_i{+}dx_i$ and writing the interval ds between them as

$$ds^2 = g_{ik}\, dx_i\, dx_k. \tag{4}$$

Following the standard definitions of Einstein and Grossmann (1913) and Einstein (1914 and 1916a), the "fundamental" or metric tensor corresponding to Eq. (4) is not the matrix g_{ik} but the equivalence class of all the matrices produced by the well-known tensor transformation law under all smooth transformations of the coordinate system. Since we seek an object peculiar to the coordinate system image set A for the second position in Eq. (3), we choose not the tensor but the matrix g_{ik}, so that the models are of the form

$$\langle A, g_{ik} \rangle. \tag{5}$$

Generalizing, Einstein's formulation of a space-time theory posits models of the form

$$\langle A, (O_1)_{ik} \cdots, (O_2)_{ik} \cdots, \cdots \rangle, \tag{6}$$

where A is an open set of R^4, the quantities $(O_n)_{ik...}$ are matrices, and each matrix occupies the position corresponding to a geometric object of equivalent rank in the modern formulation.

4.2 Coordinate Transformations

We have seen that the set of models of a space-time theory in the modern formulation is divided into equivalence classes of diffeomorphic models. Two models $\langle M, g_{ab}, T_{ab}\rangle$ and $\langle M', g'_{ab}, T'_{ab}\rangle$ of general relativity, for example, belong to the same equivalence class just in case there is a diffeomorphism h for which $M' = hM$, $g'_{ab} = h^* g_{ab}$, and $T'_{ab} = h^* T_{ab}$.

Smooth maps from open sets of R^4 to open sets of R^4—that is, coordinate transformations—serve the same function for Einstein's models. In specifying a model Eq. (6), one must also specify the transformation law for each of the quantities $(O_n)_{ik...}$ (covariant tensor, contravariant tensor, mixed tensor, etc.) A model of general relativity is the triple $\langle A, g_{ik}, T_{ik}\rangle$, where g_{ik} and T_{ik} transform as covariant tensors. Two models, $\langle A, g_{ik}, T_{ik}\rangle$ and $\langle A', g'_{ik}, T'_{ik}\rangle$, belong to the same equivalence class just in case there is a smooth map from A to A' under which g_{ik} and T_{ik} transform into g'_{ik} and T'_{ik}.

4.3 Special Relativity and General Relativity

The model theoretic version of Einstein's formulation of general relativity looks very much like the modern version. Its model set is the set of all triples $\langle A, g_{ik}, T_{ik}\rangle$ that was defined previously and satisfies the field equation

$$G_{ik} = kT_{ik}, \tag{1'}$$

corresponding to Eq. (1). The formulation of special relativity looks a little less like its modern counterpart. In his space-time formulations of special relativity, for example, Einstein (1922), introduced the metrical structure through the line element

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + dx_4^2,$$

which corresponds only to a matrix of metrical components:

$$h_{ik} = \mathrm{diag}\,(-1, -1, -1, +1), \tag{7}$$

so that, assuming the space-time is topologically R^4, special relativity has a single model:

$$\langle R^4, h_{ik}\rangle. \tag{8}$$

The more interesting case of multiple models arises if there are further matter fields defined on the space-time, such as a Maxwell field or dust cloud. To avoid unnecessary detail, I shall represent these matter fields by their stress-energy matrix T_{ik} so that the model set is the set of all triples of the form

$$\langle R^4, h_{ik}, T_{ik} \rangle, \tag{8'}$$

such that the T_{ik} satisfy the laws of the relevant matter theory and transform as a covariant tensor. A comparison of Einstein's formulation and the modern formulation of a space-time theory highlights the advantages of the modern formulation. In particular, if the space-time in question is not topologically R^n, then Einstein's formulation cannot represent it by a single model, but must combine several in a patchwise fashion, with all the attendant complications.

5. The Problem of Superfluous Manifold Structure

A general differentiable manifold is structurally a quite impoverished representation of a space-time. In modern formulations of space-time theories, if we wish to represent such familiar entities as frames of reference, we need to introduce them as further structures defined on the manifold. We represent a frame of reference by adding a congruence of curves. But, Einstein's use of number manifolds to represent space-time raised a quite different problem for him. Number manifolds have too much structure. Take some model $\langle R^4, g_{ik}, T_{ik} \rangle$ of a relativistic space-time. Since the model represents the space-time, the straightforward reading is that each of its mathematical properties represents a property of the space-time. Some of these properties are:

1. Inhomogeneity: each point of R^4 is distinct, so every space-time event is intrinsically different to every other;
2. Absolute simultaneity: x_4 is a time coordinate, so the hypersurfaces of constant x_4 represent hypersurfaces of simultaneity.
3. Absolute rest: the natural rest frame of the space-time is represented by the congruence of x_4 curves;
4. Set of inertial frames: each parallel congruence of straights with constant velocity V, where $V^2 = (dx_1/dx_4)^2 + (dx_2/dx_4)^2 + (dx_3/dx_4)^2$ represents a frame of reference of uniform velocity; and

5. Metrical significance of the coordinates: the x_1, x_2, x_3 coordinates are measures of length, and the x_4 coordinate is a measure of time.

The idea that number manifolds have such default physical interpretations played an important role in Einstein's developments of space-time theories. For example, it was not uncommon for him to introduce the notion of a space-time coordinate system by means of property (5), that is, by specifying the space and time measuring operations needed to define the coordinate values (see, for example, Einstein 1917, Chapters 2 and 3). But then, proceeding to general relativity and its wider class of coordinate systems, Einstein went to great pains to convince the reader that coordinate systems must forfeit their direct metrical significance (see, for example, Einstein 1916a, Section 4; 1917, Chapters 23–25).[8]

6. The Solution: Einstein's Covariance Principles Are Physically Significant Principles

What Einstein needed was some systematic method of denying physical significance to all of the superfluous structures imported into his space-time theories by his use of number manifolds. Einstein was not the only one to face this problem. Felix Klein had faced and solved it brilliantly some 40 years earlier. The central idea of his celebrated *Erlangen* program was to characterize geometric structures as the invariants of groups. Einstein applied this same idea to his space-time theories. Associated with each theory was a covariance group of coordinate transformations. The physically significant mathematical properties were those that remained invariant under the transformations of the group.

