On The Harmonious Colouring of Trees

A. Aflaki[a], S. Akbari[a,b], D.S. Eskandani[a], M. Jamaali[a], H. Ravanbod[a],

[a]*Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran*

[b]*School of Mathematics, Institute for Studies in Theoretical Physics and Mathematics,*

P.O. Box 19395-5746, Tehran, Iran *

Abstract

Let G be a simple graph. A harmonious colouring of G is a proper vertex colouring such that each pair of colours appears together on at most one edge. The harmonious chromatic number $h(G)$ is the least number of colours in such a colouring. In this paper it is shown that if T is a tree of order n and $\Delta(T) \geq \frac{n}{2}$, then $h(T) = \Delta(T) + 1$, where $\Delta(T)$ denotes the maximum degree of T. Let T_1 and T_2 be two trees of order n_1 and n_2, respectively and $F = T_1 \cup T_2$. In this paper it is shown that if $\Delta(T_i) = \Delta_i$ and $\Delta_i \geq \frac{n_i}{2}$, for $i = 1, 2$, then $h(F) \leq \Delta(F) + 2$. Moreover, if $\Delta_1 = \Delta_2 = \Delta \geq \frac{n_i}{2}$, for $i = 1, 2$, then $h(F) = \Delta + 2$.

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1. Introduction

Let G be a simple graph. We denote the edge set and the vertex set of G by $E(G)$ and $V(G)$, respectively. A vertex of degree 1 in G is called a *pendant vertex*. A *star* is tree with a vertex adjacent to all other vertices and with no extra edge. In this article $d(u, v)$ denotes a distance between u and v. Also, $\Delta(G)$, $N_G(v)$ and $d_G(v)$ denote the maximum degree of G, the neighbor set of v and the degree of v in G, respectively. A *harmonious colouring* of G is a proper vertex colouring of G in which every pair of colours appears on at most one pair of adjacent vertices. The *harmonious chromatic number* of G, $h(G)$, is the

* *E-mail addresses*: arian.aflaki@gmail.com, s_akbari@sharif.edu, denizeskandani@gmail.com, mohsen_djamali@yahoo.com, hajar_ravanbod@yahoo.com.

minimum number of colours needed for any harmonious colouring of G. The first paper on harmonious colouring was written in 1982 by Frank et al. [4]. However, the proper definition of this notion is due to Hopcroft and Krishnamoorthy [5]. If G has m edges and can be harmoniously coloured with k colours, then clearly, $\binom{k}{2} \geq m$. Paths and cycles are among the first graphs whose harmonious chromatic numbers have been established [4]. It was shown by Hopcroft and Krishnamoorthy that the problem of determining the harmonious chromatic number of a graph is NP-hard. Also it was shown that the problem remains hard even when we restricted to trees, see [3]. The following result was proved in [2]:

Let d be a fixed positive integer. There is a positive integer N such that if T is any tree with $m \geq N$ edges and maximum degree at most d, then the harmonious chromatic number $h(T)$ is either k or $k+1$, where k is the least positive integer such that $\binom{k}{2} \geq m$. Harmonious colouring have been studied extensively by several authors. For more information interested reader is referred to [1].

In this paper we obtain the exact value of the harmonious chromatic number of a tree when its maximum degree is at least the half of its order.

Theorem 1. *Let T be a tree of order n. If $\Delta(T) \geq \frac{n}{2}$, then $h(T) = \Delta(T) + 1$.*

Proof. We prove the theorem by induction on n. For $n = 2$, the assertion is trivial. Let u be a pendant vertex of T and $uv \in E(T)$. If T is a star, then the assertion is clear. Thus we can assume that there exists a pendant vertex u such $T \setminus u$ is a tree and $\Delta(T \setminus u)=\Delta(T)$. We have $\Delta(T \setminus u) \geq \frac{n-1}{2}$, so by induction hypothesis, $h(T \setminus u) = \Delta(T) + 1$. Consider a harmonious colouring for $T \setminus u$ using $\Delta(T) + 1$ colours. Suppose that the colour of v in this colouring is i. We claim that if $x \in V(T \setminus u)$ and $d(x) = \Delta(T \setminus u) = \Delta(T)$, then the colour of x is not i. To see this we note that if the colour of x is i, then all $\Delta(T)$ pairs of colours containing i have appeared on the edges incident with x. Thus there exists a pair containing i which appears twice, a contradiction and the claim is proved. Now, the number of edges with one end point with colour i in $T \setminus u$ is at most

$$|E(T \setminus u)| - (\Delta(T) - 1) \leq (n-2) - \frac{n}{2} - 1 = \frac{n}{2} - 1.$$

