

Field that initiates equation numbering:

DRAFT

Chapter from a book, *The Material Theory of Induction*, now in preparation.

Indeterministic Physical Systems

John D. Norton

Department of History and Philosophy of Science

University of Pittsburgh

<http://www.pitt.edu/~jdnorton>

1. Introduction

The indeterministic systems to be investigated in this chapter share the common characteristic that determining one aspect of the system leaves others open. The most familiar cases are ones in which the present state of the system fails to fix its future state. We shall see several such systems here in Section 3. The most important are systems with infinitely many degrees of freedom, for this sort of determinism is generic amongst them. Rather than delve into the details of the physics of such systems, the mechanism that generates the indeterminism will be illustrated by the simplified system of the infinite domino cascade.

A different sort of indeterministic system will be explored in Section 4. At the risk of abusing the term, I will also describe as indeterministic systems in which, at the same moment of time, one component fails to fix others, contrary to normal expectations. The examples will be drawn from Newtonian gravitation theory.

Each instance of indeterminism poses a problem in inductive inference. From the known aspect, what strengths of inductive support are provided to the remaining underdetermined aspects? Given this present, what support is provided to the various possible futures? Given this mass distribution, what support is given to the various possible Newtonian potential fields? As

explained in Section 5, each of the problems has been chosen so that the complete background physics is transparent and transparently provides no probabilities over the various underdetermined possibilities. The problem for inductive analysis is to find the strengths of inductive support for the different possibilities, without altering or adding to this physics. For to do otherwise is to change the problem posed.

We shall see in Section 6 that probabilities can only be assigned as strengths of inductive support if we add to the background facts. Normalization of a probability measure, for example, requires that the probabilities of different times of spontaneous excitation in a temporally indeterministic system diminish to zero as the times grow large. This diminution must happen at some rate: quickly or slowly; and fitting a probability measure to the process requires that some speed be chosen. To make that choice, however, is to add to the physics provided.

This is just the first of a series of problems that preclude the use of probabilities as strengths of support. The final example requires the adaptation of a uniform probability measure to an infinite dimensional space of Newtonian potentials. The infinity of the dimensions present especially intractable problems.

Section 7 then describes how the material theory of induction solves the inductive problems. We are to look to the background physical facts to provide the strengths of inductive support. By design, these facts provide very little. They allow us to say of various processes or components that they are necessary, possible and impossible. These three evaluations become the values of a spare, three-valued inductive logic. Its strengths of support coincide with those of “completely neutral support” described elsewhere, including Chapter 10 here. This completely neutral support can be fixed by certain invariances in space of possibilities; and we shall see that they are realized in this case as well.

We proceed first with a preliminary in Section 2 on the project now undertaken.

2. Why Take Simple, Unrealistic Physical Systems Seriously?

The illustrations to come involve simple, physically unrealistic systems that, mostly,¹ we will not encounter in the ordinary practice of science. So why pay any special attention to them in investigations of inductive inference? There is a simple pragmatic reason for considering them.

¹ The exception is the example of the quantum spin of electrons.

If the analysis of the warranting relations is to be transparent, we need simple systems. We need systems in which the full set of background facts is easy to comprehend, so that their full import can be seen clearly and unequivocally.

This pragmatic reason, however, is not the principal one. The deeper reason for taking these simple systems seriously pertains to the range of applicability of inductive inference. We do not balk at reasoning deductively about fictitious systems, no matter how bizarre we may find them. Correspondingly, I see no reason to prohibit inductive inference over such systems. There is no guarantee, of course, that every system will admit rich inductive inferences. Just what is possible inductively will be determined by the background facts that obtain, as the material theory of induction asserts. When we ask which inductive inferences are warranted in the simple systems below, we will find that their strengths of inductive support cannot be probability measures. That is, we will find through counterexamples that the probability calculus does not provide a universally applicable logic of induction.

It may be tempting to block the counterexamples by insisting that the scope of inductive inference is limited to ordinary physical systems of the type we normally encounter in science. This would be an unnecessary restriction on the reach of inductive methods. Worse, it would be of no help in protecting the probability calculus as the universally applicable logic of inductive inference. For the restriction to ordinary systems gives up universal applicability at the outset. Moreover the restriction itself would conform with the material theory of induction, for the range of applicability of probabilistic inductive logic would be circumscribed by the factual restriction to ordinary systems.²

3. Temporally Indeterministic Systems

The general idea of determinism is that the fixing of one aspect of a system fixes some other. This section will address the case of temporal aspects. In a (temporally) deterministic physical system, the present state of the system determines its future states. With the notable exception of quantum measurement, physical systems are generally assumed to be deterministic.

² I set aside here the further problem of delineating just what will count as “ordinary.” Many of the systems ordinarily considered in science are highly idealized and thus highly unrealistic.

The present state of the planetary system fixes the future movements of the planets and whether there will be an eclipse on any nominated time.

Systems that violate temporal determinism have attracted considerable attention in recent decades in philosophy of physics, with the modern era marked by the publication of John Earman's *Primer* (1986). Once we start to look for indeterministic systems, we find them in many places.

3.1 The Dome

One of the simplest indeterministic systems is the "dome." Since it has been discussed extensively elsewhere (Norton 2003, §3; 2008), it needs only a brief recapitulation. A unit point mass slides frictionless over the surface of a dome in a vertical gravitational field with acceleration due to gravity g , as shown in Figure 1.

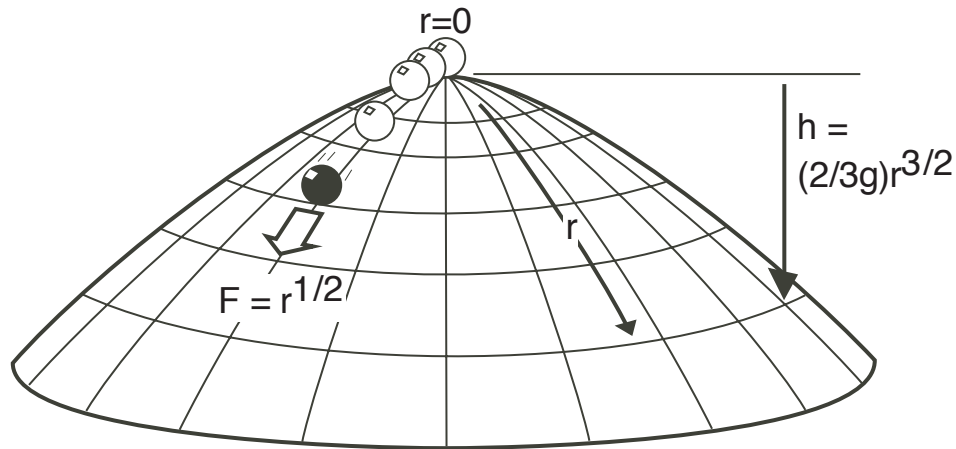


Figure 1. The Dome

The dome has a vertical axis of rotational symmetry about its apex and the surface is depressed below the apex by a (negative) height $h = (2/3g)r^{3/2}$, where r is the radial distance to the point from the apex along the surface. The force F on the point mass along the surface of the dome is

$$F = (d/dr) gh = r^{1/2}$$

and is directed outward from the apex. The motion of the point mass is governed by the equation of motion

$$\frac{d^2r}{dt^2} = r^{1/2} \quad (1)$$

where t is time. Initially, at time $t=0$, the point mass is located at the apex $r=0$ at rest. Since the force at the apex is $F = 0^{1/2} = 0$, one solution to the equation of motion is that the mass remains at the apex for all time:

$$r(t) = 0 \quad \text{all } t$$

However there is a second family of solutions, in which the particle moves spontaneously at time $t=T$ for any time $T \geq 0$:

$$\begin{aligned} r(t) &= (1/144) (t - T)^4 & \text{all } t \geq T \\ &= 0 & \text{all } t \leq T \end{aligned}$$

In this second solution, the particle remains quiescent up to and including time $t=T$. Then it moves away from the apex in any direction.

This spontaneous excitation results entirely from the equation of motion. There is no hidden triggering event, such as a slight bump to the dome that may dislodge the point mass from the apex. If there is no spontaneous motion, it is so because the equations of motion allow it. If there is spontaneous motion at time T , it happens just because the equation of motion also allow it.

The dome is a Newtonian system with only finitely many degrees of freedom. That is, its state can be specified fully just by specifying a finite list of magnitudes: the position of the particle on the dome, its speed and its direction of motion. The dome is unusual in its indeterminism in that, generally, Newtonian systems with finitely many degrees of freedom are deterministic. It was devised originally to display an unusual exception to this generality. Because of its exceptional character, the indeterminism of the dome is highly sensitive to changes in the physical system and its indeterminism can be eliminated by small adjustments to it.

3.2 Masses and Springs

Matters change, however, once we consider Newtonian systems with infinitely many degrees of freedom. An important example is a system of infinitely many interacting particles. It has infinitely many degrees of freedom since its state can only be specified by specifying infinitely many magnitudes, such as the mass, position and velocity of each particle. Such systems are generically indeterministic. While circumstances need to be specially contrived to induce indeterminism among the systems with finitely many degrees of freedom, indeterminism

is simply the standard, generic behavior of these systems with infinitely many degrees of freedom. There are many examples in the literature. Often they arise in the supertask literature, as reviewed in Manchak and Roberts (2016).

The masses and springs example consists of an infinite chain of mass-spring-mass-spring-... shown in Figure 2.



Figure 2. Masses and Springs

Its temporal behavior is recovered from an application of Newton's laws along with Hooke's laws for the springs. If the system is set initially in equilibrium with all the masses at rest and the springs unextended or uncompressed, then the system can remain in this quiescent state indefinitely. However, at any later moment, it can spontaneously self-excite with all the masses set in motion. The system is noteworthy for the ease with which a full mathematical description can be given and for what it represents physically. It is a standard model of a one-dimensional crystal, extended to infinite size. It indicates that more complex solids, such as infinite three-dimensional crystals, will exhibit similar indeterminism.³

In all these systems, the infinity of the number of degrees of freedom is essential. A finite system, no matter how large, will not manifest the indeterministic behavior as freely. A finite chain of mass-spring-mass-spring-..., once quiescent, remains so for all time, no matter how large it is.