Einstein's solution admits a precise statement in a way that mirrors the requirements of general covariance/Leibniz equivalence, defined in Section 2 for the modern formulation. The main difference is that the modern view only needs to define covariance with respect to the group of smooth transformations, whereas for Einstein, the group varies in size with the theory.

Covariance of a theory under a group G of transformations:
If $\langle A, (O_1)_{ik}\cdots, (O_2)_{ik}\cdots, \cdots\rangle$ is a model of a space-time theory, then any tuple $\langle A', (O_1)'_{ik}\cdots, (O_2)'_{ik}\cdots, \cdots\rangle$ related to it by a transformation within G is also a model of the theory.

Leibniz equivalence: If two models $\langle A, (O_1)_{ik}\cdots, (O_2)_{ik}\cdots, \cdots\rangle$ and $\langle A', (O_1)'_{ik}\cdots, (O_2)'_{ik}\cdots, \cdots\rangle$ of a space-time theory are related by a

> transformation within G, then the two models represent the same actual or physically possible space-time.

Following the pattern in Section 2, I have distinguished covariance from Leibniz equivalence, although the two are usually lumped together and simply labeled as a covariance principle. In particular, I shall read Einstein's assertions that a space-time theory is or should be covariant under some group as the assertion that it does or should satisfy both of these requirements. Einstein clearly understood Leibniz equivalence to be part of a covariance requirement, as he showed in his autobiographical notes. There, he imagined that one treats some generally-covariant field theory as if it were only Lorentz covariant. Recognizing its general covariance, he urged, led to the

> level of understanding corresponding to the general principle of relativity. For, from the standpoint of the Lorentz group, two solutions would incorrectly have to be viewed as physically different if they can be transformed into each other by a nonlinear transformation of coordinates, i.e., if from the point of view of the wider group they are merely different representations of the same field. (Einstein 1949, pp. 70–73)

The Lorentz and generally covariant approaches of his example have identical covariance in the strict sense of my definition above because they have identical sets of models ("solutions"), generated by the one set of equations. They differ just on the issue of the Leibniz equivalence requirement associated with each group, that is, on which models are to represent the same physical space-time.

Einstein also predicated the covariance property not directly to the model set but to the equations that define the model set. The two are clearly equivalent; if the equations defining a model set are covariant under a group G then the model set must also be covariant under that group and vice versa.

The covariance/Leibniz equivalence requirements provide a systematic method of depriving some of the mathematical properties of a theory's models of physical significance. If two intertransformable models represent the same space-time, then all mathematical properties upon which they differ can have no physical significance. That is, only those properties shared by the models—the invariants of the theory's covariance group—can be physically significant.

Finally and most important, notice that the satisfaction of a covariance/Leibniz equivalence requirement by a space-time theory is not merely a matter of mathematical definition. The requirement involves a judgment of whether certain mathematical properties of models are physically significant and such judgments are necessarily matters of physical contingency. In this regard, the status of the requirement is exactly the same as that of active covariance/Leibniz equivalence requirements for the modern formulation of space-time theories, so that the considerations of Section 2 apply equally here. The conclusion that Einstein's covariance/Leibniz equivalence principles have physical content after all is a direct result of the new reading that I urge here for Einstein's "coordinate systems." I stress that this conclusion is not possible under the standard modern reading of Einstein's coordinate systems as the coordinate charts of a differentiable manifold. For example, taking some model $\langle M, g_{ab}, T_{ab}\rangle$ of general relativity, and transforming between different coordinate charts of its manifold M simply transforms between what are *by mathematical definition* different component representations of the same model.[9]

7. Covariance Principles as Relativity Principles

The covariance/Leibniz equivalence properties of space-time theories formulated in the modern way usually attract little attention. The fact that they deprive certain mathematical properties of the space-time models of physical significance is rarely mentioned or even noticed since these mathematical properties are usually of minimal interest.[10]

For Einstein, however, the situation was very different. The covariance/Leibniz equivalence properties of his space-time theories contain the relativity principles of his theories. The mathematical properties of his models that are deprived of physical significance include those associated with preferred states of motion, properties 1–5 listed in Section 5. The extension of the principle of relativity in the transition from special to general relativity is embodied in a sequence of three covariance requirements: Lorentz covariance, its extension through the principle of equivalence, and general covariance.

7.1 Special Relativity

Special relativity is Lorentz covariant, that is, it is covariant under the transformation of the extended Lorentz group. By definition, the extended Lorentz group is the set of all transformations that map $\langle R^4, h_{ik}\rangle$ onto itself. To be a special relativistic matter theory, the laws that govern the T_{ik} of the models Eq. (8') must be Lorentz covariant; therefore, if $\langle R^4, h_{ik}, T_{ik}\rangle$ is a model of the theory, then so is any Lorentz transform of it, $\langle R^4, h_{ik}, T'_{ik}\rangle$ and under Leibniz equivalence, the two models must represent the same physically possible space-time.

Applying the discussion of Section 6, we can now read some of the properties of the models deprived of physical significance by Lorentz covariance. They are those that are not invariant under Lorentz transformation and include (1) inhomogeneity, (2) absolute simultaneity, and (3) absolute rest. The assertion that there is no physical significance accorded an absolute state of rest is the principle of relativity of special relativity.

But, those structures that are invariant under Lorentz transformation remain physically significant. The most notable of these is property 4, Section 5, the set of inertial frames. While each inertial frame of R^4 will in general be mapped onto a different frame, the set of inertial frames is mapped onto itself. This feature of the coordinate systems of special relativity makes Einstein's description of them as "inertial systems" a natural one.

7.2 General Relativity

An extension of the principle of relativity to accelerated motion requires a theory in which not just the state of rest but also the inertial frames of the number manifolds are deprived of physical significance. This result is obtained in general relativity through its general covariance. Its covariance group is the group of all smooth transformations. Under this group, none of properties 1–5 are invariant, including in particular the set of inertial frames, property 4. Thus, throughout his writings, Einstein characterized general relativity as the theory that achieved the elimination of the preferred inertial system (see, for example, Einstein 1913, p. 1260, footnote; Einstein 1953).

