A HORIZONTAL EQUIVARIANT MAIN CONJECTURE AND APPLICATIONS

Abhiram Kumar

Abstract

Let p be an odd prime, F a totally real number field, and H/F a finite CM abelian extension of order prime to p. We construct a horizontal Iwasawa module ∇_∞ and a horizontal p-adic L-function Θ_∞ along a tower of ray class fields of growing squarefree conductor, and we prove an equivariant main conjecture $\mathrm{Fitt}_\Lambda(\nabla_\infty) = (\Theta_\infty)$ over the horizontal Iwasawa algebra Λ, which is non-Noetherian. Combining this with a density theorem of Kriz–Nordentoft, we show that for a positive proportion of cyclic degree-p extensions K/F in the tower, the χ_0-component of the class group of $H \cdot K$ has the smallest size achieved by any such K/F. In particular, this size does not depend on the number of ramified primes in K.

Contents

1 Introduction 1

2 Preliminaries 4

3 Ritter–Weiss modules and the result of Dasgupta–Kakde 6

4 Construction of the horizontal Iwasawa module and the horizontal L-function 8

5 A size formula and the arithmetic statistical application 13

1 Introduction

1.1 Motivation

Let F a totally real field, and H/F be a finite CM abelian extension with $G = \mathrm{Gal}(H/F)$. The Brumer–Stark conjecture, proved away from $p = 2$ by Dasgupta and Kakde [DK23] and for all primes p in [DKSW23], is deduced from a statement about the Ritter–Weiss module $\nabla_S^T(H)$, which packages the (S,T)-class group of H together with a divisor group supported on S, where S and T are finite sets of places of F. The statement they prove is that this module admits a quadratic presentation, and its Fitting ideal is generated by a Stickelberger element $\Theta_S^T(H)$ (see §3.2 for a definition):

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}\left(\nabla_S^T(H)_p^-\right) = \left(\Theta_S^T(H)\right). \tag{1}$$

An arithmetic invariant of H is thus computed by an element $\Theta_S^T(H)$ interpolating L-values. One can ask whether such an identity can itself be interpolated in an Iwasawa-theoretic tower. Along the cyclotomic $\mathbb{Z}_p$-extension of H/F, this has been carried out by Gambheera and Popescu [GP25], where they prove an equivariant main conjecture over the classical Iwasawa algebra $\mathbb{Z}_p[\![\mathbb{Z}_p]\!] \simeq \mathbb{Z}_p[\![T]\!]$.

In this paper, we fix an odd prime p and carry out the interpolation in a different direction. Instead of a tower consisting of a single $\mathbb{Z}_p$-direction, we add more and more tame primes to the conductor. The resulting tower is indexed by the number of primes that are allowed to ramify. The relevant Iwasawa algebra is the completed group ring $\Lambda = \mathbb{Z}_p[\![\prod \Delta_i]\!]$ of an infinite product of non-trivial finite abelian p-groups Δ_i. Unlike classical Iwasawa algebra, this ring is neither Noetherian nor a domain, and its modules have no special structure theory.

Our analytic input comes from [KN24], where Kriz and Nordentoft develop a theory of horizontal p-adic L-functions, which can be applied to elements of such a horizontal Iwasawa algebra Λ. They show that for an elliptic curve E, there is a positive proportion of characters χ for which the twisted L-value $L(E,\chi,1)$ does not vanish. This relies on their key density theorem, which states that given any non-zero element $\nu \in \Lambda$, there is a finite set of characters $\{\chi_1,\dots,\chi_n\}$ such that every character χ has a translate $\chi\chi_i$ at which ν attains its minimal p-adic valuation (see Theorem 2.4.1).

The goal of the present work is to horizontally interpolate all of the objects appearing in the equality (1), so that the theory of Kriz and Nordentoft can be applied to (S,T)-class groups. Working with Ritter–Weiss modules, we construct a horizontal Iwasawa module ∇_∞ and a horizontal p-adic L-function $\Theta_\infty \in \Lambda$, assembled from the modules ∇_S^T and the Stickelberger elements Θ_S^T along the horizontal tower. Our main theorem identifies the Fitting ideal of ∇_∞ with the ideal generated by Θ_∞, thereby establishing an equivariant main conjecture. This is the content of Theorem 1.