3.3 The Infinite Domino Cascade

Rather than work through the technical details of the examples, I will display a toy example, shown in Figure 3., that illustrates the mechanism that brings about indeterminism in all these infinite cases. In a domino cascade, dominoes or slender tiles are set on their edges in a

³ I have argued in Norton (2012) that this fact ensures that the infinite component, thermodynamic limit of thermal physics cannot involve examination of a system that consists of infinitely components. Through their indeterminism, such infinite systems have qualitatively different properties from the real target of analysis, systems with many, but finitely many, components.

row, such that when one falls, it strikes another, leading it to fall; that falling domino strikes yet another, leading it to fall; and so on down the row.

Consider a very large row of dominoes, finite in number. We assume no external perturbing effects. There are no slight vibrations from passing trucks, no thermal agitation from air molecules, and so on. If it is set up at rest initially, it will remain so indefinitely.

Consider an infinite row of dominoes with the same provisions. As with the finite case, it *can* remain at rest indefinitely. However, it is also possible for it to be set into motion spontaneously. The final stages of this spontaneous motion are:

- the first domino falls, because it was struck by the second domino that started falling earlier;
 - the second domino fell, because it was struck by the third domino that started falling earlier;
 - the third domino fell, because it was struck by the fourth domino that started falling earlier;
- and so on.

As we proceed through the falling of the first, second, third, ... dominoes, we trace the process back through time and eventually consider the falling of all infinity of the dominoes.

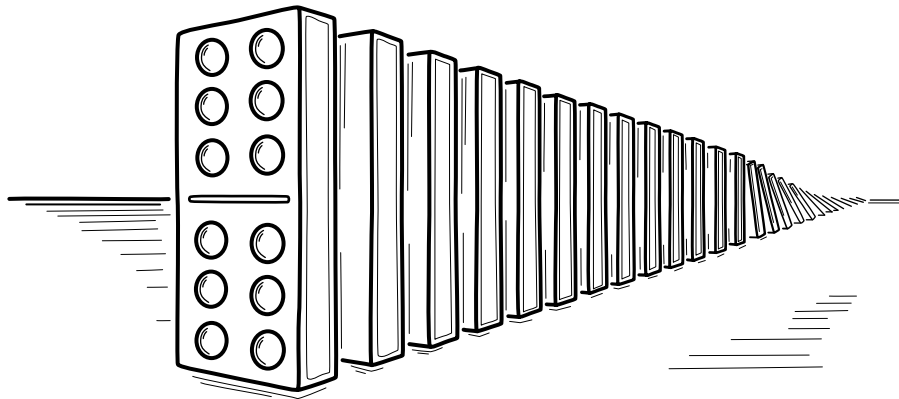


Figure 3. Infinite Domino Cascade

This cascade of falls could not happen spontaneously if there were finitely many dominoes. For, as we trace back through the finite cascade, we would eventually come to the last domino. It would not fall because there are no further dominoes to fall on it. There is nothing to start the cascade. In the infinite case, we never come to the end of the cascade. For any domino, there is always a next domino to fall on it. So every domino falls. There is no first fall to initiate the cascade and no need for one.

All that remains now is to close a loophole. If each domino takes the same amount of time to fall onto the next, then the infinity of domino falls needed to complete the cascade

requires an infinite time. That does not make the process impossible. Rather it makes it uninteresting for our purposes, for it is simply a process that has been underway for all of an infinite past time. If each fall takes one second, then the N th domino fell N seconds ago; and so on for N indefinitely large.

We close the loophole by contriving the geometry of dominoes such that each time of fall is successively shorter as we proceed along the cascade. If the successive dominoes require $1/2$, $1/4$, $1/8$, $1/16$, ... seconds to fall, then all infinity of them will have fallen after $1/2 + 1/4 + 1/8 + 1/16 + \dots = 1$ second. To an observer, the motion would appear as follows. The initially quiescent dominoes remain so for some time. Then, off in the distance of the infinite end of the row of dominoes, at the moment of spontaneous excitation, there is a disturbance that rapidly propagates towards the beginning of the row and leaves all the dominoes toppled.

Some delicacy is needed to arrange all the dominoes so that they can behave this way. The time each takes to fall on the next will depend on how hard it is struck and how close is the next domino. Under plausible assumptions, computed in Appendix A, the time each domino needs to fall onto the next scales in direct proportion to the distance between the dominoes.

Thus we secure the above schedule of acceleration of the falls by shrinking the distance between the dominoes in proportion to the times $1/2$, $1/4$, $1/8$, $1/16$, ... If we assume that the widths of the dominoes are scaled similarly, then the cascade can be completed in finite time just if the length of the domino row is finite.

One outcome of this scaling is that the dominoes will become arbitrarily thin. One might imagine that this means that the dominoes become pseudostable rather like a pencil balanced on its infinitely sharpened tip. However none of the dominoes will be pseudostable, since a pseudostable system is one which is toppled by an arbitrarily small perturbation. Each domino will have a finite width, even if small, which forms a stable base. Toppling it requires some non-zero work to lift its center of mass past its edge.

This is a toy model. However it illustrates how indeterminism arises generically in systems with infinitely many degrees of freedom. In such systems there are many cascades of excitation processes that cannot arise spontaneously in finite systems, since the finite system requires some initiating event to get the process started. In a system with infinitely many of degrees of freedom, these processes can happen spontaneously without need of some initiating event, for they are comprised of infinite cascades of events that have no first member.

These general remarks can be made more precise. For a synopsis of the analysis for a more general case and for the quantitative analysis of the masses and springs example specifically, see Norton (2012, Appendix).

4. Indeterminism Among Components of a System

In the indeterminism of the last section, the present state of the system fails to fix its future state. It may also happen that, at the same time, the state of some components of a system may fail to fix the state of other components, contrary to our expectations. The problem in inductive inference is then to determine the strengths of support afforded to these incompletely determined components.

4.1 Gauge Systems

There is a simple recipe for generating many problems of this type by injecting a small fiction into physics. Modern physical theories are replete with gauge freedoms. They arise when one has two descriptions that appear to be of distinct physical systems, but it turns out that the differences are merely artifacts of the descriptions used. It is “the Eiffel tower” and “la tour Eiffel.” The two systems are the same physically. They just differ in their names.

Imagine, however, that through some novel physics we do find a way to distinguish the two. Then we would have a difference that makes a difference; and learning which is the correct one would become a problem in inductive inference. Since there are many gauge freedoms in modern physics, this stratagem can create many new inductive inference problems of just the type sought here.

Sometimes fact can mimic fiction. The gauge field associated with magnetism is the vector potential A . In classical physics, it is merely a useful adjunct in computing magnetic field strengths, but not a physically significant quantity in its own right. The coming of quantum theory initially showed promise of changing this circumstance. Bohm and Aharonov (1959) found a quantum effect that arose when there was an A field present, but no magnetic field. They initially offered it as evidence that the A field is physically significant. Later analysis showed the situation to be more complicated.

For concreteness, I will elaborate one of the simplest gauge freedoms. In ordinary Newtonian gravitation theory, the physically significant quantity is the gravitational force on a unit test mass and the associated quantities of work. The distribution of all such possible forces over all space is the Newtonian gravitational force field f . For the case of the sun, the force field is given by the familiar inverse square law

$$f(r) = \frac{GM}{r^2} \quad (2)$$

where a force of magnitude $f(r)$ on a unit test mass is directed towards the center of the sun. M is the mass of the sun, r the radial distance from the center of the sun to the test mass and G the universal constant of gravitation.

The Newtonian gravitational potential field $\varphi(r)$ is defined through the work $W(r_0, r_1)$ needed to be performed against this force field when we move a unit test mass from one position r_0 to another r_1 . That is, the potential fields $\varphi(r_0)$ and $\varphi(r_1)$ are related by

$$W(r_0, r_1) = \varphi(r_1) - \varphi(r_0) = \int_{r=r_0}^{r_1} f \, dr = \int_{r=r_0}^{r_1} \frac{GM \, dr}{r^2} = -\frac{GM}{r_1} - \left(-\frac{GM}{r_0} \right) \quad (3)$$

We usually infer from (3) that $\varphi(r) = -\frac{GM}{r}$. However we are really only authorized to infer to something weaker:

$$\varphi(r) = -\frac{GM}{r} + K \quad (4)$$

where K can be any number, positive or negative, large or small.

The choice of K leaves the physically significant quantities unaltered. That is, for all K we end up with the same work term $W(r_0, r_1)$ in (3) since

$$\left(-\frac{GM}{r_1} + K \right) - \left(-\frac{GM}{r_0} + K \right) = \left(-\frac{GM}{r_1} \right) - \left(-\frac{GM}{r_0} \right)$$

and the same force field $f(r)$ in (2) since

$$f(r) = -\frac{d}{dr} \left(-\frac{GM}{r} + K \right) = \frac{GM}{r^2}$$

The freedom in selection of different K 's is a gauge freedom and transforming between different, physically equivalent expressions for $\varphi(r)$ by changing the value of K is a gauge transformation.

The inductive inference problem posed is this. We introduce the fiction that some new physics will enable us to detect and distinguish among the gravitational potentials of (4). Given the gravitational force field $f(r)$ of the sun (2), what is the inductive strength of support given to the gravitational potential fields $\phi(r)$ of (4) with different values of K ?

4.2 Newtonian Cosmology

Indeterminism among components in a physical theory can arise without need for any fictitious physics. A simple example, inspired by Wallace (2016), arises in Newtonian gravitation theory. We expect that the specification of the position and masses of all bodies in the universe will fix the gravitational force on a test body and the gravitation potential field at any point in space.