Since $h(T \setminus u) \geq \frac{n}{2} + 1$, so there are at least $\frac{n}{2}$ pairs of colours of the set $\{1, \ldots, h(T \setminus u)\}$, containing i. Thus there exists a pair, say i and j, appears in no edge of $T \setminus u$. Now, colour the vertex u by j to obtain a harmonious colouring for T. Thus $h(T) \leq \Delta(T) + 1$. Since $h(T) \geq \Delta(T) + 1$, the proof is complete. □

Remark 1. We note that the lower bound for the maximum degree in the previous theorem is sharp. For instance consider the following tree:

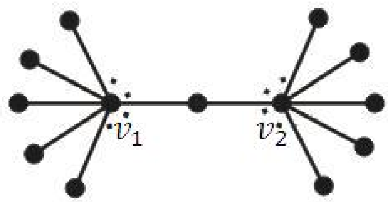

Figure 1. A tree with maximum degree $\frac{n-1}{2}$ and $h(T) > \Delta + 1$.

Theorem 2. *Let T_1 and T_2 be two trees of order n_1 and n_2, respectively and $F = T_1 \cup T_2$. Assume that $\Delta(T_i) = \Delta_i$. If $\Delta_i \geq \frac{n_i}{2}$, for $i = 1, 2$, then $h(F) \leq \Delta(F) + 2$. Moreover, if $\Delta_1 = \Delta_2 = \Delta \geq \frac{n_i}{2}$, for $i = 1, 2$, then $h(F) = \Delta + 2$.*

Proof. First we prove the second part of the theorem. Clearly, every graph with at least two non-adjacent vertices of maximum degree has harmonious chromatic number at least $\Delta + 2$. First suppose that F is the following graph in which $T_1 = T_2$ are following trees:

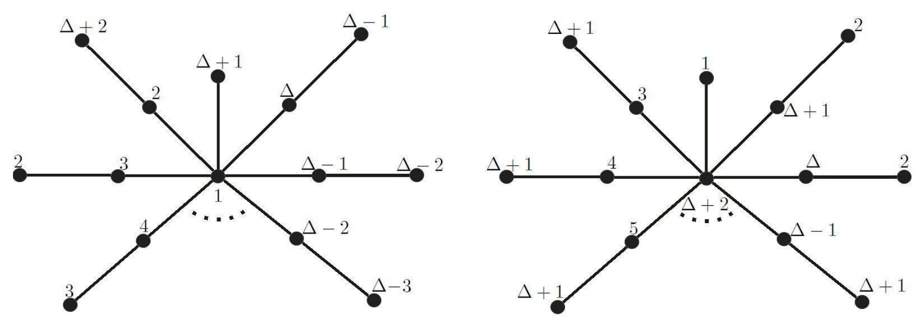

Figure 2

One can easily check that the above colouring is a harmonious colouring for F with the desired property then clearly we have the desired colouring.

Let $v_i \in V(T_i)$ and $d(v_i) = \Delta$, for $i = 1, 2$. By induction on $|V(F)|$, we prove that there exists a harmonious colouring c of F with $\Delta + 2$ colours in which $c(v_1)$ appears as colour of a pendant vertex adjacent to v_2.

Suppose that at least one of the trees T_1 and T_2 is star. If T_1 is a star, then there exists a harmonious colouring of T_1 with colours $\{1, \ldots, \Delta + 1\}$ such that $c(v_1) = 1$. By Theorem 1, there exists a harmonious colouring of T_2, say c', with colours $\{2, \ldots, \Delta + 2\}$. By permutation of colours one can assume that $c'(v_2) = \Delta + 2$. Since $d(v_2) = \Delta$, there

exists a pendant vertex z adjacent to v_2. Recolour z by colour 1 to obtain the desired colouring for F.
Now, assume that T_2 is a star. By Theorem 1, there exists a harmonious colouring of T_1 with colours $\{1, \ldots, \Delta+1\}$ such that the colour of v_1 is 1. Also, we colour T_2 with the colours $\{2, \ldots, \Delta+2\}$ such that the colour of v_2 is $\Delta+2$. Now, we change the colour of one of the pendant vertices adjacent to v_2 to 1 to obtain the desired colouring for F.Thus suppose that none of the T_1 and T_2 is star.