7.3 Principle of Equivalence

The basic strategy of the extension of the principle of relativity in the move from special to general relativity is the expansion of the covariance group of the theory. To motivate this expansion, Einstein routinely included the case of the principle of equivalence as an intermediate between special and general relativity.[11] The principle is nothing other than an expansion of the covariance group of special relativity. As formulated so far, the intertransformable models of special relativity are always related by a Lorentz transformation. The principle of equivalence requires that we add to the set of models all those models produced by transformations that map inertial frames into uniformly accelerated frames. Under these new transformations, a model

$$\langle R^4, h_{ik} \rangle \quad (8)$$

will be mapped to a model

$$\langle A, g_{ik} \rangle \tag{9}$$

where A is some open subset of R^4 and g_{ik} no longer has the simple constant values of $h_{ik} = \text{diag}(-1, -1, -1, +1)$.

The covariance/Leibniz equivalence requirement asks that Eqs. (8) and (9) represent the same physically possible space-time. Now, Eq. (9) has an interesting physical interpretation. The trajectories of free test particles do not coincide with the curves of the inertial frames, as canonically defined for A via (property 4), Section 5. But, these accelerated trajectories of the free particles are governed solely by the matrix of coefficients g_{ik} and are independent of the particles' masses. So, in a much celebrated argument dependent on the equality of inertial and gravitational mass, Einstein urged that we interpret the acceleration of the particles as due to a homogeneous gravitational field represented by g_{ik}. The covariance/Leibniz equivalence requirement amounts to the assertion of the physical equivalence of Eqs. (8) and (9), one interpreted as gravitation free and the other as with a homogeneous gravitational field. If we use K to label the coordinate system of the inertial model, Eq. (8) and K' for the coordinate system of the accelerated model, Eq. (9), the covariance/Leibniz equivalence requirement becomes exactly Einstein's statement of the principle of equivalence:

> The assumption of the complete physical equivalence of the systems of coordinates K and K', we call the "principle of equivalence"... (Einstein 1922, p. 56)

Finally, since the principle of equivalence is an intermediate between Lorentz and general covariance, we can understand why Einstein would claim:

> The requirement of general covariance of equations embraces the principle of equivalence as a quite special case." (Einstein 1916b, p. 641)

This is a claim that has hitherto resisted coherent interpretation by modern commentators.

8. Kretschmann's Objection

One of the earliest objections to Einstein's equating of general covariance with a general principle of relativity was raised in Kretschmann 1917.[12] Kretschmann's paper is remarkable not so much for this objection, for which the paper is usually remembered, or, as I urge in Section 8.2, *mis*remembered. Rather, as I explain in Section 8.3, it is remarkable for its prescient treatment of relativity principles and other ideas in space-time theories.

8.1 The Popular Version

Kretschmann's objection is now routinely represented (see, for example, Graves (1971, p. 137).) as the remark that one can take the equations expressing any given law and, with sufficient mathematical ingenuity, modify them until they take on generally-covariant forms. Thus, the achievement of general covariance is merely a challenge to our mathematical ingenuity and fails to express anything physical, let alone to express a relativity principle.

The objection, in this simplified form, has been widely accepted. But, it is not at all clear that *this* form of the objection is correct. Of course, one can always take a law-like equation and modify its form until it becomes generally-covariant. One can then *assert* that the modified equation has the same physical content as the original. But, this assertion might require physically contingent hypotheses. In modifying the original equations, one might, for example, introduce further mathematical structures to enable achievement of general covariance. To retain physical equivalence with the original equation, one must make the physically contingent assumption that the added structure is merely an auxiliary and has no independent physical significance.

A clear instance of this arises in the expansion of the covariance of special relativity by the principle of equivalence. The expansion was effected by adding models of the form Eq. (9). However, the assertion that the expanded theory has the same physical content as the original theory required an assumption. It was that a model of the form Eq. (8) represents the same space-time as a model of the form in Eq. (9) to which it could be transformed, even though the second model was considerably different in mathematical structure. The assumption needed, that these mathematical differences have no physical significance, is Leibniz equivalence, which, as I have repeatedly stressed, is a physically contingent hypothesis. Its physical character is shown very clearly in this case, because, in the guise of the principle of equivalence, it makes the assertion that a gravitation-free model, Eq. (8), represents the same space-time as the model in Eq. (9), which contains a homogeneous gravitational field.

Finally, if one accepts this form of the objection, I see no reasonable way that one can avoid analogous objections against the physical significance of just about any property of any given theory. To establish the physical vacuity of some given property of a theory, one need only represent it as a purely *formal* property and establish that the presence of that formal property can be guaranteed in formulations of a wide range of incompatible theories, a task so loosely defined that it is trivially achievable. Thus, the presence of $E = mc^2$ in the laws of special relativity can be shown to be devoid of physical content. If one modifies the formulation of special relativity so that the string $E = mc^2$ becomes a stipulation, the definition of a quantity m in terms of the energy E of some body, then the physical content formerly carried by $E = mc^2$ is transferred to the physically contingent $m' = m$, where m' is defined as the inertial mass of the body. With only a little mathematical ingenuity and the introduction of a few auxiliary terms, one can similarly guarantee the presence of $E = mc^2$ in a formulation of Newtonian mechanics[13] or just about any other theory.

8.2 Kretschmann's Version

What Kretschmann actually said was more cautious and more interesting than the simplified report of section 8.1. At the outset of his paper, after recalling the connection claimed by Einstein between covariance and relativity principles, Kretschmann summarized his first worry:[14]:

> . . . imagine that all physical observations consist in the last analysis of the determination of pure topological relations (coincidences[a]) between space-time objects of perception and, hence immediately that no coordinate system is privileged[b] above any other by them, so that one is forced to the conclusion that each physical theory can be brought into accord with every arbitrary relativity postulate, including the most general, without alteration of the theory's freely chosen and observationally testable content, by a means associated at worst with mathematical difficulties: a purely mathematical transformation of the representing equations.[c] (Kretschmann 1917, pp. 575–576)

Kretschmann claimed that the laws of any space-time theory can be brought into generally-covariant form purely by mathematical manipulation and thus, by Einstein's lights, into accord with a generalized principle of relativity. But, the claim is dependent on an assumption: that "all physical observations consist in the last analysis of the determination of pure topological relations (coincidences[a])..." This assumption is immediately recognizable as the central assertion of Einstein's so-called point-coincidence argument, and of course, Kretschmann's footnote (a) to the word coincidences is to Einstein's best-known published version of the argument (Einstein 1916a, p. 776).