That things are not simple precipitated an acute problem in Newtonian cosmology in the late 19th and early 20th century. Newtonian cosmology assumes that infinite Euclidean space is filled with a uniform matter distribution of constant density ρ . The expectation is that there is a unique gravitational force acting on any test body in such a universe. That force is calculated by summing all the gravitational forces acting on the test body from the uniformly distributed cosmic matter. The trouble is that there are many ways to sum these forces. Pick any resultant force you like and there is a way to carry out the sum so that the net force on the test body is just that force. For a survey of this period and for an example of the simple calculations that lead to the multiplicity of forces, see Norton (1999a).

In retrospect, the difficulty is all too easy to see. Contrary to expectations, the cosmic matter distribution does not fix the net gravitational force on the test body. Many fields are compatible with the one matter distribution and thus we can compute many forces on the test body simply by drawing quantities from different possible fields.

At the time, however, this possibility was overlooked since the loss of uniqueness of the force was unthinkable. Instead, many physicists found it obvious and even compelling that the symmetries of the problem must force a unique solution: there can be no preferred directions in a homogeneous, isotropic cosmology. So the net force can point in no direction. Hence there is no net force on the test body and, as a result, the gravitational potential field must everywhere be a constant.

We shall return below to this risky idea that physical intuition can override what well-established equations say. Before we do, it is interesting to note that a favored resolution was to modify Newton's inverse square law of gravity until it returned the expected constant gravitational potential. This computation was used by Einstein in 1917 as a foil to motivate his introduction of the cosmological constant into general relativity.

We can develop the difficulty as follows. A curious result of Newtonian gravitation theory concerns an infinite flat plate of matter of density ρ and thickness Δx . The gravitational force exerted by this plate on a test body turns out to be independent of the distance from the plate. It is just

$$f = 2\pi G\rho \Delta x. \tag{5}$$

directed along the line of shortest distance to the plate. (See Appendix B for a justification and demonstration of this result and further analysis of this example.) We can use this result to determine the gravitational force on a test body in a Newtonian cosmos. We divide the uniform matter distribution into infinitely many flat plates of thickness Δx and infinite area, arranged parallel to the y and z axes of a Cartesian coordinate system (x, y, z) .

Consider a unit test mass at some fixed x -coordinate position, say $x'=x$. We can divide the matter distribution that acts gravitationally on it into two parts. As shown in Figure 4, the first consists of all those infinite plates between $x'=-x$ and $x'=x$. The second consists of all the remaining infinite plates.

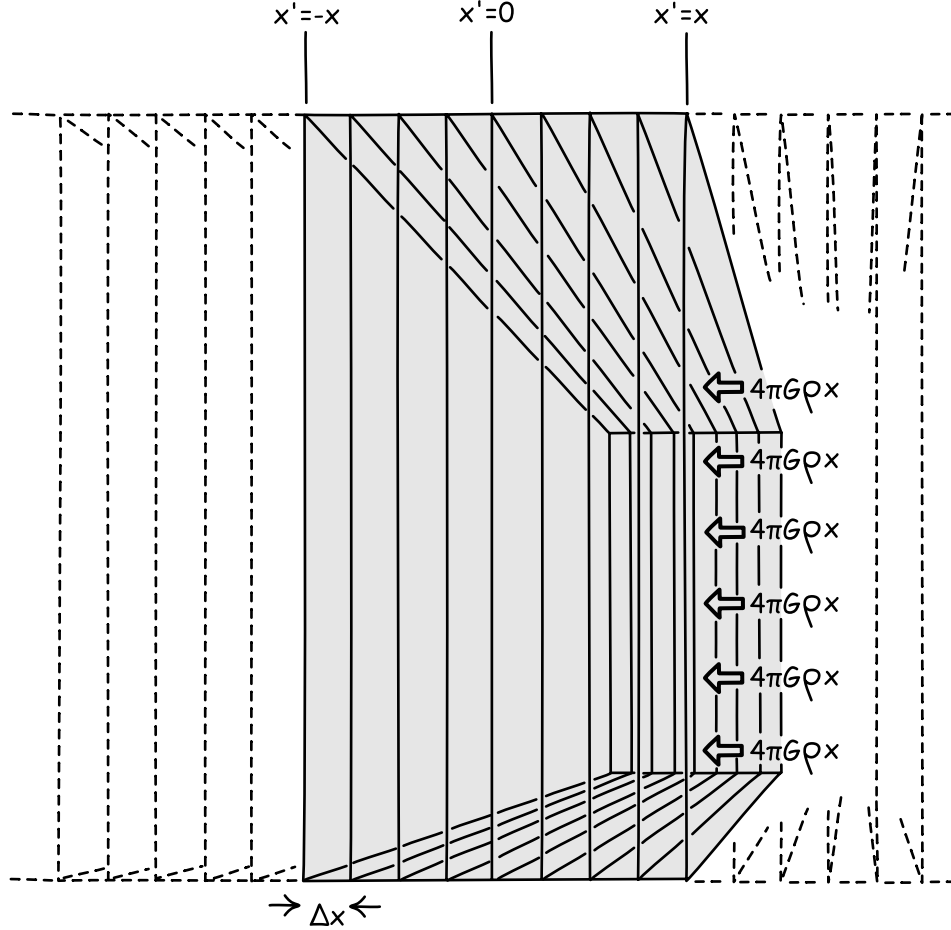


Figure 4. Unbalanced Forces in Newtonian Cosmology

We have from (5) that each plate of thickness Δx contributes force $2\pi G\rho \Delta x$. Hence the force on the test body from the plates between $x'=-x$ and $x'=x$ is just their sum

$$f(x, y, z) = 4\pi G\rho x. \quad (6)$$

and is directed along the x -axis towards $x=0$. The remaining plates each exert the force $2\pi G\rho \Delta x$ on the test body. The force will be in the $+x$ direction if the plate is located at $x'>x$ and it will be in the $-x$ direction if the plate is located at $x'<-x$. Hence we can pair up the plates at coordinate positions $+x'$ and $-x'$, matching one that exerts a force in the $+x$ direction with one that exerts a force in the $-x$ direction, so the net force from the pair is zero. This pairing exhausts all the matter of the second part, as shown in Figure 5. The net result is that the force on the test body is given by (6).

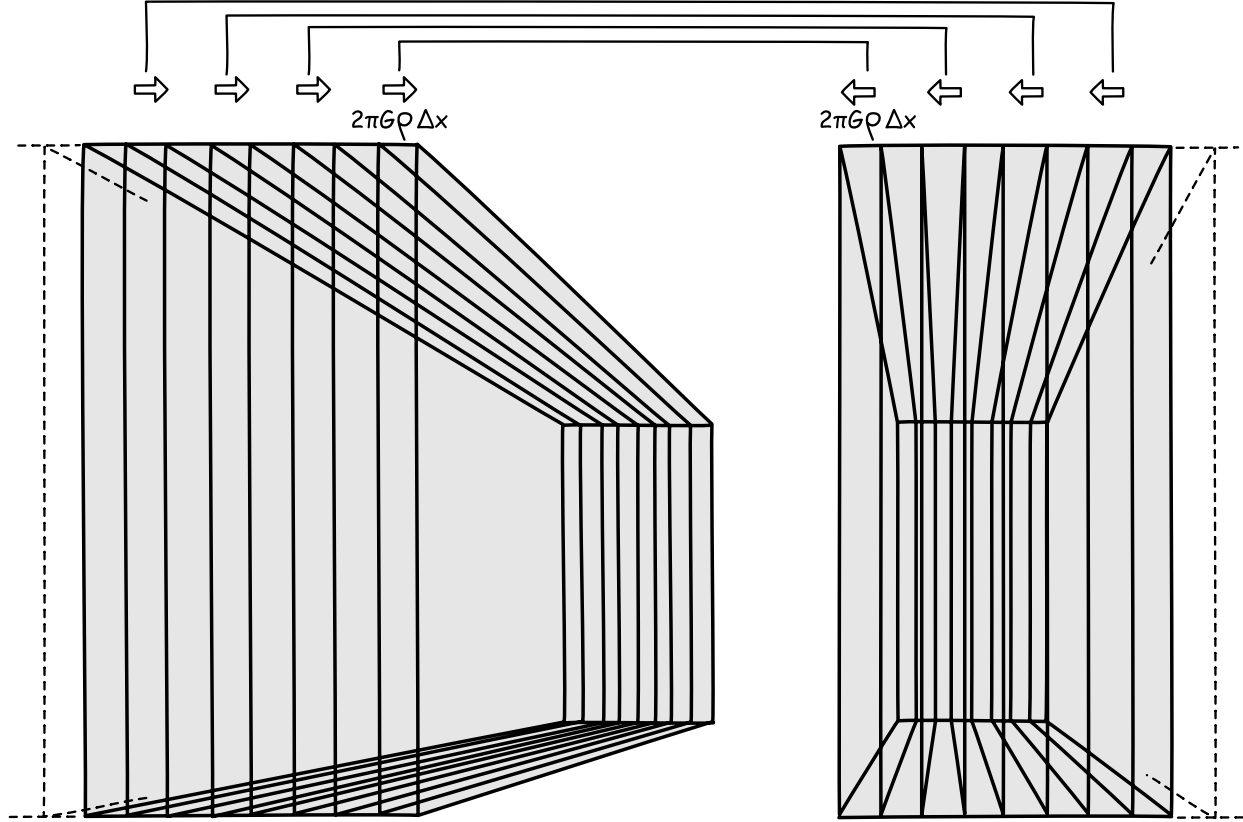


Figure 5. Balanced Forces in Newtonian Cosmology

We can repeat this construction for every point in space, so that the expression (6) represents the gravitational force field due to the cosmic matter. This force field induces a gravitational potential through a relation analogous to (3) as

$$\varphi_x(x, y, z) = \int_{x'=0}^{x'=x} f(x', y, z) dx' = 2\pi G\rho x^2 \quad (7a)$$

The problem should now be obvious. The division of the cosmic matter into plates perpendicular to the x axis was arbitrary. We could also have divided it into plates perpendicular to the y or the z axes. We could then replicate the above analysis and recover two distinct potential fields⁴

$$\varphi_y(x, y, z) = 2\pi G\rho y^2 \quad (7b)$$

$$\varphi_z(x, y, z) = 2\pi G\rho z^2 \quad (7c)$$

⁴ For experts: the potentials (7a, b, c) derive from physically distinct gravitational systems and not gauge equivalent along the lines of Malament (1995). For more, see Appendix B.

