The étale fundamental group

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The goal of these notes is to motivate and formulate the notion of the étale fundamental group, which is the algebraic analogue of the usual fundamental group of a topologiacal space. We will see some examples of the étale fundamental group and develop the necessary background to state the Grothendieck-Riemann existence theorem [1], which relates the étale fundamental group to the usual fundamental group for varieties over the complex numbers. We have mainly referred to [3] for these notes. We have also used [2] and [4] for some concepts. Special thanks to Eugene Eyeson for useful conversations and his personal notes on the topic.

Contents

1 Motivation **2**

2 Fundamental groups and covering spaces **2**

2.1 The profinite fundamental group 3

3 Étale morphisms and étale covers **3**

4 The étale fundamental group **4**

4.1 Varieties over the complex numbers 5

1 Motivation

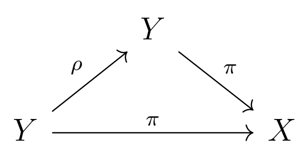
Let k be a field. In algebraic geometry, the space k^n is given the Zariski topology, where the algebraic sets are declared to be the closed sets. It makes sense, then, to talk about the fundamental group, among other algebraic invariants, of an algebraic variety with respect to this topology. But the following theorem says that this does not produce interesting invariants in general.

Theorem 1.1. *Let X be an uncountable set with the cofinite topology. Then X is contractible.*

If X is an algebraic curve over an uncountable field like $\mathbb{C}$, then the Zariski topology on X matches with the cofinite topology (a polynomial in one variable has only finitely many roots). The above theorem then implies that the fundamental group of X at any base point is 0. Therefore, the fundamental group cannot distinguish between algebraic curves over $\mathbb{C}$. The problem with the Zariski topology is that it is too coarse to have good algebraic invariants. We want to think about a purely algebraic analogue of the fundamental group. In order to do this, we first reinterpret the usual notion of the fundamental group in terms of covering spaces.

2 Fundamental groups and covering spaces

Let (X, x) be a path-connected, locally path-connected and locally simply-connected pointed topological space. Under these assumptions, the universal cover $\tilde{X}$ exists. Let $\mathcal{C}_X$ denote the category of covering spaces $\pi : (Y, y) \to (X, x)$, where Y has finitely many connected components and let $\mathcal{C}_X^{fin}$ denote the category of covering spaces whose fibers are finite.
The universal cover of (X, x) need not exist in $\mathcal{C}_X^{fin}$. For example, look at $X = \mathbb{C} \setminus \{0\}$ with any base point. For each $n \geq 1$, we have the n^{th}-power map $X \xrightarrow{n} X$, which is an n-cover of X. Hence, the universal cover of X does not exist in $\mathcal{C}_X^{fin}$ in this case.
Given a cover $\pi : (Y, y) \to (X, x)$, a deck transformation is a homeomorphism $\rho : Y \to Y$ such that the following diagram commutes



Also, we have the fiber functor

$$F_x : \mathcal{C}_X \to \textbf{Sets}$$

which maps a cover $\pi : (Y, y) \to (X, x)$ to $\pi^{-1}(x)$.

We have the following theorem (see [3])

Theorem 2.1. *The following groups are isomorphic*

- *The group of homotopy classes of loops in X based at x.*
- *The group of deck transformations of the universal cover $\tilde{X}$.*
- *The automorphism group of the fiber functor F_x.*

To recall, the automorphism group of a functor F is the set of all natural isomorphisms $F \to F$, with the group operation of composition. The (topological) fundamental group $\pi_1(X, x)$ is defined to be any one (hence all) of the above three groups.

2.1 The profinite fundamental group

We can restrict our attention to $\mathcal{C}_X^{fin}$ and consider the restriction of the fiber functor F to $\mathcal{C}_X^{fin}$ to get the functor $\tilde{F} : \mathcal{C}_X^{fin} \to \mathbf{Sets}$. Again, the universal cover need not exist in $\mathcal{C}_X^{fin}$ and so if we take the automorphism group of $\tilde{F}$, we get what is called the profinite fundamental group $\tilde{\pi}_1(X, x)$.

