De Sitter spacetime in Static Coordinates

5-D
Minkowski spacetime
in ordinary
coordinates

$$ds^2 = -dX_1^2 - dX_2^2 - dX_3^2 - dX_4^2 + dX_5^2$$

Transform to
uniformly accelerated
coordinates

$$X_5 = x \sinh t$$
$$X_1 = x \cosh t$$

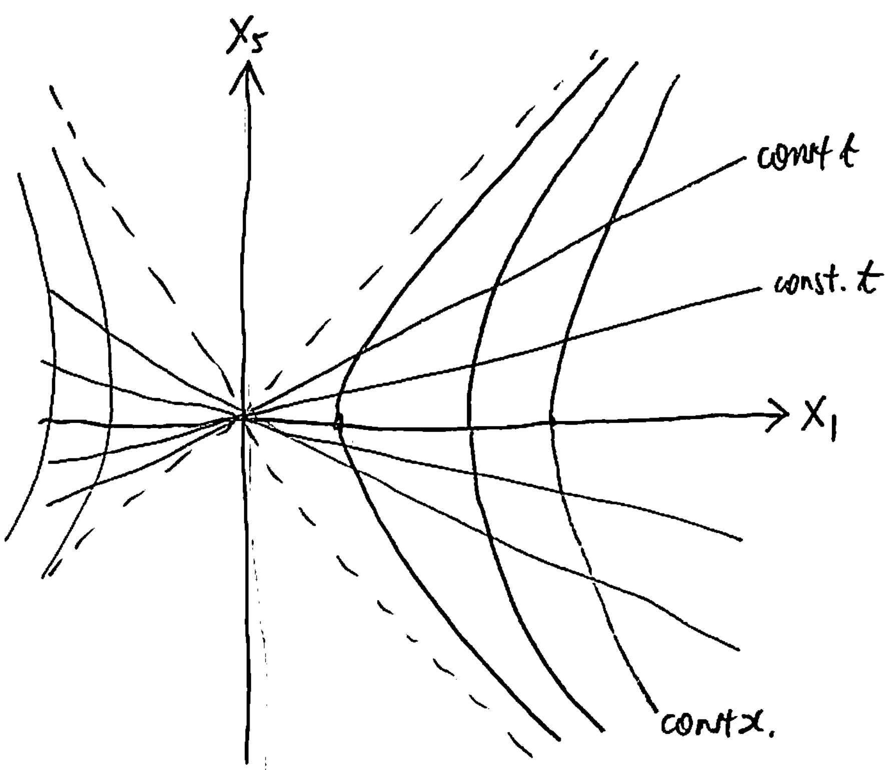
leave X_2, X_3, X_4

$$dX_5 = \sinh t \, dx + x \cosh t \, dt$$
$$dX_1 = \cosh t \, dx + x \sinh t \, dt$$

$$-dX_1^2 = -\cosh^2 t \, dx^2 - 2x \cancel{\cosh t \sinh t \, dx dt} - x^2 \sinh^2 t \, dt^2$$
$$+dX_5^2 = \sinh^2 t \, dx^2 + 2x \cancel{\cosh t \sinh t \, dx dt} + x^2 \cosh^2 t \, dt^2$$

$$-dX_1^2 + dX_5^2 =$$
$$-\underbrace{(\cosh^2 t - \sinh^2 t)}_{1} dx^2 + x^2 \underbrace{(\cosh^2 t - \sinh^2 t)}_{1} dt^2$$

$$ds^2 = -dx^2 - dX_2^2 - dX_3^2 - dX_4^2 + x^2 dt^2$$

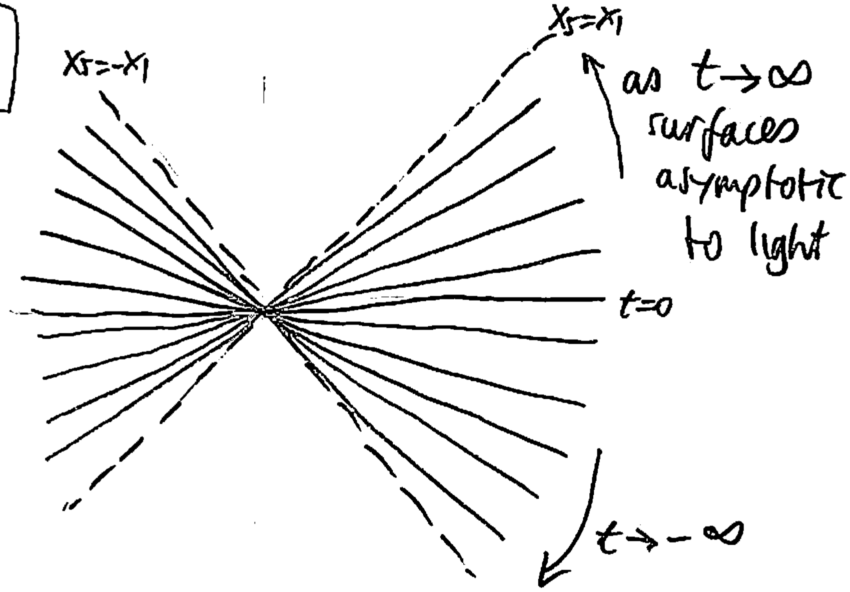


new
coordinates
cover only
half of spacetime
for $-\infty < t < \infty$
$-\infty < x < \infty$