We can generate still further potential fields. Another arbitrary choice was to locate the center of the plates of the first part at x -coordinate 0. We could equally have chosen any x -coordinate, such as x_0 . We would then have arrived at the gravitational potential fields

$$\Phi_{x,x_0}(x, y, z) = 2\pi G\rho (x - x_0)^2 \quad (8a)$$

$$\Phi_{y,y_0}(x, y, z) = 2\pi G\rho (y - y_0)^2 \quad (8b)$$

$$\Phi_{z,z_0}(x, y, z) = 2\pi G\rho (z - z_0)^2 \quad (8c)$$

Taken together, we have many potentials compatible with the cosmic matter distribution. One might well suspect at this point, quite correctly, that we have only begun to explore the gravitational potential fields compatible with the cosmic matter distribution.

These potential fields form a large space and we can navigate through them by the following artifice. We start with any admissible potential, such as (7a). We arrive at another simply by adding a “harmonic function” to it. (A harmonic function is one that satisfies Laplace’s equation $\nabla^2\Phi = 0$. For more, see Appendix B.) It turns out that

$$\Phi = 2\pi G\rho (y^2 - x^2)$$

is a harmonic function. Adding it to (7a) moves us to (7b):

$$\Phi_x(x, y, z) + \Phi = 2\pi G\rho x^2 + 2\pi G\rho (y^2 - x^2) = 2\pi G\rho y^2 = \Phi_y(x, y, z)$$

Another harmonic function is

$$\Phi = 2\pi G\rho ((z-z_0)^2 - x^2)$$

Adding it to (7a) moves us to (8c).

The remarkable fact is that there are infinitely many harmonic functions and they are linearly independent. That means that we cannot reduce the set by expressing some as linear combinations of others. If we represent an infinite set of linearly independent harmonic functions as $\Phi_1, \Phi_2, \Phi_3, \dots$, then adding any linear combination of them to an admissible potential produces another. Thus we arrive at an infinite dimensional space of gravitational potentials

$$\Phi_x + a_1\Phi_1 + a_2\Phi_2 + a_3\Phi_3 + \dots \quad (9)$$

where the space is parameterized by infinitely many parameters a_1, a_2, a_3, \dots which can each independently take on all values, positive and negative, large and small. The potentials of (7a, b, c) and (8a, b, c) are just some of the simplest potentials in the space.

The inductive problem to be addressed shortly is to determine the support for each of the solutions in the space of potentials defined by (9), given the spatial geometry and matter distribution of Newtonian cosmology.

Since both the spatial geometry and the matter distribution are isotropic and homogeneous, it is natural to assume that the gravitational potential will share some or all of these symmetries. One may even have a strong intuition, as did the physicists of the past, that the potential must share these symmetries. Imposing them would have the effect of greatly reducing the size of the space of potentials (9). While the resulting reduced problem is interesting in its own right, it is not the one to be addressed here. We do not assume homogeneity and isotropy of the potential field, for there is no compulsion to assume either. It is not an assumption that can be derived from the corresponding symmetries of the geometry and the matter distribution and, as the viability of the potentials (9) show, it is not enforced on individual potentials of Newtonian gravitational theory.

5. Inductive Analysis of Temporally Indeterministic Systems

The indeterministic systems of Sections 3 and 4 above each pose a problem in inductive inference. Take certain fixed aspects of a system: its present state or certain of its components. Find the strength of inductive support that aspect provides to some other aspect: the system's future state or certain others of its components. The systems have been chosen so that all share the following two properties:

- The physics described is an exhaustive account of the totality of background facts. There are no further hidden background facts.
- The physics leaves one aspect of the system underdetermined, but provides no probabilities for the different possibilities.

An essential condition to be placed on the inductive analysis is that it merely extracts and displays the relations of inductive support already present in the fully specified systems. That is, setting off the controlling idea for emphasis:

The analysis may not impose new physics.

For to impose new physics is to introduce new facts that alter the problem posed. What would result might well be a cogent analysis of some problem, but it would not be an analysis of the problem originally posed.

6 A Probabilistic Analysis

Let us attempt to represent the strengths of inductive support as probabilities. We shall see that this analysis inevitably imposes new physical facts on the systems.

6.1 Temporally Indeterministic Systems⁵

The temporally indeterministic systems of Section 3 all involve systems that remain quiescent until some time $t=T$ of spontaneous excitation. The inductive problem is to determine the strengths of support for various times T . Initially, this looks like a problem tailor-made for probabilistic analysis, for it is similar to the problem of radioactive decay: a radioactive atom remains quiescent until the moment of decay. This moment is governed by the familiar law of radioactive decay. The probability $P(T)$ of decay in the time interval from 0 to T is

$$P(T) = 1 - \exp(-T/\tau) \quad (10)$$

where the time constant τ of the decay is related to the empirically determined half-life of the element by $T_{1/2} = \tau \ln 2$.

This law of radioactive decay is the natural probabilistic law adapted to these cases, for it is the unique law with “no memory” of what happened in the past. That is, whether the atom will decay in the moments immediately to come is independent of how long the atom has survived so far, without decaying. It has no memory of whether that past survival was long or short.

If we write $Q(T) = 1 - P(T)$ for the probability that the atom does *not* decay in the initial time T , then this no memory property is expressible as

$$Q(T+u) = Q(T).Q(u) \quad (11)$$

That is, the probability that the atom survives undecayed for a total time $T+u$ is given by the probability that it survives first for time T and then, given no decay, that it then survives for a further time u . The no memory property says that these last two probabilities are independent, so

⁵ The analysis of this section draws on Norton (2010a).

the probability of the conjunction of their outcomes is just the product of (11). This relation entails the exponential decay law (10).⁶

The probability distribution (10) expresses a physical chance. It is immediately and naturally converted into a logic of induction through the conditional probabilities it induces on pairs of hypotheses concerning the time of decay. For example, write:

$H(T_1, T_2)$: the hypothesis that the time T of spontaneous excitation occurs in the interval $T_1 \leq T < T_2$

If we take as our background B the physical description of the radioactive atom, then the support accrued to the hypothesis from B that the atom will decay sometime up to time T is just given by

$$P(H(0, T)|B) = P(T) = 1 - \exp(-T/\tau)$$

The support for the hypothesis of decay between T_1 and T_2 , from the evidence that decay happens by time $T > T_2 > T_1$ is

$$P(H(T_1, T_2) | H(0, T)) = \frac{\exp(-T_1/\tau) - \exp(-T_2/\tau)}{1 - \exp(-T/\tau)}$$

All this is unremarkable and it seems to be the natural analysis to apply to the spontaneous excitations of Section 3. Here, however, our familiarity with radioactive decay is leading us astray. For the probabilistic law (10) includes a time constant τ . The magnitude of the time constant has a profound effect on the dynamics, as shown in Figure 6.

⁶ Differentiate (11) with respect to u and find $\frac{dQ(T+u)}{d(T+u)} \cdot \frac{d(T+u)}{du} = \frac{dQ(T+u)}{d(T+u)} = Q(T) \cdot \frac{dQ(u)}{du}$.

Evaluate this expression at $u=0$ and recover $dQ(T)/dT = k Q(T)$, where $k = dQ(u)/du|_{u=0}$ is a constant independent of T . The solution is $Q(T) = \text{constant} \cdot \exp(kT)$. Since the atom must eventually decay, $P(T) = 1 - Q(T)$ must go to unity as T goes to infinity. Hence we must have “constant” = 1 and $k = -1/\tau$, for any $\tau > 0$.

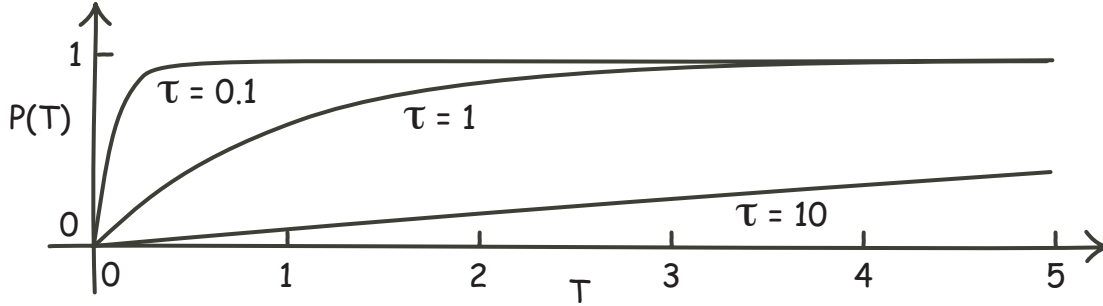


Figure 6. Effect of Different Time Constants τ on the Probability of Spontaneous Motion

A small time constant entails that spontaneous excitation is all but sure to happen soon. If τ is one millisecond, then there is a probability of 0.999 of spontaneous excitation in time $\tau \ln 1000 = 6.91\tau = 6.91$ milliseconds.⁷ A large time constant entails that spontaneous excitation is very unlikely to happen soon. If τ is one thousand years, then there is a probability of only 0.001 of spontaneous excitation in $\tau \ln 1.001 = 0.001\tau =$ one year.

Since use of the probabilistic law (10) requires selection of a time constant τ , it can only be employed if we, in effect, make some judgment about how soon the spontaneous excitation will occur. We already have the complete physics of the systems of Section 3. There is no time scale provided and no judgments of sooner or later. All the physics tells us is that spontaneous excitation is possible.

Thus to apply the probabilistic law (10) is to introduce new physics. That is, it is to change the problem posed to a new one to which probabilistic methods happen to be well-adapted.

The analysis above is just a beginning. There are many ways to apply probabilistic analysis to this problem of spontaneous excitation. While some are quite ingenious, none succeed. Here are a few of the possibilities.