The point-coincidence argument is Einstein's version of argument (1) for Leibniz equivalence given in Section 2.[15] In brief and translated into the model theoretic terms of this chapter, he urged that all observables are, in the end, space-time coincidences, such as the world line intersections of particles in a simplified universe containing only particles or, more generally, that all measurement reduces to coincidences of material systems and measuring instruments. All such coincidences are preserved under coordinate transformation, so that intertransformable models must represent the same physically possible space-time, which is Leibniz equivalence (and, in turn, is included automatically in Einstein's requirement of general covariance).

This assumption, upon which the argument and Kretschmann's claim are based, amounts to a significant physical assumption. It amounts to requiring that

> (PC) *the physical content of a space-time theory is fully exhausted by the catalog of its space-time coincidences,*

so that we are enjoined to accord no physical significance to any property of a model of a space-time theory, if the property is not fully recoverable from this catalog of coincidences. That such a dramatic assumption is at the heart of Einstein's argument is seen more clearly in the versions of the argument written to his correspondents. To Besso in a letter of January 3, 1916, he explained the core of the argument:

> *Reality* is physically nothing other than the totality of space-time point coincidences. If, for example, all physical occurrences were constructed from the motion of material points alone, then the meetings of the points, i.e., the intersections of their world lines, would be the only reality, i.e., that which is in principle observable. (Speziali 1972, p. 64)

Following the pattern of his other presentations of the argument,[16] Einstein continued to argue that this assumption leads directly to general covariance:

> Naturally, these intersection points are preserved under all transformations (and nothing new is added), as long as certain uniqueness conditions remain true. Therefore it is most natural to require of the laws that they do not determine any *more* than the totality of time-space coincidences. According to what I have said above, this has already been achieved through generally covariant equations.

Thus, we have neither Einstein's nor Kretschmann's arguments warranting the claim that general covariance is physically vacuous. Rather, for both, it is a consequence of a profound physical assumption (PC) about the world, that the physical content of a space-time theory is exhausted by the catalog of its space-time coincidences.

I have been unable to develop a clear picture of Einstein's own attitude toward Kretschmann's arguments and, in particular, whether Einstein realized that Kretschmann had failed to demonstrate the physical vacuity of general covariance. Einstein's published response (Einstein 1918) to Kretschmann is clearest. There we find that Einstein conceded a little less than is usually thought. He listed the three principles upon which his general theory depends: (1) the principle of relativity, (2) the principle of equivalence and (3) Mach's principle. The principle of relativity *becomes* the assumption (PC) and the remark that general covariance follows from it:[17]

> *Principle of relativity*: The laws of nature are only assertions about time-space coincidences; therefore, they find their only natural expression in generally covariant equations.

On the following page (p. 242) he conceded to Kretschmann that general covariance "is only a requirement relating to the mathematical formulation [of laws]," but this concession is clearly dependent on the prior assumption of (PC). The concession is immediately explained by Einstein with a restatement of the assumption (PC) and the fact that it leads directly to general covariance.[18]

Einstein's later treatments of general relativity uniformly stress the fundamental role played by a requirement of general covariance and seem to remain essentially unaffected by whatever concession he might have made to the claimed physical vacuity of general covariance. Such a concession is not mentioned, for example, in his textbook-like exposition (Einstein 1922, see especially p. 60) or in a lengthy, unpublished exposition of special relativity and general relativity (Einstein 1920). It is mentioned in passing elsewhere (such as in Einstein 1924, pp. 90–91) and in greater detail in Einstein (1949, pp. 64–65). But, it is difficult to see that he took the objection seriously given, in the latter instance (Einstein 1949), the discussion that follows and especially his insistence on (Einstein 1949, p. 73) that "we have already given *physical* reasons for the fact that in physics invariance under the wider [general] group has to be required..." (Einstein's emphasis—not mine!).

8.3 Kretschmann's Formulation of Relativity Principles

That generally-covariant formulations should be available for all space-time theories (given PC) was only the first point of Kretschmann's critique. The bulk of his paper was devoted to the question of determining just what relativity principle was satisfied by any given space-time theory, given that this judgment could not simply be made from the covariance group of the theory's equations. The answer he gave and many of the results he derived turn out to be characteristic of much later work.

In modern terms, he recommended that we identify the relativity principle of a space-time theory with the symmetry group of the geometric structure of the theory. This viewpoint emerges most clearly when he gives a "geometric determination" of the theory's relativity principle. In essence, he sought that group of transformations that maps the conformal and affine structure of the theory back into itself, that is, their symmetries. His approach depended on the idea that these are the observable structures of the theory. For special relativity (Kretschmann 1917, pp. 581–583), he sought only the symmetries of the conformal structure, that is, of the "bundle of the light-like world lines" and arrived at a group of transformations that is independent of the equations describing the theory. He provided (Kretschmann 1917, pp. 607–611) a similar analysis for general relativity, first seeking the symmetries of the conformal structure and then the (time-like) affine structure to conclude that these structures jointly have no non-trivial symmetries, so that general relativity is "a completely absolute theory" and, moreover, that special relativity has the largest relativity group of space-time theories employing a Lorentz signature metric.

It is remarkable that as a part of his analysis of the extent to which observation can determine the mathematical structures of special relativity and general relativity, Kretschmann asked and answered the question of how much conformal and affine structures determine the metrical structure. He found (Kretschmann 1917, pp. 585–590) that the conformal structure determines the metric up to a conformal factor and that the affine structure forces this factor to take a constant value. These results and the general approach are usually attributed to a later tradition initiated by Weyl (Torretti 1983, pp. 192–193). Kretschmann's paper is also notable for its treatment of invariant methods of defining specialized coordinate systems in general relativity in terms of curvature invariants (Kretschmann 1917, pp. 591–606).

The precise response (Einstein 1918) to this latter and most significant part of Kretschmann's critique is less clear. Presumably, it is embodied in his distinction of Mach's Principle[19] from the principle of relativity, a distinction that he conceded in a footnote (Einstein 1918, p. 241) he had not previously made and that, he allowed, had caused confusion. Presumably, Einstein hoped that the satisfaction of Mach's principle by general relativity would distinguish it from other generally-covariant space-time theories and justify the idea that the theory embodies some kind of extension of the relativity of motion. It is now well known that these hopes were never fulfilled and that Einstein abandoned Mach's principle in later years. The story is told in many places; see, for example, Torretti 1983, pp. 194–202.