Our result has applications to the asymptotics of the p-part of the class group of degree-p extensions, which is a well-studied problem in arithmetic statistics. The p-torsion conjecture says that for any $\epsilon > 0$, one has $\#\operatorname{Cl}(E)[p] = O_{\epsilon,n,p,F}(D^\epsilon)$ for every extension E/F of degree n and absolute discriminant D. In [KW20], Klüners and Wang proved this conjecture when E/F varies over p-extensions. To be more precise, let t be the number of ramified primes in the degree-p extension E/F, and let $t' = \max\{1,t\}$. They prove the rank bound

$$\operatorname{rk}_p\big(\operatorname{Cl}(E)\big) \;\le\; p\big(t' - 1 + \operatorname{rk}_p(\operatorname{Cl}(F))\big) \tag{2}$$

which grows linearly in the number of ramified primes. Beyond upper bounds, the distribution of p^∞-class groups and Selmer groups in degree p-twist families has been determined in the work of Smith [Smi23], which verifies the predictions of Cohen–Lenstra–Gerth for this family.

Our arithmetic-statistical applications of Theorem 1 go in a similar direction. A distinctive feature of our approach is that it applies the p-adic methods of Iwasawa theory to study a classical question in arithmetic statistics about p-parts of class groups. Given an odd character χ_0 of G, we study the size $\#\operatorname{Cl}(H\cdot K)_{\chi_0}$ of the χ_0-component of the class group as K ranges over cyclic degree-p extensions of F in a family $\mathcal{K}$ ordered by conductor. We use Kriz–Nordentoft's density theorem to produce a constant e which only depends on p, H, and χ_0 (and not on K), such that for a positive proportion of the fields in the family $\mathcal{K}$ (made precise in Theorem 2a),

$$v_p(\#\operatorname{Cl}(H\cdot K)_{\chi_0}) = e.$$

In fact, e is the smallest possible size achieved by any K/F in $\mathcal{K}$. Unlike the bound in (2), our bound does not depend on the number of ramified primes in K. Moreover, we analyze the entire p-part of the class group as opposed to just the p-rank. In contrast to [KW20], rather than controlling the entire class group for all cyclic p-extensions, we obtain finer information by specializing to a character component and to a positive proportion of fields in $\mathcal{K}$.

We expect that our methods can also be used to treat some families of degree p^n-cyclic extensions of F, and even non-cyclic abelian extensions of degree p^n. We hope to address these cases in future work.

1.2 Main results

We fix some more notation and state our main results. We assume that $G = \operatorname{Gal}(H/F)$ has order prime to p. Let $c \in G$ be the complex conjugation and let $\chi_0 \in \operatorname{Hom}(G, \overline{\mathbb{F}}_p^\times)$ be an odd character. There is a unique lift of χ_0 which takes values in $\bar{\mathbb{Q}}_p^\times$, which we also denote by χ_0. We set $\mathcal{O} := \mathbb{Z}_p[\chi_0]$, a finite unramified extension of $\mathbb{Z}_p$ of residue degree $f := [\mathcal{O}/\mathfrak{m}_\mathcal{O} : \mathbb{F}_p]$, where $\mathfrak{m}_\mathcal{O}$ is the maximal ideal of $\mathcal{O}$. For a finite extension H'/H and a $\mathbb{Z}[\operatorname{Gal}(H'/F)]$-module M, we write M_{χ_0} for the χ_0-component (see §4.2 for the definition). Since χ_0 is odd, M_{χ_0} automatically lies in the (-1)-eigenspace M^- of the action of c on M. Thus we need not carry the minus superscript of (1) once we specialize to the χ_0-component. Finally, let S and T be auxiliary finite disjoint sets of places of F satisfying the standard conditions (A1)–(A4) of §3.1. Additionally, assume that S and T satisfy the smoothing and depletion conditions of §5. The primes we allow in the conductor of K/F in the horizontal Iwasawa tower are the following.

Definition 1.2.1. A prime $\mathfrak{l}$ of F is *orderly* for the data (p, H, χ_0, S, T) if

(O1) $\mathfrak{l} \nmid p$, $\mathfrak{l} \nmid \operatorname{cond}(H/F)$, and $\mathfrak{l} \notin S \cup T$,

(O2) $\mathrm{N}\mathfrak{l} \equiv 1 \pmod{p}$,

(O3) $\chi_0(\mathrm{Frob}_{\mathfrak{l}}) \neq 1$.

The terminology *orderly* was introduced in [KN24]. Conditions (O2) and (O3) are both conditions on the Frobenius class of $\mathfrak{l}$ in the single finite abelian extension $F(\mu_p) \cdot H_{\chi_0}$, where H_{χ_0} is the fixed field of $\ker \chi_0$. The definition of orderly primes are reminiscent of Taylor–Wiles primes that appear in the Taylor–Wiles method [TW95]. By the Chebotarev density theorem, the set of all orderly primes has a positive density δ computed in §4. Fix once and for all an infinite set $\mathcal{L} = \{\mathfrak{l}_1, \mathfrak{l}_2, \dots\}$ of orderly primes of density δ. For $n \geq 1$, let F_n be the maximal p-power-degree sub-extension of the ray class field of F of conductor $\mathfrak{l}_1 \cdots \mathfrak{l}_n$, and put

$$H_n := H \cdot F_n, \qquad \Delta_n := \mathrm{Gal}(H_n/H), \qquad H_\infty := \bigcup_n H_n, \qquad \Delta_\infty := \varprojlim_n \Delta_n.$$