The physics is indifferent to which is the time T of spontaneous excitation. So a natural choice is a uniform distribution of probability over all values of T from zero to infinity. The immediate difficulty is that the probabilities of such a uniform distribution cannot sum to unity. We set equal the probability of equal intervals

$$\varepsilon = P(H(0,1) | B) = P(H(1,2) | B) = P(H(2,3) | B) = P(H(3,4) | B) = \dots \quad (12)$$

⁷ To arrive at these estimates, invert (10) to recover $T = \tau \ln [1/(1-P)]$.

Since there are infinitely many of these intervals, the total probability is

$$\varepsilon + \varepsilon + \varepsilon + \varepsilon + \varepsilon + \dots = \infty \times \varepsilon = \infty.$$

This is a failure of the probability distribution to normalize: these probabilities should sum to the unit probability required by the axioms of probability theory for the entire outcome space.

While this failure is usually treated as fatal, the normalization condition is sometimes dropped, under the expectation that conditionalization may lead to a normalized probability distribution. However, even if this expectation is sometimes met, the real problem with the distribution (12) is that it still adds to the physical facts. It assures us that, for example, $H(0,2)$ is twice as probable as $H(0,1)$. If we make the usual connections to frequencies, that means that we should expect $H(0,2)$ to arise roughly twice as often as $H(0,1)$ in many repeated trials. The physical facts for these systems include no such provision. They simply allow that any of the times in these hypotheses may be the time of spontaneous excitation; and nothing more.

Another possibility was explored more fully in the earlier Chapter: Infinite Lottery Machines. It is that we drop the requirement of countable additivity that allows us to sum the infinitely many ε 's above. Instead, we are allowed to sum finitely many only, that is, we are restricted to finite additivity. The result is that we can set $\varepsilon=0$ in (12) without breaching normalization. All the individual hypotheses of (12) are assigned zero probability

$$0 = P(H(0,1) | B) = P(H(1,2) | B) = P(H(2,3) | B) = P(H(3,4) | B) \dots$$

but their infinite disjunction is assigned unit probability.⁸ Finite disjunctions of them are also assigned zero probability

$$\begin{aligned} P(H(0,3) | B) &= P((H(0,1) \vee H(1,2) \vee H(2,3)) | B) \\ &= P(H(0,1) | B) + P(H(1,2) | B) + P(H(2,3) | B) = 0 + 0 + 0 = 0 \end{aligned}$$

This is promising initially, since all finite intervals of times are treated equally, even if as zero probability outcomes.

The difficulty is that the finitely additive measure is still adding significantly to the physics. For even finitely additive measures must assign unit probability to some set of outcomes; and these become privileged as the events we expect to happen. There is no way to assign this privileged set without adding to the physics. For example, the above measure assures

⁸ Or, more carefully, one less whatever probability is assigned to the hypothesis that there is never a spontaneous excitation.

us that the time of spontaneous excitation is, with probability 1, greater than or equal to $T=1$: $P(H(1,\infty) | B) = 1$. The physics is equally indifferent to the times of spontaneous excitation as it is to the inverse times of spontaneous excitation, $1/T$. If the finitely additive measure is a reasonable way to represent complete indifference, then it should work equally well when it is applied to the inverse times $1/T$. In that application, by parallel reasoning, we arrive at the result that, with probability one, $P(H(1/1,\infty) | B) = 1$. But $H(1/1,\infty) = H(1,\infty) = H(0,1)$,⁹ so that we have a contradiction with the earlier probability assignment $P(H(0,1) | B) = 0$.

The escape from the contradiction is to decide that only one of the two finitely additive measures may be used. That, however, amounts to selecting a privileged subset of probability one times of excitation: the times between 0 and 1, or between 1 and infinity. The physics makes no such distinction. It is an addition forced on us by the probabilistic measure.

Two further probabilistic embellishments have been treated elsewhere in Norton (2010a) and in earlier chapters. First, one might try to escape the need to select a single time constant τ in (10) by adopting the complete set of measures (10), for all values of τ , as the representation of the strength of support. The motivation is correct in that it seeks a representation weaker than a single probability measure. However it is too indirect in that it seeks to preserve probability measures by using them to simulate a different, non-additive logic. The better approach is simply to write down that logic directly, as is done in Section 7 Below.

Second, one might adopt the measure of (10) as a subjective degree of belief. The earnest but possibly unrealizable hope is that repeated conditionalization will wash away the subjective opinion and leave behind the objective bearing of evidence, or at least some approach to it. Once again, the motivation is good, but the execution poor. Again, the better approach is merely to write down the warranted logic directly.

6.2 Probabilities, Empirically?

While we may not be able to recover probabilities from the physics governing these indeterministic systems, might we introduce them through an empirical artifice? To take a concrete case, imagine that somehow we are able to physically realize a dome. We might then set up very many of them and just observe what happens. Might we find that that the frequencies for

⁹ Aside from the inclusion of $T=1$ in $H(1/T,\infty)$, but not in $H(0,1)$.

different times of spontaneous excitation stabilize towards limiting values? We could then introduce probabilities, set in value to those empirically determined, limiting, relative frequencies.

Dawid (2015) considers an even simpler case in the same spirit. What if we have 100 domes and find that they all excite spontaneously at exactly 16.8 seconds? Might we then infer to a deterministic rule: spontaneous excitation occurs at 16.8 seconds for all domes?

How we treat these proposals will depend on how certain we are of the background, governing physics. Are we certain of the background physics or are we not?

In the first case, we remain certain that the Newtonian physics specified is the totality of the physics governing the processes. That all excitations occur at 16.8 seconds is compatible with the indeterministic physics, but it is not something we could predict from that physics, at the exclusion of many other possibilities. Correspondingly, the background physics authorizes no further predictions, even after we have seen all 100 domes excite at 16.8 seconds. We should remain as uncertain of the next excitation time as we were prior to seeing the first dome in the imagined experiment.

This situation is quite similar to that of a gambler in a casino at a roulette wheel. Neglecting 0 and 00, the chance of a black on a properly functioning wheel is $1/2$. Imagine, however, that the gambler steps up to the table with the wheel and finds 20 successive spins to yield black. Assume the gambler is confident of the background theory: the wheel is functioning properly. All the gambler can properly conclude is that an extremely unlikely event has occurred. Twenty successive black outcomes is possible, just improbable.

What the gambler should not now think is that the wheel is on some sort of “streak” so that, contrary to the physical construction of the wheel and the laws of probability, the next outcome is more likely to be black. To think that is to commit a notorious gambler’s “streak” fallacy.

It is the same with the dome. As long as we remain convinced that the Newtonian physics described is the totality of the physics that governs the dome, repeated excitations at 16.8 seconds is merely a coincidence. In a similar vein, the indeterministic physics does not support the existence of stable limiting frequencies for different excitation times. Any appearance of such stability is mere coincidence that cannot be expected to persist.

That was the first case. In the second case, we become uncertain that the Newtonian physics described is all that governs the actual domes of our experiment. We suspect that some further or some other physics is at play. What physics it might be is hard to say, since the entire scenario is built from multiple layers of fiction. I leave it to the reader's imagination. Whatever alternative physics we may suspect here is what will guide the inferences.

Once again, the situation is similar to that of the gambler. The probability of 20 black outcomes is exceedingly small: $1/2^{20}$, which is roughly $1/1,000,000$. Having seen such an improbable occurrence, the gambler would reasonably suspect that something odd is afoot. Perhaps the wheel has some ingenious cheating device that is malfunctioning and delivering all black outcomes. If the gambler believes that to be the case and that the cheating device will continue to operate well, the gambler would be well warranted to conclude that the next outcome will be black.

In short, as long as we retain the presumptions made at the outset of the totality of the physics governing the indeterministic systems, any empirically observed regularities of the type suggested will be of no help to us inductively. To expect otherwise is to commit a fallacy analogous to the gambler's "streak" fallacy.

6.3 Systems with Indeterminism Among their Components

The inductive problems posed in Section 4 are to find the inductive strengths of support afforded to underdetermined components of a physical system by those that are fixed by the problem specification. Much of the analysis of Section 6.1 can be carried over to the probabilistic analysis of these problems. Probabilistic analysis fails in the same way. In addition, the infinite dimensionality of the space of underdetermined potentials (9) in Newtonian cosmology raises more problems.

The simplest problem was posed in Section 4.1. We are to choose among the infinitely many gauge equivalent fields of (4). This choice amounted to selection of a value of the constant K , which can take any real value, positive or negative, large or small.

The straightforward approach is to represent strength of inductive support by a probability distribution over K . However, since K has an infinite range, the distribution must be attenuated towards zero for large positive and large negative values of K . Otherwise it will not normalize to unity. Here the difficulty is like that faced by the probabilistic law (10). The rate of

attenuation will be represented by some parameter or some characteristic of the distribution that is akin to the selection of the time constant τ in (10). Any choice of a rate of attenuation, however, is an addition to the physics of the gauge system.

One might also try to avoid the problem by employing an unnormalizable probability distribution akin to (12). Once again, this will add to the physics, for it requires us to assign higher probability to larger intervals of K , even though the physics does not authorize it. Finally the difficulties of the finitely additive measure can be replicated here as well.

The still harder case for probabilistic analysis is that of Newtonian cosmology in Section 4.2. For now we are to distribute probabilities uniformly over the space of potentials (9). Its individual solutions are picked out by specifying values for the infinitely many parameters a_1, a_2, a_3, \dots . That is, it is an infinite dimensional space. The familiar problem is that we cannot easily assign an additive measure over such spaces since the parameter values range from minus infinity to plus infinity. In the examples so far, it is the requirement of normalization of the measure of the full space to unity that forces the problem. The new problem with an infinite dimensional space is there is still no well behaved, uniform measure over this space, even if we drop the requirement of normalization.

To see this, recall that probabilities behave like volumes in space. So, for continuity with familiar notions, let us continue to call them volumes. First consider a space of parameters a_1, a_2, \dots, a_n of finite dimension n . The set of all points for which $0 < a_i < 2$, all i , forms a cube of side 2. This cube consists of 2^n cubes of unit side. In a three dimensional space, the side 2 cube consists of $2^3 = 8$ unit sided cubes. If we assign unit volume to each unit cube, the side 2 cube just has volume 2^n .