Now, assume that at least one of the trees T_1 and T_2 is not isomorphic to the tree shown in Figure 2. With no loss of generality assume that T_2 is that tree. Obviously, for $i = 1, 2$, there exist a pendant vertex, v'_i adjacent to v_i and another pendant vertex u''_i not adjacent to v_i. Also, T_2 has at least one pendant vertex $w \neq v'_2$ adjacent to v_2. Let $F' = F \setminus \{v'_1, v'_2, u''_1, u''_2\}$. By induction hypothesis there exists a harmonious colouring c' of F' by the colours $\{1, \ldots, \Delta+1\}$, such that $c'(w_1) = c'(v_1)$, where w_1 is a pendant vertex adjacent to v_2 in F'. Now, switch the colours of w_1 and w in the forest F'. Next, using the harmonious colouring of F', we like to find a desired harmonious colouring for F.

Let $u'_i \in V(T_i)$ and $u''_i u'_i \in E(T_i)$, for $i = 1, 2$. Four cases can be considered:

Case 1. $d(u''_1, v_1) = d(u''_2, v_2) = 2$. Consider three following subcases:

(i) $c'(u'_1) \neq c'(u'_2)$. Define the harmonious colouring c for F as follows:

$$c(u''_1) = c(u''_2) = c(v'_2) = c(v'_1) = a,$$

where a is a new colour and for any other vertex x, let $c(x) = c'(x)$.

(ii) $c'(u'_1) = c'(u'_2)$ and there exists i, $1 \leq i \leq 2$ such that $d_F(u'_i) \geq 3$. With no loss of generality assume that $d_F(u'_1) \geq 3$. Now, define $c(u''_1) = c(v'_1) = c'(u'_1)$ and $c(u'_1) = c(v'_2) = a$, where a is a new colour. Let $y \in N_F(u'_1) \setminus \{v_1, u''_1\}$. Now, define $c(u''_2) = c'(y)$.

(iii) $c'(u'_1) = c'(u'_2)$ and $d_F(u'_1) = d_F(u'_2) = 2$. Define the harmonious colouring c as follows:

$$c(v'_1) = c'(u'_1), c(u'_1) = c(v'_2) = c(u''_2) = a,$$

where a is a new colour and keep the colour of other vertices. In the colouring c, a appeared 3 times. Since $\Delta \geq 3$, there exists a colour t such that pair $\{a, t\}$ does not appear in the end points of the edges of F. Now, define $c(u''_1) = t$.

Case 2. $d(u_1'', v_1) = 2$ and $d(u_2'', v_2) \geq 3$. Let $u_2 \in V(T_2)$, $u_2'u_2 \in E(T_2)$ and $d(u_2', v_2) = d(u_2, v_2) + 1$. Consider four following subcases:

(i) $c'(u_2') = c'(v_2)$ and $c'(u_1') \neq c'(u_2)$. Define the harmonious colouring c as follows: $c(u_2'') = c'(v_2), c(v_2') = c'(u_2)$ and $c(u_1'') = c(v_1') = c(u_2') = a$, where a is a new colour and keep the colour of other vertices.

(ii) $c'(u_2') = c'(v_2)$ and $c'(u_1') = c'(u_2)$. Define the harmonious colouring c as follows: $c(v_2') = c(v_1') = c'(u_1'), c(u_2'') = c'(v_2)$ and $c(u_2') = c(u_1') = a$, where a is a new colour and keep the colour of other vertices. In this case clearly, $d(u_2'', v_2) \geq 4$. Let $y \in N_{T_2}(u_2)$ and $y \neq u_2'$. Then define $c(u_1'') = c'(y)$.

(iii) $c'(u_1') = c'(u_2')$. Define $c(u_1'') = c'(u_2)$, $c(u_2'') = c'(u_2')$ and $c(u_2') = c(v_1') = c(v_2') = a$, where a is a new colour and keep the colour of other vertices.