9. Conclusion

What has been established here is that the covariance principles of Einstein's formulations of space-time theories have physical content in a manner precisely analogous to the less important active covariance principles of the modern formulation of space-time theories. We have also seen that Einstein's covariance principles have the character of relativity principles since they deny physical significance to a series of mathematical structures interpreted as representing rest and inertial frames of reference. But, this relativity principle character is peculiar to the formulation of space-time theories chosen by Einstein. It depends on his use of number manifolds to represent space-time, which introduces considerably more mathematical structure into his models than is present in the models of a modern formulation of the same theory. The corresponding covariance principles of the modern formulations, then, have physical content, but they do not have the character of a relativity principle. Just as Kretschmann urged in 1917, the relativity principles of modern formulations are defined in terms of the symmetries of the geometrical structures defined on the manifold, and perhaps with the exception of special relativity, the identification of these principles has considerably less importance to the theory's formulation.

Perhaps the most important heuristic outcome of the comparison of Einstein's and the modern formulations of space-time theories is the recognition that the latter uses a considerably richer repertoire of mathematical structures. I argued that the failure to recognize this or to account for it properly had led to a systematic mistranslation of many of Einstein's ideas into the modern context. I have argued elsewhere, in a longer treatment of the issues raised here (Norton 1989), that another instance of this problem is the great difficulty modern commentators have faced in deciding whether to read Einstein's coordinate transformations actively or passively in the context of his hole and point-coincidence arguments, and I urge that the methods introduced here provide a complete solution to this problem.

Appendix: Two Traditions
Absolute Differential Calculus and Vector Analysis

Einstein and Grossmann drew their mathematical techniques for general relativity from two traditions so that these techniques are actually an amalgamation of the methods and standard procedures of these two traditions. A review of these traditions and how they were combined explains some of the apparent idiosyncrasies of Einstein and Grossmann's techniques and, to some extent, their neglect to define the nature of their manifolds.

First and most important was the tradition in differential geometry and the invariants of quadratic differential forms that began with Gauss's theory of surfaces and extended through the work of Riemann, Christoffel, Lie, Ricci, and Levi-Civita. For the advent of general relativity, the tradition's most important product was the Ricci and Levi-Civita (1901) review article of the absolute differential calculus upon which Einstein and Grossmann drew heavily. There were two distinct parts to this tradition. On the one hand was the investigation initiated by Gauss of surfaces characterized in the simplest two-dimensional case by the line element

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \qquad \text{(A1)}$$

where u and v are the coordinates and E, F, and G are some functions of them. The study of such surfaces led directly to the investigation of the invariants of quadratic differential forms such as Eq. (A1). However, the theory of these invariants could in principle be developed as an autonomous theory; one need only think of u and v as variables subject to transformations and suppress the surface theoretic interpretation. The literature in the theory of invariants of quadratic differential forms, such as Wright (1908, p. 4), did go to some pains to point out that differential geometry provided just one interpretation of the theory.

Einstein and Grossmann's debt to this tradition has been frequently recalled (see, for example, Pais 1982, pp. 210–213) and is much celebrated as an instance of pure mathematics anticipating the needs of the physicist. This tradition is usually designated as the primary mathematical source for general relativity. Einstein (1915, p. 779) had introduced one of the final versions of the theory as "a veritable triumph of the method of general differential calculus founded by Gauss, Riemann, Christoffel, Ricci, and Levi-Civiter [sic]." More soberly in the introductory page of his 1916 review of the theory, he remarked:

> The mathematical aids necessary for the general theory of relativity lay ready made in the "absolute differential calculus," which rests on the

> research of Gauss, Riemann, and Christoffel on non-Euclidean manifolds and had been brought into a system by Ricci and Levi-Civita and already applied to the problems of theoretical physics. (Einstein 1916a, p. 769)

(Lamentably this page was omitted from the standard English translation, Einstein 1916b.)

It is easy to misunderstand Einstein's remarks here. What he pointed out are the mathematical methods required for general relativity that were not *already* a part of the standard repertoire of physicists and of relativists in particular. There was a second mathematical tradition upon which Einstein also drew heavily, the tradition of work in vector analysis. Einstein did not mention it since it was already incorporated into relativity theory by 1912 and 1913. (We shall see that Grossmann was clearer on the role of vector analysis in general relativity.)

This second tradition grew from the work of Hamilton and Grassmann, and Gibbs and Heaviside, and its methods and concepts were developed with a very careful eye on their applications in physics. (For a general historical survey, see Crowe 1967). Vector analysis entered the German physics literature by means of its prime application, the theory of electricity and magnetism, when August Föppl decided to use vector methods in his text (Föppl 1894) on Maxwell's theory. This text, and its later incarnations, soon became the most popular text in electricity and magnetism in the German physics community. Föppl's text offered a self-contained exposition of vector analysis, a practice that was repeated in the later incarnations, such as in Abraham and Föppl (1904) and the last version of Becker (1964).

Since there were close historical connections between special relativity and the theory of electricity and magnetism, it was natural for vector methods to become associated with special relativity, even though Einstein's 1905 article "On the Electrodynamics of Moving Bodies" (Einstein 1905) did not employ the latest vector methods. That association was made most firmly by Minkowski (1907), when he showed that the four-dimensional space-time formulation of special relativity enabled an especially simple formulation of the theory of electrodynamics. The vehicle of this new formulation of electrodynamics was the extension of the standard vector analysis of the period to a four dimensional vector analysis at whose heart lay (among others) the two new quantities "space-time vectors of the first kind" and "space-time vectors of the second kind," which soon came to be known as four- and six-vectors. Introductions to the mathematical techniques required for the space-time formulation of special relativity soon leaned heavily toward an exposition of four-dimensional vector analysis. Thus, Sommerfeld's (1910a, 1910b) introduction to these techniques is entirely devoted to an exposition of four-dimensional vector analysis, and the association between vectorial concepts and special relativity had become so strong that Sommerfeld formulated the principle of relativity simply as the assertion that "only *space-time vectors* are allowed to appear in physical equations" (1910a, p. 749; Sommerfeld's emphasis).