For any finite n this relation is unproblematic. That ceases to be so when we take the case of the infinite dimensional space. For then, the sided 2 cube consists of an uncountable infinity 2^∞ of unit cubes. Since the measure is uniform, all the unit cubes have the same volume. There are two cases: the unit cubes have non-zero volume; and the unit cubes have zero volume.

If the unit cubes have some finite, non-zero volume, then it follows that the side 2 cube must have infinite volume. This follows using only finite additivity of the volumes. For if we suppose any finite volume for the side 2 cube, then we need only sum finitely many of the unit

cubes to recover a summed volume greater than it. Of course, if the unit cubes have infinite volume, then so must also the side 2 cube.

The other possibility is that the unit cubes have zero volume. Then the side 2 cube can also have zero volume. However it may also have a finite, non-zero volume or an infinite volume. This may seem odd, since we are supposing the side 2 cube to consist of nothing but zero volume unit cubes. Why not add up all these zeroes and get zero volume? The problem is that there are an *uncountable* infinity 2^∞ of zeroes and adding uncountable infinity of them is an undefined operation.¹⁰ The volume of the side 2 cube must merely be greater than the sum of the volumes of finitely many unit cubes; or (if countable additivity is assumed) of a countable infinity of them. So its volume can be non-zero.¹¹

These results can be applied to a cube anywhere in the space. Every cube can be decomposed into 2^∞ half-sided cubes; and every cube is itself a component cube of a doubled-sided cube. What results are three possibilities for the uniform measure. The two simple ones are just that all cubes have either zero volume or infinite volume. The complicated case is that there is some value L such that an L sided cube has finite, non-zero volume. Since the measure is uniform, all cubes of side L will have this volume. It follows by replicating the above reasoning that all smaller cubes that can be compounded to form cube of side L must have zero volume; and all larger cubes that can be built from cubes of side L must have infinite volume.

This third option violates the requirement that we add nothing to the physics, for it singles out quite particular, preferred sets of parameters as just those that reside in the cubes of side L . Since parameter values correspond to gravitational potentials, this is a privileging of certain sets of potentials.

Combining the three possibilities, cubes in this space will almost everywhere have either zero volume or infinite volume. One can see this result informally by noting what happens when

¹⁰ This is a familiar result. Each point in the unit intervals of reals is of zero length. Since there are an uncountable infinity of them, we cannot add them to find the length of the unit interval of reals, which is not zero, but one.

¹¹ This is an uncommon possibility. In discussions of measures on infinite dimensioned spaces, it is usually assumed that the spaces are separable, which allows that each region can be composed of a countable infinity of equal volume subregions. Separability fails in this case.

we scale up or scale down any region by a factor M . That is, we multiply all the parameter values in the set specifying the region by M . The volume of the region will scale by a factor $M^{\text{dimension of space}} = M^\infty$. This factor is zero if $M < 1$ and infinity if $M > 1$. This suggests that almost all volumes will be zero or infinity. For a finite, non-zero volume cannot stay finite and non-zero under any scaling, either up or down. It becomes an infinite or a zero volume respectively. However employing this factor M^∞ directly in a more thorough argument is not straightforward since it leads to indeterminate arithmetic forms. For example, scale up a zero volume by an infinite factor M^∞ , when $M > 1$. The new volume is “ $0 \times \infty$,” which is an expression that cannot be evaluated.

Note that these troubles arise *without* assuming that the volume of the total space normalizes to unity. If we retain countable additivity, the possibilities above admit only two values for the volume of the entire space: zero or infinity.

It might be tempting to drop countable additivity, assign zero volume to any bounded region and unit volume to the whole space. One does not escape the difficulty already developed above for finitely additive measures in the case of spontaneous excitations. Briefly, the measure ought to be indifferent to whether we parameterize the space with the original parameters a_i or their inverses, $1/a_i$. Then we would assign zero volume to the side 2 cube in the inverse parameterization $1/a_i$ for which $0 < |1/a_i| < 1$, all i . But this region corresponds to the entirety of the space in the original parameterization, $1 < |a_i| < \infty$, excepting a zero volume cube $0 < |a_i| < 1$. In the original parameterization, this region is assigned unit volume.

7. The Inductive Logic Warranted

7.1 The Logic

The material theory of induction directs us to look to the background facts to determine which logic is warranted. In the cases of this chapter, the background facts are, by careful contrivance, such as to support essential no non-trivial inductive inferences at all. They allow us merely to say that certain outcomes are possible but to provide no discriminations of the nature

of “more possible” or “less possible.” This lack of discrimination can be codified into a formal calculus with three values:¹²

nec = necessary

poss = possible

imp = impossible

These values are assigned to strengths of inductive support, written as “[*A|B*],” where this symbol represents the strength of inductive support afforded to proposition *A* by proposition *B*. The little structure these strengths have is induced by deductive relations among the propositions; or, in other terms, by set theoretic containment amongst the sets of possibilities. That is, we have:

$$\begin{aligned}
 [A|C] &= \textit{nec}, \text{ if } C \text{ deductively entails } A. \\
 &= \textit{imp}, \text{ if } C \text{ deductively entails not-}A. \\
 &= \textit{poss}, \text{ otherwise.}
 \end{aligned}
 \tag{13}$$

The logic is empty until we specify the propositions to which it applies. Many choices are possible here. One convenient choice arises in the context of the spontaneously exciting systems of Section 4.1. The propositions over which this logic is defined are: $H(T_1, T_2)$, as defined in Section 6.1; *B*: the proposition that describes the background physical facts of the system; and, for completeness, $H(\infty)$: the time of spontaneous excitation $T = \infty$. Proposition $H(\infty)$ corresponds to the case in which there is no spontaneous excitation.

The logic now authorizes us to assign strengths of support such as

$$[H(T_1, T_2) | B] = \textit{poss}, \text{ for any } T_2 > T_1.$$

$$[H(\infty) | B] = \textit{poss}$$

$$[H(1, 2) | H(0, 4)] = \textit{poss}$$

$$[H(0, 4) | H(1, 2)] = \textit{nec}$$

$$[H(0, 4) | H(10, 20)] = \textit{imp}$$

There is a natural and obvious generalization to the systems of Section 4 with indeterminism among the system components.

¹² This logic has been developed in various forms in Norton (2008a, 2010a and 2010b) and in Chapter 10.

An important property of this logic is that it is not additive, in contrast with the probability calculus. That is, if A_1 and A_2 are mutually exclusive propositions, such that $[A_1|C] = [A_2|C] = poss$, then it is possible that $[A_1 \vee A_2 | C] = poss$. Overall, we violate additivity since

$$[A_1|C] = [A_2|C] = [A_1 \vee A_2 | C] \quad (14)$$

The additivity of a probability measure would require in this case that

$$P(A_1|C) + P(A_2|C) = P(A_1 \vee A_2 | C)$$

so the probabilities assigned to A_1 , A_2 and $A_1 \vee A_2$ cannot be equal unless we have the exceptional case of all probability zero outcomes.

7.2 Invariances

Norton (2008a, 2010b) and Chapter 10 argued that this logic (13) represents the case of completely neutral support; that is, the case in which we have no reason at all to favor any of the contingent propositions in any degree. It was shown that the logic can be derived in two ways from two invariance properties. We shall see below that these invariances are respected to a great extent in these systems. However, do recall that the logic (13) of Section 7.1 was not derived from these invariances, but directly from the possibilities allowed by the background physical facts.

Redescription

The first invariance is invariance under redescription. This invariance is commonly employed in the context of the principle of indifference. It arises when we redescribe a system in a way that preserves our indifferences.

Take, for example, the value of the parameter K in the Newtonian gauge system of Section 4.1. Represent a useful set of hypotheses by:

$$H_K(k_1, k_2): \text{ the parameter } K \text{ lies in the interval } k_1 \leq K < k_2$$

On the basis of the background facts B , we are indifferent to K lying in equal ranges of values, so we have

$$\begin{aligned} poss &= [H_K(0, 1) | B] = [H_K(1, 2) | B] = [H_K(2, 3) | B] = [H_K(3, 4) | B] \\ &= [H_K(4, 5) | B] = [H_K(5, 6) | B] = [H_K(6, 7) | B] = [H_K(7, 8) | B] \end{aligned}$$

Now replace the parameter K by $L = K^3$. Since L is an equally good parameter to use in (4), we can also write

$$poss = [H_L(0, 1) \mid B] = [H_L(1, 2) \mid B]$$

However $H_L(1, 2) = H_K(1, 2^3) = H_K(1, 8)$

$$= H_K(1, 2) \vee H_K(2, 3) \vee H_K(3, 4) \vee H_K(4, 5) \vee H_K(5, 6) \vee H_K(6, 7) \vee H_K(7, 8)$$

Combining with $H_L(0, 1) = H_K(0, 1)$ we recover

$$\begin{aligned} poss &= [H_K(1, 2) \mid B] = [H_K(2, 3) \mid B] = [H_K(3, 4) \mid B] \\ &= [H_K(4, 5) \mid B] = [H_K(5, 6) \mid B] = [H_K(6, 7) \mid B] = [H_K(7, 8) \mid B] \\ &= [H_K(1, 2) \vee H_K(2, 3) \vee H_K(3, 4) \vee H_K(4, 5) \vee H_K(5, 6) \vee H_K(6, 7) \vee H_K(7, 8) \mid B] \end{aligned}$$

This is an example of the failure of additivity of the type of (14).

Negation

The second invariance is invariance under negation. If the support for some proposition A is completely neutral, then we have no grounds to assign it more or less support than its negation $\text{not-}A$. We must assign the two equal support. That is, the strength of support remains unchanged under the negation map that sends hypotheses to their negations.