$\boxed{\text{curves of constant } t}$

$X_5 = x \sinh t$
$X_1 = x \cosh t$

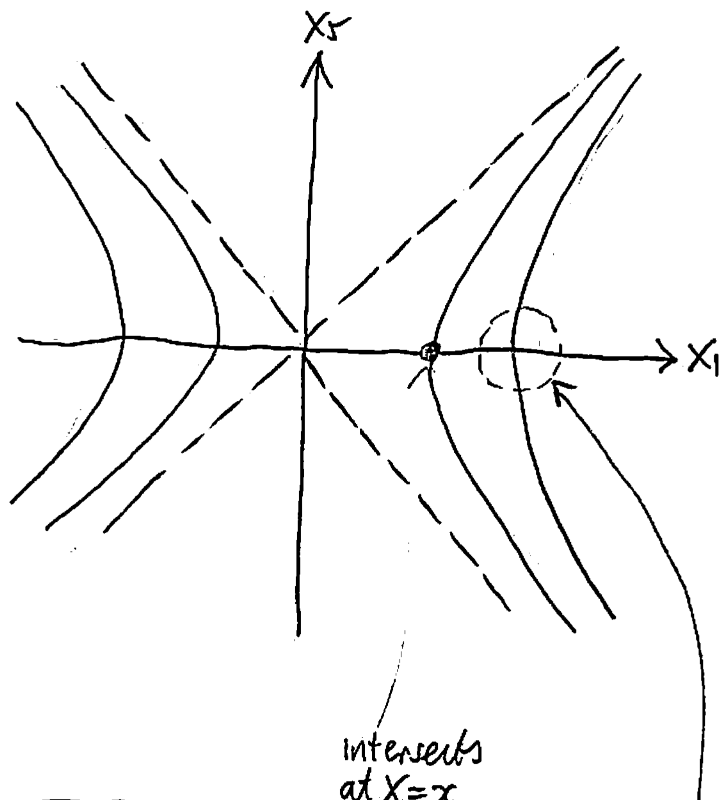
$$\boxed{X_5 = X_1 \underbrace{\tanh t}_{\substack{\text{as } t \text{ goes} \\ -\infty \text{ to } +\infty, \\ \tanh t \text{ goes} \\ -1 \text{ to } +1}}}$$



$\boxed{\text{curves of constant } x}$

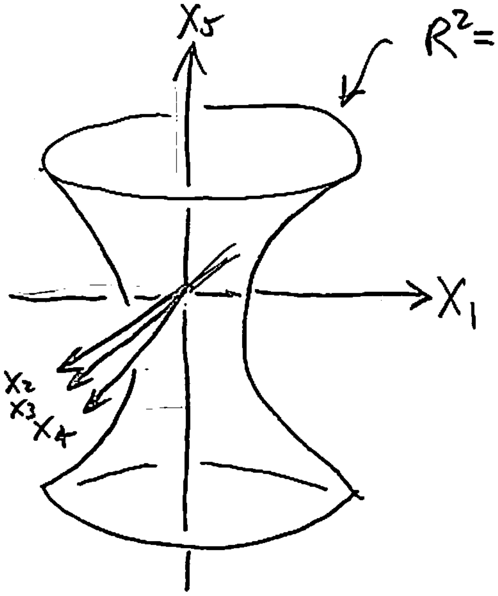
$X_1 = x \cosh t \to x$ as $t \to 0$

$X_1 = \dfrac{X_5}{\tanh t} \to X_5$ for large t
$\to -X_5$ for t very negative



uniform acceleration?
For $\underset{\text{small } t}{X_5 \ll x}$, $X_5/x = \sinh t \approx t$
then $X_1 = x \cosh t$
$\approx x(1+\frac{1}{2}t^2)$
$\approx x(1+\frac{1}{2}\left(\frac{X_5}{x}\right)^2)$
$= x + \frac{1}{x}\cdot\frac{1}{2}(X_5)^2$ ← i.e. parabola for small t, fixed x

NOW add the de Sitter hyperboloid !



$$R^2 = X_1^2 + X_2^2 + X_3^2 + X_4^2 - X_5^2$$

$$ds^2 = -dX_1^2 - dX_2^2 - dX_3^2 - dX_4 + dX_5^2$$

$X_5 = x \sinh t$
$X_1 = x \cosh t$

$$ds^2 = \underbrace{-dx^2 - dX_2^2 - dX_3^2 - dX_4^2}_{} + \underbrace{x^2}_{} dt^2$$

Restricted to hyperboloid. so This is the line element of a spherical space $d\omega(R)^2$

Restricted to hyperboloid x varies between $-R$ and $+R$ Hence introduce coordinate r where $x = R\cos r/R$ and $-R < r < R$

$$\boxed{ds^2 = -d\omega(R)^2 + R^2\cos^2(r/R)\,dt^2}$$

"static" since g_{ik} independent of t

static coordinates only cover a part of the hyperboloid