The two traditions described developed in essentially autonomous literatures, although it was clearly obvious to the researchers in both traditions that they shared common problems and that their methods could be applied in the other tradition. For example, the review article by Ricci and Levi-Civita (1901) concludes with a discussion of the application of the absolute differential calculus in physics to vectorial fields and the example of the theories of electrodynamics, heat, and elasticity in general coordinates. Conversely, Abraham in his 1901 *Teubner Encyklopädie* article on vector analysis, noted in passing that the absolute differential calculus could be used to represent vectors and tensors in curvilinear coordinates (Abraham 1901, p. 38). That a more substantial connection should be effected between the traditions became all the more likely after Minkowski (1907) stressed the importance to the Lorentz group of the invariance of the space-time interval

$$x^2 + y^2 + z^2 - t^2, \tag{A2}$$

an expression quadratic in the space-time coordinates x, y, z, t. This emphasis was close to the focus of the differential geometry of the time, the invariants of quadratic differential forms.

It was Einstein and Grossmann who effected the first systematic amalgamation of the two traditions. In developing general relativity, Einstein began with special relativity in the four-dimensional vector formulation given to it by Minkowski. In order to incorporate gravitation into the theory, Einstein replaced Minkowski's requirement of the invariance of Eq. (A2), by the invariance of the quadratic differential form:

$$dx^2 + dy^2 + dz^2 - dt^2,$$

or, more generally in the arbitrary curvilinear coordinate x^i (i = 1, 2, 3, 4),

$$g_{ik}\, dx^i dx^k. \tag{A3}$$

Through Grossmann, Einstein found that the mathematical techniques needed to understand the invariants of this quadratic differential form were already collected in the Ricci and Levi-Civita (1901) review article on the absolute differential calculus. Einstein and Grossmann took the techniques of this calculus and grafted them onto the vectorial methods of four-dimensional special relativity to yield a generalized vector analysis that became the standard for work in general relativity.

Grossmann explicitly acknowledged this procedure in the introduction to his mathematical part of Einstein and Grossmann (1913). He noted that the mathematical aids for developing a "vector analysis of the gravitational field" derive from the work of Christoffel and Levi-Civita and continued:

> However, the vector analysis of Euclidean space related to arbitrary curvilinear coordinates is formally identical to the vector analysis of an arbitrary manifold given through its line element. Therefore, there are no difficulties in extending the vector analytic conceptual system, as it had been developed in recent years by Minkowski, Sommerfeld, Laue, and others for relativity theory, to the general theory of Einstein given here.
>
> The *general vector analysis*, which one then recovers, proves with some practice to be just as simple to manage as the special vector analysis of three- or four-dimensional Euclidean spaces;. . . (Einstein and Grossmann 1913, p. 23; Grossmann's emphasis)

Elsewhere, Grossmann (1913) introduced the new mathematical system with an equation by equation comparison of the "usual vector analysis" of Minkowski and Sommerfeld, which was limited to orthogonal coordinate systems, with the "general vector analysis" of the new theory to which it had been generalized by introduction of the work of Christoffel, and of Ricci and Levi-Civita and now admitted arbitrary curvilinear coordinate systems.

As a result of this procedure of generalization, aspects of vector methods and concepts were accorded a prominence that they otherwise might not have enjoyed. The clearest illustration of this prominence lies in the nomenclature used. The Ricci and Levi-Civita (1901) review article is now commonly known by such titles as "Ricci and Levi-Civita's Tensor Analysis Paper" (Hermann, 1975). However, the remarkable fact is that the term tensor appears nowhere in Ricci and Levi-Civita's article. The quantities we would now call tensors (and vectors) are there called "contravariant and covariant systems." Moreover, the term tensor is not used in the modern way in standard texts in differential geometry and the invariants of quadratic differential forms at the time of Einstein's development of general relativity (e.g., Wright 1908, Bianchi 1910, Darboux 1914).

The term tensor comes directly from vector analysis. Tensors were routinely but certainly not prominently defined in the standard developments of vector analysis. However, in the older tradition, the scope of the definition was narrower than the modern definition. Thus, Abraham (1901, p. 28) defined as a tensor only those quantities whose components transformed like the components of what we would now call the outer product of two vectors. That is, Abraham's tensors were what we would now call second rank, symmetric tensors. The reason for this narrowness was simple. As the word tensor betrays, the quantities were defined with a specific application in mind, the stress tensors of the theory of elasticity, which were second rank and symmetric. The generality of the modern definition would simply have been superfluous. The term tensor found its way into the four-dimensional vector analysis of special relativity—eventually. Minkowski (1907) did not use the term, although he did present a matrix representation for the components of the four-dimensional analogs of tensors. Sommerfeld (1910a, p. 767) applied the term tensor in the four-dimensional vector analysis in as narrow a way as Abraham and for similar reasons. Sommerfeld's tensors were again what we would call second rank, symmetric tensors, and the word tensor was still entirely appropriate to their principal application as a stress energy tensor. Following Minkowski (1907, p. 82), Sommerfeld (1910a, p. 754) recognized that his six-vectors had components that formed a second rank, antisymmetric matrix, but he did *not* apply the term tensor or antisymmetric tensor to the quantity. Finally, Laue's (1911) exposition of special relativity offered as restricted a definition of his four-dimensional "world tensors" as Sommerfeld's definition of his "tensor," although deeper in his exposition, Laue admitted an "unsymmetric," three-dimensional stress tensor (p. 151).

It is easy to see why Einstein or Grossmann would categorize the coefficients g_{ik} in the fundamental form Eq. (A3) as representing a tensor within their generalized vector analysis, even though there is no longer any immediate connection between the quantity represented and the theory of elasticity. These coefficients transform exactly as required by the definition of the tensors of Sommerfeld and Laue. Yet, for a reason that is not clear to me, the term tensor was defined by Grossmann in Einstein and Grossmann (1913, p. 25) as applying not just to the quantity represented by the coefficients g_{ik}, but to all those represented by Ricci and Levi-Civita's contravariant and covariant systems. This broader usage justifies the labeling of the absolute differential calculus as a tensor calculus. It became the standard not just in the generalized vector analysis used in relativity theory, but in differential geometry and the theory of the invariants of quadratic differential forms as well. In 1925, Levi-Civita published a treatise called *Lezioni di Calcolo Differenziale Assoluto* (Levi-Cevita 1925). Its expanded English translation was published the following year [(Levi-Civita 1926) (see Levi-Civita 1925 in references)] with the appropriate title *The Absolute Differential Calculus*. But, to it, presumably in deference to the new nomenclature grafted onto it from vector analysis, was added parenthetically the new subtitle "Calculus of Tensors."