This negation map can be implemented in the case of systems that can spontaneously excite as follows. Write

$H_T(T_1, T_2)$: the time of spontaneous excitation T lies in the interval $T_1 \leq T < T_2$

Hypothesis $H_T(0, 1)$ says that this time lies in $0 \leq T < 1$. Its negation, $\text{not-}H_T(0, 1)$, asserts that that the time of spontaneous excitation lies in $1 < T \leq \infty$. Negation invariance of the strengths of support requires the equality

$$[\text{not-}H_T(0, 1) \mid B] = [H_T(0, 1) \mid B] \tag{15}$$

We can see that this equality obtains according to the rules of (13). For

$$\text{not-}H_T(0, 1) = H_T(1, \infty) \vee H(\infty)$$

and from the rules

$$[\text{not-}H_T(0, 1) \mid B] = [H_T(1, \infty) \vee H(\infty) \mid B] = poss$$

as well as

$$[H_T(0, 1) \mid B] = poss$$

All these hypotheses accrue equal support *poss* from the background *B* since none are deductively entailed by *B*.

We can also derive negation invariance from redescription invariance. Consider the support, not for various times *T*, but for the inverse times $1/T$. If we are indifferent to the two parameterizations of the time, *T* and $1/T$, then we would have, under description invariance:

$$[H_T(0, 1) \mid B] = [H_{1/T}(0, 1) \mid B]$$

The interval $1 < T \leq \infty$ is the same $0 \leq 1/T < 1$. That is,

$$\text{not-}H_T(0, 1) = H_{1/T}(0, 1)$$

Combining we infer

$$[H_T(0, 1) \mid B] = [H_{1/T}(0, 1) \mid B] = [\text{not-}H_T(0, 1) \mid B]$$

This is just negation invariance (15).

8. Conclusion

According to the material theory of induction, there is no logic or calculus of inductive inference that applies universally to all problems in inductive inference. It follows that there are problems in inductive inference in which strengths of support cannot properly be represented by probability measures. This chapter illustrates this claim with examples of indeterministic physical systems contrived to be resistant to a representation of strengths of inductive support as probabilities. The contrivance depends on finding simple physical systems in which a full specification of the background physical facts can be given and their burden easily discerned. An inductive analysis must determine strengths of inductive support without requiring alteration of or addition to these background facts. In the examples presented, using probabilities to represent strengths of supports requires just such additions. For this reason their use fails.

The material theory of induction asserts that the applicable logic of induction is determined by these background facts. Their paucity supports a very weak, three-valued inductive logic that happens to coincide with the completely neutral strengths of inductive support elaborated elsewhere.

The inductive problems of this chapter all involve problems of indeterminism in which certain aspects of a system fail to fix certain other aspects. Problems of this sort do arise in recent science. The most obvious involves singularities in general relativity. Singular spacetimes can

develop in many ways into the future. The possibilities are not determined and there are no probabilities provided by general relativity to weight the different possibilities.

A white hole is the temporal inverse of a black hole. When systems fall into a black hole, their structures are obliterated by the black hole, whose properties are merely mass, charge and angular momentum. If we now take the time reverse of the falling in, anything that can fall into a black hole can also be ejected by a white hole. The possibilities are not determined.

In relativistic cosmology, the big bang is a spacetime singularity in our common past, out of which the entire universe issued. The long-standing puzzle has been to explain why this singularity issued in a universe that is so nearly spatially homogeneous and isotropic and with spatial curvature very close to zero. Here is a problem in inductive inference. Given the background facts of general relativity and that there is an initial singularity, what support do we have for the various possible cosmologies that may arise? There are very many possible configurations other than the particular one manifested in our universe; and there are no good reasons provided in pre-inflationary cosmology¹³ that we should have just these initial conditions and not others.

It is tempting to convert these last facts into the claim that it is very improbable that we have the initial conditions we do. But such a claim, if read literally, solves the inductive problem by means of a probability measure. Since the background facts listed provide for no probabilities, their introduction is as illicit as in the contrived examples of this chapter.

The moral of the chapter is that we should be prepared for problems in inductive inference in which strengths of support are not well-represented by probability measures. To do otherwise, to persist in representing strengths of inductive support universally as probability measures, risks unwittingly importing new facts that change the problem posed to a new one amenable to probabilistic representation. The outcome is that we will not have solved the problem actually before us but a different one that we wished we had.

¹³ The once common claim that inflationary cosmology does provide these reasons is now challenged. See for example Holland and Wald (2008).

Appendix A: Toppling Dominoes

A domino has width W , height H and mass m and is separated from the next domino by an inter-domino distance L . To be toppled, a small impulse is needed to push the domino from its vertical position until it strikes the next domino, as shown in Figure 7.

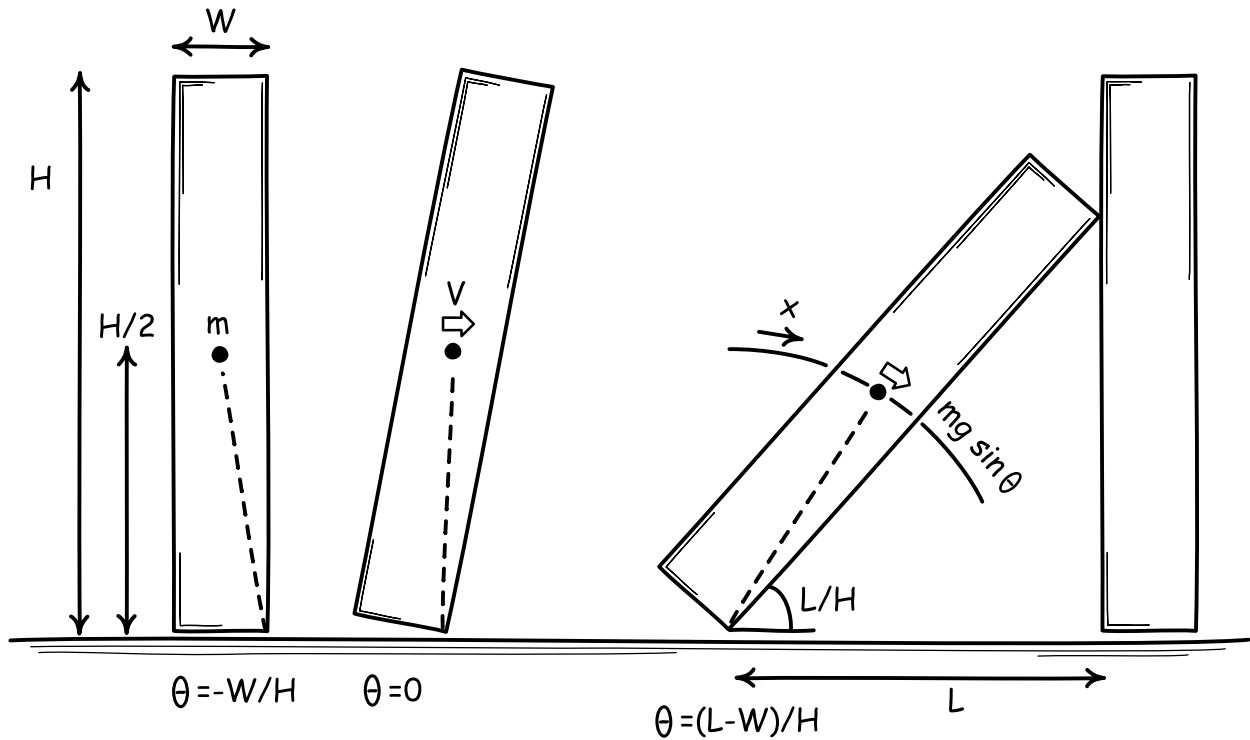


Figure 7. Geometry of a Toppling Domino

As the center of mass of the domino pivots on one edge, it forms an inverted pendulum. Call the angular position of the center of mass θ as it pivots around the edge and set $\theta=0$ when the center of mass is directly over the edge. If the distance along the circular arc traced by the center of mass of the domino is x and the center of mass is located in the geometric center of the domino, then the gravitational force on the center of motion in the direction of the arc is $mg \sin\theta$, for g the

acceleration due to gravity. The equation of motion in time t is $m \frac{d^2x}{dt^2} = mg \sin\theta \approx mg\theta$, where

$\sin\theta$ is approximated as θ for the small angles we encounter here. Since $\theta = x/(H/2)$, we have

$$\frac{d^2\theta}{dt^2} = k^2\theta$$

where $k^2 = g/(H/2)$. This inverted pendulum equation of motion admits the general solution $\theta(t) = A \sinh(kt) + B \cosh(kt)$, for undetermined constants A and B . We set $\theta(t) = 0$ when $t=0$, so that $B=0$, and arrive at:

$$\theta(t) = A \sinh(kt)$$

In toppling, the center of mass of the domino is first lifted by the rotational pivot about the edge and then falls under gravity once past the edge.

It would be convenient if there were some longest time this motion could take. One might imagine that, if the domino were given just the right, minimal push, it would pivot slowly and its center of mass would momentarily have zero speed as it passes over the edge at the apex of its motion. This cannot happen. A longer computation shows that this motion would require infinite time. (For more, see Norton, 2003, pp. 11-12.)

The best we can secure is that the center of mass, at the moment of passing over the edge, has some small linear speed V . Since the angular speed is $d\theta(t)/dt = Ak \cosh(kt)$, we require $V/(H/2) = d\theta(0)/dt = Ak \cosh(k0) = Ak$. Thus the solution is

$$\theta(t) = V/(kH/2) \sinh(kt) \approx Vt/(H/2)$$

since, for small times, $\sinh(kt) \approx kt$.

The domino center of motion must move from its initial angular position $\theta = -W/H$ to its collision with the next domino at angular position $\theta = (L-W)/H$. Substituting into the last equation for $\theta(t)$, we have $L/H = Vt/(H/2)$ for the time t required by the domino to fall. That is

$$t = L/2V$$

Thus the time t_n for the n th domino to fall is given by $L_n/2V$, where L_n is the distance between dominoes n and $(n-1)$. Thus:

Total time for cascade

$$= \sum_n t_n = (1/2V) \sum_n L_n = (1/2V) \text{ Total distance between dominoes}$$

If we assume that the domino width scales in the same way as the distance between the dominoes, the condition that the cascade completes in finite time reduces to the condition that the domino row be of finite spatial length. (Informally, this condition follows if we imagine that the falling propagates through the chain at roughly a constant speed V .)