(iv) $c'(u_1')$, $c'(u_2')$ and $c'(v_2)$ are distinct. Define c as follows:

$$c(u_1'') = c(u_2'') = c(v_1') = c(v_2') = a,$$

where a is a new colour and keep the colour of other vertices.

Case 3. $d(u_1'', v_1) \geq 3$ and $d(u_2'', v_2) = 2$. Let $u_1 \in V(T_1)$, $u_1'u_1 \in E(T_1)$ and $d(u_1', v_1) = d(u_1, v_1) + 1$. Now, consider three following subcases:

(i) $c'(u_1') = c'(v_2)$. Define the harmonious colouring c as follows:

$$c(u_1'') = c'(v_2), c(v_2') = c'(u_1), c(u_2'') = c(u_1') = c(v_1') = a,$$

where a is a new colour and keep the colour of other vertices.

(ii) $c'(u_1') = c'(u_2')$. Define the harmonious colouring c as follows: $c(u_1'') = c'(u_1')$, $c(u_2'') = c'(u_1)$, $c(u_1') = c(v_2') = c(v_1') = a$, where a is a new colour and keep the colour of other vertices.

(iii) $c'(v_2), c'(u_1')$ and $c'(u_2')$ are distinct. Define the harmonious colouring c as follows:

$$c(u_1'') = c(u_2'') = c(v_1') = c(v_2') = a,$$

where a is a new colour and keep the colour of other vertices.

Case 4. $d(u_i'', v_i) \geq 3$, for $i = 1, 2$.
Let $u_i \in V(T_i)$, $u_i'u_i \in E(T_i)$ and $d(u_i', v_i) = d(u_i, v_i) + 1$, for $i = 1, 2$. Now, consider five following subcases:

(*i*) $c'(u_1') = c'(u_2')$. In this case define the following colouring:

$$c(u_1'') = c'(u_2), c(u_1') = c(u_2'') = c(v_1') = c(v_2') = a,$$

where a is a new colour and for any other vertex x, let $c(x) = c'(x)$.

(*ii*) $c'(u_1'), c'(u_2')$ and $c'(v_2)$ are distinct. Now, define

$$c(u_1'') = c(u_2'') = c(v_1') = c(v_2') = a,$$

where a is a new colour and keep the colour of other vertices.

(*iii*) $c'(v_2) = c'(u_1')$ and $c'(u_2') \neq c'(u_1)$. Clearly, $c'(u_2') \neq c'(v_2)$. Define $c(u_1'') = c'(v_2)$, $c(v_2') = c'(u_1)$ and $c(u_1') = c(u_2'') = c(v_1') = a$, where a is a new colour and keep the colour of other vertices.

(*iv*) $c'(v_2) = c'(u_1')$ and $c'(u_2') = c'(u_1)$. Define $c(u_1'') = c'(u_2), c(u_2'') = c'(v_2)$ and $c(u_1') = c(v_1') = c(v_2') = a$, where a is new colour and keep the colour of other vertices.

(*v*) $c'(v_2) = c'(u_2')$, then similar to the subcases (*iii*) and (*iv*) we can obtain a harmonious colouring of F.

For the first part of theorem, suppose that $\Delta(T_1) \geq \Delta(T_2)$, $d(v_1) = \Delta(T_1)$ and $d(v_2) = \Delta(T_2)$. Add $d(v_1) - d(v_2)$ pendant vertices to v_1. Now, by the previous part, the assertion is obvious and the proof is complete. □

Remark 2. The tree given in Remark 1 shows that the lower bound of Theorem 2 is sharp. To show this, note that one can not colour one of the trees, say T_1, by $\Delta + 1$ colours harmoniously. Hence every colour $\{1, \ldots, \Delta + 2\}$ should be used at least once in colouring of T_1. To colour the other tree, we begin from the vertices with the maximum degree. One colour can be used in these vertices if and only if in the colouring of T_1, it appears in at most one pair; Since we need Δ colours to colour the neighbors of the vertex with maximum degree and if we use a colour which appeared in two pairs we only have $\Delta - 1$ colours which are not appeared with this colour. Note that we have at most one colour in the colouring of T_1 which appeared in one pair while we have two vertices with the degree of Δ. So we can not colour the second tree with these $\Delta + 2$ colours.

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