Remark 2.1. *The profinite fundamental group gets its name for a good reason: if G is a subgroup of $\pi_1(X, x)$ of finite index, then the Galois correspondence associates to G a finite pointed covering space $Y \to X$. If G is a normal subgroup, then the group of pointed automorphisms of Y over X is just $\pi_1(X, x)/G$, and there is a canonical map $\pi_1(X, x) \to \pi_1(X, x)/G$. Taking the inverse limit over all normal subgroups G of finite index defines the profinite completion of $\pi_1(X, x)$. The resulting group is isomorphic to the profinite fundamental group $\tilde{\pi}_1(X, x)$.*

In order to define the fundamental group in the algebraic setting, we want to mimic one of the three definitions in theorem 2.1. As we know, loops do not make much sense in the Zariski topology and so we would like to have an algebraic analogue of covers. To this end, we define étale morphisms and étale covers, and define the étale fundamental group in terms of these objects. The étale fundamental group will then classify all the étale covers similar to how the fundamental group classifies all the covers.

3 Étale morphisms and étale covers

We define étale maps between schemes, but one could assume them to be just varieties for simplicity. First, we define a morphism of schemes $f : X \to Y$ to be flat if the induced map on each stalk is flat (as a ring map). That is,

$$f_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is flat. An important property of a flat morphism is that it is always open.
The next ingredient to define an étale map is the notion of an unramified morphism. Recall that a map of local rings $\phi : R \to S$ is called unramified if

$$\phi(\mathfrak{m}_R)S = \mathfrak{m}_S,$$

and $k(S)$ is a separable extension of $k(R)$. We upgrade this definition to schemes as

Definition 3.1. *A morphism* $\phi : X \to Y$ *of schemes is called unramified if it is locally of finite type and for all* $x \in X$ *the induced map on local rings*

$$\phi_X^{\#} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is unramified.

We say that $f : X \to Y$ is locally of finite type if for all $x \in X$ there exists an affine open neighbourhood $Spec(A) = U \subset X$ of x and an affine open $Spec(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite type.

There are other equivalent ways to define an unramified morphism, one of which is

Definition 3.2. *A morphism* $f : X \to Y$ *is unramified at* $x \in X$ *if and only if the module of Kälher differentials* $\Omega_{X/Y,x} = 0$*.*

Example 3.1. *Let* L/K *be an extension of number fields with ring of integers* $\mathcal{O}_L$ *and* $\mathcal{O}_K$*. We have the inclusion map* $i : \mathcal{O}_K \to \mathcal{O}_L$*. Let* $\mathfrak{p}$ *be a prime ideal of* $\mathcal{O}_K$ *and* $\mathfrak{q}$ *be a prime ideal of* $\mathcal{O}_L$*. We have the residue field extension* $\mathcal{O}_K/\mathfrak{p} \to \mathcal{O}_L/\mathfrak{q}$*, which is finite and separable. Therefore, the inclusion map* i *is unramified if and only if*

$$\mathfrak{p}\mathcal{O}_{L,\mathfrak{q}} = \mathfrak{q}$$

This is equivalent to the condition that the ramification index of $\mathfrak{p}$ *in* L *is* 1*. Therefore, the definition of an unramified morphism between schemes generalizes the notion coming from algebraic number theory.*

We are now ready to define an étale morphism of schemes.

Definition 3.3. *A morphism of schemes* $f : X \to Y$ *is said to be étale if it is flat and unramified.*

A finite étale cover is an étale morphism that is finite. The category of finite étale covers of X is denoted by FEt_X. Let X be a scheme with a choice of a geometric point $x \in X$, i.e., a choice of a morphism $x : Spec(\bar{k}) \to X$. An important fact about an étale cover Y of X is that the pullback x^*Y is finite and the cardinality does not depend on the choice of x. Therefore, the notion of a finite étale cover is a good generalization of the notion of a finite covering.
In particular, if $F_x : FEt_X \to \mathbf{Sets}$ is the fiber functor sending $(p : Y \to X)$ to $p^{-1}(x)$, then as a set, $p^{-1}(x) = \coprod_{i=1}^{n} Spec(\bar{k})$ for some $n \in \mathbb{N}$.

4 The étale fundamental group

Let X be a connected, locally noetherian scheme with a choice of a geometric point $x \in X$. Consider the fiber functor $F_x : FEt_X \to \mathbf{Sets}$.