An assumption of this analysis is that each domino has the same speed V as its center of mass passes its apex. One might wonder whether the system can provide each domino sufficient energy. Some qualitative considerations show that this will not be a problem. Each domino by supposition has speed V at its apex and thus kinetic energy $(1/2)mV^2$. Assuming elastic collisions, it will pass this much energy to the next domino as well as the extra energy released when the domino center of mass falls to a lower height overall.

Indeed the problem will not be a lack energy to sustain the cascade, but the danger of a surfeit. For there are infinitely many dominoes of the same mass, each falling through a height in a finite time. If each domino falls to the same prone position, that will result in release of an infinite amount of energy.

Appendix B: Newtonian Cosmology

The force (15) exerted by an infinite, flat plate of density ρ and thickness Δx is independent of the distance to the plate is easy to see qualitatively. Consider the portion of the plate subtended by a very small angle Ω at the location of unit test mass. The volume and thus the mass of this portion is proportional to Ωr^2 . However the force exerted by this mass on the test mass diminishes with $1/r^2$. Hence the force is proportional just to Ω and independent of distance.

The full expression for the force is computed as follows. The distance r from the unit test mass to each part of the plate satisfies $r^2 = x^2 + s^2$ where x the shortest distance to the plate and s the distance from the closest point on the plate to the part at issue. A circular ring of width ds at radius s in the plate exerts a force on the unit test mass of

$$\frac{G\rho 2\pi s ds \Delta x}{r^2} \cdot \frac{x}{r} = G\rho 2\pi x \Delta x \frac{s ds}{(s^2 + x^2)^{3/2}},$$

where x/r is the cosine of the half angle at the base of the cone subtended by the ring. Integrating over all s , we recover (15) as

$$f = 2\pi G\rho x \Delta x \int_{s=0}^{s=\infty} \frac{s ds}{(s^2 + x^2)^{3/2}} = 2\pi G\rho x \Delta x \left. \frac{-1}{(s^2 + x^2)^{1/2}} \right|_{s=0}^{s=\infty} = 2\pi G\rho x \Delta x \frac{1}{x} = 2\pi G\rho \Delta x$$

We can also compute the Newtonian gravitational potential field ϕ directly from Poisson's equation

$$\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi = 4\pi G\rho \quad (16)$$

For constant ρ , the solutions (7a, b, c) and (8a, b, c) follow immediately. For example, we recover (7a) as

$$\nabla^2\phi_x = \nabla^2(2\pi G\rho x^2) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (2\pi G\rho x^2) = 2\pi G\rho \frac{\partial^2}{\partial x^2} x^2 = 4\pi G\rho$$

That $\Phi = 2\pi G\rho (y^2 - x^2)$ is harmonic follows since

$$\nabla^2(y^2 - x^2) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) y^2 - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) x^2 = \frac{\partial^2}{\partial y^2} y^2 - \frac{\partial^2}{\partial x^2} x^2 = 2 - 2 = 0$$

That adding a harmonic function to a solution of Poisson's equation (16) takes us to another solution follows from the linearity of the operator ∇^2 . If Φ is a harmonic function, which satisfies Laplace's equation $\nabla^2\Phi = 0$, and we add it to an existing solution ϕ of Poisson's equation (16), their sum ($\phi + \Phi$) also satisfies Poisson's equation, for

$$\nabla^2(\phi + \Phi) = \nabla^2\phi + \nabla^2\Phi = \nabla^2\phi + 0 = 4\pi G\rho$$

The full set of harmonic functions is a linearly independent set. There is no simple way to write this set. In spherical coordinates (r, φ, θ) , the harmonic functions are

$$\Phi(r, \varphi, \theta) = (A_j r^j + B_j r^{j+1}) P_j^m(\cos \theta) (a_m \cos m\varphi + b_m \sin m\varphi),$$

for A_j, B_j, a_m, b_m arbitrary constants; $m = -j, -(j-1), \dots, (j-1), j$; and $j = 0, 1, 2, 3, \dots$;

and $P_j^m(\cos \theta)$ are the associated Legendre functions of $\cos \theta$. (From Bronshtein and Semendyayev, 1985, p. 463, after correction of apparent typographical errors.)

Digression for Experts

Since this problem of Newtonian cosmology has attracted considerable attention in the philosophy of physics literature, I include a short digression for experts.

Among the solutions to (16) is one that is formed as the equally weighted sum of the three solutions $(1/3)\phi_x + (1/3)\phi_y + (1/3)\phi_z$ and is called by Malament (1995) a canonical solution centered at the origin

$$\Phi_{can} = (2/3)\pi G\rho (x^2 + y^2 + z^2) = (2/3)\pi G\rho r^2 \quad (17a)$$

where the radial coordinate r satisfies $r^2 = x^2 + y^2 + z^2$. This solution has a special status as a solution with maximum isotropy: it is isotropic about the origin $\mathbf{r} = (x, y, z) = \mathbf{0}$. That falls well short of the full homogeneity and isotropy that the early physicists expected. It has a preferred center at the origin of coordinates. Infinitely many more, distinct canonical solutions are possible, each centered at different points in space, $\mathbf{r}_0 = (x_0, y_0, z_0) \neq \mathbf{0}$.

$$\Phi_{can} = (2/3)\pi G\rho (r-r_0)^2 \quad (17b)$$

Malament showed, however, that the differences among these canonical solutions were only apparent. He adopted the natural assumption that the physically real properties of a Newtonian cosmology manifest in the relative accelerations of point masses in free fall. It turned out that all the canonical solutions give the same relative accelerations. That is, the choice among them was merely the exercising of a gauge freedom. For further motivation for this choice of what is physically significant, see Norton (1995).

Malament's analysis gave a satisfactory answer to this question: which isotropic, homogeneous Newtonian cosmologies are there? The answer is given uniquely by the canonical solutions.

Our present question is a different one. It is: which potential fields are fixed by a uniform matter distribution through Poisson's equation (16). The answer to this question, as has been emphasized by Wallace (2016), is that there are infinitely many such fields and they form the infinite set (9). Only very few of them prove to be physically equivalent after the manner of (17a) and (17b). Solutions (7a), (7b) and (7c) are *not* physically equivalent. It follows from (6) that masses in free fall in (7a) ϕ_x experience relative accelerations in the x -direction but not in the y - or z -directions. Similarly masses in free fall in ϕ_y and ϕ_z experience relative accelerations respectively in the y - and z -directions only.

A natural way to block this failure of the mass distribution to determine the gravitational potential, as Wallace (2016) has emphasized, is to impose boundary conditions. All but the canonical solutions are eliminated if we require isotropy in the physically significant properties, as do Malament (1995, p. 492, p. 501) and Norton (1995, p. 513, footnote 2). However the imposition of this condition must be understood as a distinct choice we make in order to prune the space of solutions to a subset that happens to interest us. We cannot derive it from the

isotropy of space and the matter distribution, for the Poisson equation does not respect this symmetry in its individual solutions.

References

- Aharonov, Yakir and Bohm, David (1959) "Significance of Electromagnetic Potentials in the Quantum Theory," *Physical Review*, **115**, pp. 485–491.
- Dawid, Richard (2015) "Turning Norton's Dome Against Material Induction," *Foundations of Physics* **45**, pp. 1101-1109.
- Earman, John (1986) *A Primer On Determinism*. Dordrecht, Holland: D. Reidel Publishing Company.
- Hollands, Stefan and Wald, Robert M. (2008) "An Alternative to Inflation," <https://arxiv.org/pdf/gr-qc/0205058.pdf>
- Malament, David (1995) "Is Newtonian Cosmology Really Inconsistent?" *Philosophy of Science*, **62**, pp. 489-510.
- Manchak, John and Roberts, Bryan W., (2016) "Supertasks", *The Stanford Encyclopedia of Philosophy (Winter 2016 Edition)*, Edward N. Zalta (ed.), <http://plato.stanford.edu/archives/win2016/entries/spacetime-supertasks/>.
- Norton, John D., (1995) "The Force of Newtonian Cosmology: Acceleration is Relative" *Philosophy of Science*, **62**, pp. 511-22.
- Norton, John D., (1999), "A Quantum Mechanical Supertask", *Foundations of Physics*, **29**, pp. 1265–1302.
- Norton, John D., (1999a) "The Cosmological Woes of Newtonian Gravitation Theory," in H. Goenner, J. Renn, J. Ritter and T. Sauer, eds., *The Expanding Worlds of General Relativity: Einstein Studies*, Volume 7, Boston: Birkhäuser, pp. 271-322.
- Norton, John D. (2003) "Causation as Folk Science," *Philosophers' Imprint* Vol. 3, No. 4 <http://www.philosophersimprint.org/003004/>
- Norton, John D. (2008) "The Dome: An Unexpectedly Simple Failure of Determinism," *Philosophy of Science*, **75**, pp. 786-98.
- Norton, John D. (2008a) "Ignorance and Indifference," *Philosophy of Science*, **75**, pp. 45-68.

- Norton, John D. (2010) "Deductively Definable Logics of Induction." *Journal of Philosophical Logic*, 39 (2010), pp. 617-654.
- Norton, John D. (2010a) "There are no Universal Rules for Induction," *Philosophy of Science*, 77, pp. 765-777.
- Norton, John D. (2010b) "Cosmic Confusions: Not Supporting versus Supporting Not," *Philosophy of Science*, 77, pp. 501-523.
- Norton, John D. (2011) "Challenges to Bayesian Confirmation Theory," *Philosophy of Statistics*, Vol. 7: *Handbook of the Philosophy of Science*. Prasanta S. Bandyopadhyay and Malcolm R. Forster (eds.) Elsevier.
- Norton, John D. (2012) "Approximation and Idealization: Why the Difference Matters" *Philosophy of Science*, 79, pp. 207-232.
- Bronshstein, I. N. and Semendyayev, K. A. (1985) *Handbook of Mathematics*. New York: Van Nostrand Reinhold.
- Wallace, David (2016) "More Problems for Newtonian Cosmology," <http://philsci-archive.pitt.edu/12036/1/newton-cosmology-phil.pdf>