We can now turn to the question of Einstein and Grossmann's failure to define the nature of their manifolds. In brief, this failure simply follows the standard practice of many of the expositions in both of the mathematical traditions upon which they drew.

Expositions of vector analysis around 1900 saw their primary burdens to be the definition of vectorial quantities and the development of their properties. Thus, the first definitions given are typically of polar and axial vectors, which are characterized in terms of their behavior under coordinate transformation. There is no discussion of the nature of the physical space in which the vectors are defined; coordinate systems and right angled coordinate systems are simply introduced as primitive notions presumed intelligible to the reader. Sommerfeld (1910a, 1910b) took a similar approach in his exposition of the four-dimensional vector algebra and analysis. After a brief introductory page, he turned immediately to his first definitions, four- and six-vectors. The concepts of four-dimensional space-time and coordinate systems are presumed to be familiar to the reader.

In their exposition of the absolute differential calculus, Ricci and Levi-Civita noted:

> . . . a manifold V_n is defined intrinsically in its metrical properties by n independent variables and by all of a class of quadratic forms of the differentials of these variables, such that any two of them are transformable from one to the other by a point transformation. (Ricci and Levi-Civita 1901, p. 482)

This definition of a manifold is different from the definition of a manifold as a number manifold, which we saw in Section 3 in the work of Klein, and Minkowski and, in 1925, of Levi-Civita.[20] Later in his 1925 exposition, Levi-Civita (1926, p. 119) defined what he called a "metric manifold," designated by V_n, as a number manifold in conjunction with a quadratic differential form, Eq. (A3). The manifolds V_n of the Ricci and Levi-Civita review article are actually equivalence classes of these 1925 V_n. Schouten (1924, p. 58) effectively gave the same definition as

Levi-Civita's 1925 definition for a V_n, a number manifold in conjunction with the specification of a fundamental tensor g_{ik}, and called it a "manifold with quadratic metric" or a "Riemannian manifold." Schouten (1924, p. 8) reserved the designation X_n for number manifolds.

For our purposes, the important point is that Ricci and Levi-Civita buried their definition of a manifold in the short preface to their paper within an account of the geometric ancestry of their absolute differential calculus. The formal exposition of their calculus begins in Chapter 1, with no mention of manifolds and in a way that seems to seek as much of a divorce from geometrical associations as possible. The first topic, for example, is the transformations of *variables*, a discussion that modern readers almost irresistibly read as "really" about the transformations of coordinate systems. They then proceeded to define their covariant and contravariant systems. Like the researchers in the theory of quadratic differential forms, Ricci and Levi-Civita clearly conceived their calculus as a very general instrument. Its obvious geometrical application in the theory of surfaces is just one of the applications alluded to in the paper's title, so that its discussion is isolated in Chapter IV. Chapters III, V, and VI deal with applications in analysis, mechanics, and physics.

Thus, by 1912, the precedent for Grossmann was clearly set. Whether he conceived his mathematical part of Einstein and Grossmann (1913) as the exposition of a generalized vector analysis, as the exposition of the absolute differential calculus of Ricci and Levi-Civita, or as the exposition of some combination of them, the concepts of manifold and coordinate system were to be taken as terms already known to the reader. At best, they were to be dismissed briefly in prefatory remarks. One is to proceed through the topic of transformations as rapidly as one can to the important definitions, the definitions of the vectorial or, correspondingly, covariant quantities. That is essentially what Grossmann did. After defining the differential form Eq. (A3), he proceeded through the topic of coordinate transformations to his first major definitions, the definitions of covariant, contravariant, and mixed tensors of arbitrary rank. Similarly the mathematical aids section of Einstein's 1914 article begins with the definitions of covariant and contravariant four-vectors and proceeds to define tensors, their algebra and the differential operations that can be applied to them (Einstein 1914, p. 1034). The section on mathematical aids in Einstein's 1916 article follows exactly the 1914 pattern with the insertion of a brief introduction to motivate the need for the excursion into tensor calculus (Einstein 1916a).

Einstein and Grossmann's precedents gave them no strong reason to define the nature of their manifolds in their expositions of the mathematical techniques needed for the new theory. But, if that definition is not given there, should it not be given elsewhere in the developments of the theory? A simple answer is that Einstein's developments of general relativity began with the presumption that the reader was already familiar with the four-dimensional formulation of special relativity and that whatever manifold concept was used there was to be carried over automatically to the new theory. While I do believe that this presumption was made, I think there is a simpler answer. That Einstein represented his physically possible space-times by number manifolds without any intervening mathematical structures is simply implicit in the way he used coordinate systems. I believe that he saw the point as such an elementary one that it bore no real discussion. The point is only not immediately obvious to modern readers because they approach Einstein's writings with a far more complicated concept of the manifold already in hand. But, how else, for example, are we to read Einstein's discussion in the early pages of his introductory relativity text, *Meaning of Relativity*, when, after presuming certain elementary properties for space, he wrote:

> ...it is easy to say what we mean by the three dimensionality of space; to each point three numbers, x_1, x_2, and x_3 (coordinates), may be associated in such a way that this association is uniquely reciprocal and that x_1, x_2, and x_3 vary continuously when the point describes a continuous series of points (a line). (Einstein 1922, pp. 3–4)

In introducing the three-dimensional coordinate system, Einstein simply gave us a recipe for expressing more precisely the topological properties of a physical space: one represents the space with the mathematical structure of the number manifold R^3.[21]

Addendum: I am grateful to John Stachel for pointing out to me an affirmation in the mathematical literature of 1914 of the novelty of the Einstein–Grossmann use of the term "tensor." E. Budde, in *Tensoren und Dyaden im Dreidimensionalen Raum: Ein Lehrbuch* (Budde 1914), concluded with a brief discussion of the extension of the subject of his book to higher dimension spaces and higher ranks (pp. 245–46). The latter extension is due to W. Voigt and invokes quantities of the "*n*th rank," which "transform as a combination of the *n*th dimension of the coordinates." He described Voigt's nomenclature:

> The quantities of the first rank then are vectors, the symmetric, second rank quantities are the tensors, those of the third rank Voigt calls trivectors [*Trivektoren*]; those of the fourth rank bitensors [*Bitensoren*].