Definition 4.1. *The étale fundamental group* $\pi_1^{et}(X, x)$ *of* X *is the automorphism group of the fiber functor* F_x*.*

For a connected scheme X with a geometric point x, as (Y, y) varies over pointed connected finite Galois étale covers of (X, x), the Galois groups $Gal(Y/X)$ form an inverse system whose inverse limit is $\pi_1^{et}(X, x)$, and so we get that $\pi_1^{et}(X, x)$ is a profinite group.

Example 4.1. *Let $X = Spec(k)$, and let $x \in X$ be a geometric point (represents an algebraic closure of k). Every finite étale cover X is a disjoint union of the spectra of finite separable extensions of k. Every separable Galois extension corresponds to a normal subgroup of $Gal(k^{sep}/k)$ and hence $\pi_1^{et}(X, x)$ will be the inverse limit of these Galois groups and hence $\pi_1^{et}(X, x) \simeq Gal(k^{sep}/k)$.*
For example, the étale fundamental group of $\mathbb{Q}$ is the absolute Galois group of $\mathbb{Q}$.

4.1 Varieties over the complex numbers

Let X be a $\mathbb{C}$-variety. The $\mathbb{C}$ points $X(\mathbb{C})$ is a topological space. Let $\mathcal{C}^{fin}_{X(\mathbb{C})}$ denote the category of finite covers of $X(\mathbb{C})$. The Grothendieck-Riemann existence theorem says the following (see [3])

Theorem 4.2. *For each $\mathbb{C}$-variety X, the natural functor (the $\mathbb{C}$-points functor)*

$$FEt_X \to \mathcal{C}^{fin}_{X(\mathbb{C})}$$

is an equivalence of categories.

From the earlier discussion of the profinite fundamental group, we get the following corollary,

Corollary 4.1. *For a variety X over $\mathbb{C}$, we have the following isomorphism of profinite groups*

$$\pi_1^{et}(X, x) \simeq \tilde{\pi}_1(X(\mathbb{C}), x)$$

Let us see some consequences of this corollary as examples

Example 4.2. *Let $X = A^1$ over $\mathbb{C}$. We have*

$$\pi_1^{et}(X, x) \simeq \tilde{\pi}_1(X(\mathbb{C}), x) = \tilde{\pi}_1(\mathbb{C}, x) = 0$$

since $\mathbb{C}$ has no non-trivial finite covers.

Example 4.3. *Let $X = \mathbb{P}^1$ over $\mathbb{C}$ with a choice of a basepoint $x \in \mathbb{P}^1$. We have $\mathbb{P}^1(\mathbb{C}) \simeq S^2$ and therefore*

$$\pi_1^{et}(X, x) \simeq \tilde{\pi}_1(X(\mathbb{C}), x) = \tilde{\pi}_1(S^2, x) = 0$$

Remark 4.1. *We end with the remark that the étale fundamental group of $\mathbb{P}^1$ over any algebraically closed field is still 0. This can be proved by using the Riemann-Hurwitz formula, which relates the Euler characteristic of a cover of X with the Euler charactersitic of X. In our case, suppose $f : C \to \mathbb{P}^1$ is a connected finite étale cover of $\mathbb{P}^1$ of degree n, where C is a genus g curve. Since f is unramified, the ramification divisor is 0. The Riemann-Hurwitz formula then gives*

$$2g - 2 = n(2g(\mathbb{P}^1) - 2) = -2n$$

This happens if only if $g = 0$ and $n = 1$, which means $C = \mathbb{P}^1$. Therefore, we get

$$\pi_1^{et}(\mathbb{P}^1, x) = 0$$

References

[1] Alexander Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*, volume 224 of *Lecture notes in mathematics*. Springer-Verlag, 1971.

[2] James S Milne. Lectures on étale cohomology. *Available on-line at http://www. jmilne. org/math/CourseNotes/LEC. pdf*, 1998.

[3] Bjorn Poonen. *Rational points on varieties*, volume 186. American Mathematical Society, 2023.

[4] Tamás Szamuely. *Galois groups and fundamental groups*, volume 117. Cambridge university press, 2009.