A sentence later, the section concludes by contrasting this usage with that of Grossmann:

> Recently, Mr. Grossmann (see bibliography) has proposed a still further reaching generalization. He denotes quantities of arbitrary rank as "tensors," so that vectors, trivectors, and bitensors are also subsumed by the term "tensor"; the generalization consists in extending his definitions to structures of *n*th rank in *m*-dimensional space.

The only work by Grossmann in Budde's bibliography is "*A. Einstein und M. Grossmann: Entwurf einer verallgemeinerten Relativitätstheorie.* Leipzig und Berlin 1913." Thus, the generalized use of the term tensor in the *Entwurf* paper was novel at least to Budde, who was sufficiently acquainted with the mathematical literature available in 1914 to write a textbook on three-dimensional tensors.

NOTES

[1] See, also, for example, Einstein 1916, p. 776 and Einstein 1917, pp. 97–98.

[2] Of course, it has proven possible to find reinterpretations of Einstein's ideas that do make sense to modern readers. The most successful of these attempts is based on the notions of absolute and dynamic objects, best known from the work of Anderson (1967), and explored most recently by Stachel (1986).

[3] For elaboration on the difference between these active versions of the requirements and the corresponding passive versions, see Norton 1987 or Norton 1989.

[4] Imagine, for example, a space-time theory that requires two manifolds with disjoint point sets to represent two different space-times, even though the models that host the manifolds may be diffeomorphic. For a concrete example, take a Euclidean three space and foliate it into a family of flat, two-dimensional hypersurfaces S_i, with i a real valued index. We can model each hypersurface by the unique manifold-metric pair $\langle N_i, h_{ab}^{(i)} \rangle$, where by stipulation, two diffeomorphic models $\langle N_i, h_{ab}^{(i)} \rangle$ and $\langle N_j, h_{ab}^{(j)} \rangle$ differ only by having disjoint point sets.

[5] A major deficiency of the tradition was that spaces that were not topologically R^n could only be represented in a patchwise fashion.

[6] I translate Minkowski's "Mannigfaltigkeit" as "manifold" where the standard translation (Minkowski 1908a, p. 76) has "multiplicity."

[7] I adopt the following convention. Indices a, b, c, ... are to be read according to the abstract index notation (Wald 1984, Section 2.4), so that g_{ab} is a second rank covariant tensor, i.e., a bilinear map from the tangent space of the manifold to the reals. Indices i, k, l, ... take real values 1, 2, 3, 4 so that g_{ik} represents a 4×4 matrix of reals, which could be, for example, the components of g_{ab} in some coordinate chart.

[8] This procedure recapitulates Einstein's historical pathway to general relativity. This fascinating story can be found in Stachel 1980.

[9] For further discussion of this problem of the modern reading, see Norton 1989.

[10] The most important exception arises in the context of the Cauchy problem in which these mathematical properties engender a gauge freedom that threatens the determinism of the space-time theory.

[11] For an extensive survey of Einstein's statements of the principle and its role in relativity theory, see Norton 1985.

[12] Kretschmann's mathematical methods also lie within the number manifold tradition of Klein and Minkowski. This fact is expressed neatly in Kretschmann's use of the term "coordinate manifold" (see, for example, Kretschmann 1917, pp. 581–82, p. 583). In 1915, he was even more explicit. He announced that he would "conceive of the space-time reference system of physics as a four-dimensional manifold of pure numbers" (Kretschmann 1915, p. 917) and he devoted considerable analysis to the "representation postulates" of physical theories, which "relate empirical space and empirical time with the spaces and time coordinates of theoretical physics" (Kretschmann 1915, p. 979).

[13] For example, guarantee the presence of the string $E = mc^2$ by defining $m = E/c^2$, where E is the kinetic energy of a body with inertial mass m_i and velocity v; m then enters the physically contingent law $m = m_i v^2/2c^2$. I thank Cory Juhl for discussion.

[14] At the point marked by my superscript letters *a, b,* and *c,* Kretschmann's text has footnote references: (a) Einstein 1916a, p. 766; (b) for details, Kretschmann (1915, p. 914–924); (c) compare Ricci and Levi-Civita (1901, p. 125). In the place cited in (b) and pages 924–26 following, Kretschmann argued at length for what is essentially just the point-coincidence argument: observation provides only "topological" results, such as the coincidence of parts of measuring instruments and subjects, so that the choice between coordinate systems is made by convention and arbitrary stipulation. Kretschmann's paper was submitted on October 21, 1915, two months before the point-coincidence argument even appeared in Einstein's correspondence. This fact leaves room for speculation on the priority and circumstances surrounding its discovery.

[15] For a more detailed treatment of Einstein's formulations of the point-coincidence argument and its role in his work on general relativity, see Norton 1987.

[16] For example, Einstein 1916a, p. 776 or Einstein to P. Ehrenfest, December 26, 1915, in Norton 1987, pp. 168–169.

[17] This is not the place to analyze the ambiguity lurking in the notion of "space-time coincidence." My view is that, in the last analysis, the assumption (PC) can only be made precise by replacing it by nothing less than the requirement of general covariance/Leibniz equivalence.

[18] He only then turned to the remark best known from his reply. *Even if,* he said, all empirical laws can take on generally-covariant forms, his relativity principle still has heuristic power because for two empirically equivalent theories, we should prefer the one whose generally-covariant formulation (absolute

differential calculus) is simpler and more transparent.

[19] This name appears here for the first time and the formulation given begins with the now familiar words "The G-field is determined *without residue* by the masses of bodies. . . ."

[20] While Grossmann gave no definition of a manifold in Einstein and Grossmann 1913, this Ricci–Levi-Civita definition is clearly the one he had in mind when he wrote of "the vector analysis of an arbitrary manifold given through its line element" (p. 23) and again in a similar remark on p. 31.

[21] I believe that Minkowski also found the point too elementary to bear sustained discussion. His technical exposition (Minkowski 1907) does not contain the definition of "the world" quoted in Section 3. He limited himself to the remark that "a single system of values x, y, z, t or x_1, x_2, x_3, x_4, is to be called a *space-time point*" (p. 57, Minkowski's emphasis), with the more elementary discussion of "the world" delegated to his popular lecture (Minkowski 1908).

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