In §4, we construct from this data the horizontal Iwasawa algebra $\Lambda \cong \mathcal{O}[\![\Delta_\infty]\!]$ and the horizontal Ritter–Weiss module ∇_∞. We also construct the horizontal p-adic L-function $\Theta_\infty \in \Lambda$ as a projective limit of χ_0-components of Stickelberger elements after an Euler-factor renormalization. Our main result can be stated as follows.

Theorem 1. *The Λ-module ∇_∞ is finitely generated, and we have an equality*

$$\mathrm{Fitt}_\Lambda(\nabla_\infty) = (\Theta_\infty)$$

of ideals of Λ. In particular, $\mathrm{Fitt}_\Lambda(\nabla_\infty)$ is principal and closed.

The proof of Theorem 1 starts with the equality in (1) and checks that the Stickelberger elements are norm-compatible after renormalization. The issue is that the Fitting ideal and projective limit need not commute in general since Λ is not Noetherian. However, we can deduce that they commute after taking closure. We can then descend via a topological argument, using the principality of (Θ_∞) and the fact that Λ is a local profinite ring.

We now turn to the arithmetic statistical applications. Consider the family

$$\mathcal{K}_p(\mathcal{L}) := \big\{K/F \ : \ K/F \text{ cyclic of degree } p,\ \mathfrak{f}(K) \text{ a non-trivial squarefree product of primes of } \mathcal{L}\big\},$$

where $\mathfrak{f}(K)$ denotes the conductor of K, and order it by the absolute norm $\mathrm{N}\mathfrak{f}(K)$. This ordering need not be unique, but we fix one and work with it. For a real number X, we write $\mathcal{K}_p(X, \mathcal{L}) = \{K \in \mathcal{K}_p(\mathcal{L}) : \mathrm{N}\mathfrak{f}(K) \leq X\}$, which a finite set. For $K \in \mathcal{K}_p(\mathcal{L})$, we put $H_K := H \cdot K$, which is a CM field of degree $p\,[H : F]$ over F.

Just to illustrate our result in the simplest case, we let $F = \mathbb{Q}$ and H be an imaginary quadratic field with $H \not\subseteq \mathbb{Q}(\mu_p)$, and let χ_0 be its quadratic character so that $\mathcal{O} = \mathbb{Z}_p$ and $f = 1$. Then $\mathcal{L}$ consists of the rational primes $\ell \equiv 1 \pmod{p}$ that are inert in H which has density $\delta = \frac{1}{2(p-1)}$. For $K \in \mathcal{K}_p(\mathcal{L})$, the group $\mathrm{Cl}(H_K)_{\chi_0}$ is the minus part of the p-part of $\mathrm{Cl}(H_K)$. Thus $\#\,\mathrm{Cl}(H_K)_{\chi_0}$ is the p-part of the relative class number $\#\,\mathrm{Cl}(H_K)/\#\,\mathrm{Cl}(K)$.

Theorem 2a. *Let $p, H, \chi_0, \mathcal{L}$ be as above, let $m := \min\{v_p(\eta(\Theta_\infty)) : \eta \in \mathrm{Hom}(\Delta_\infty, \mu_p)\}$, and set*

$$e := v_p\big(\#\,\mathrm{Cl}(H)_{\chi_0}\big) + (p-1)\,m.$$

Then e is a non-negative integer, and a positive proportion of the fields of $\mathcal{K}_p(\mathcal{L})$, ordered by the absolute norm of the conductor, satisfy $v_p(\#\,\mathrm{Cl}(H_K)_{\chi_0}) = e$; that is,

$$\liminf_{X\to\infty} \frac{\#\big\{K \in \mathcal{K}_p(X, \mathcal{L}) \ : \ v_p\big(\#\,\mathrm{Cl}(H_K)_{\chi_0}\big) = e\big\}}{\#\mathcal{K}_p(X, \mathcal{L})} \ > \ 0.$$

We note that the value m is an element of $\frac{1}{p-1}\mathbb{Z}$. The idea of the proof of Theorem 2a is as follows: Theorem 1 lets us read off the size of each χ_0-component from the values of Θ_∞: if η is the character of Δ_∞ cutting out K, an explicit size formula expresses $v_p(\#\,\mathrm{Cl}(H_K)_{\chi_0})$ in terms of the valuations $v_p(\eta^j(\Theta_\infty))$ for $j = 0, \dots, p-1$. By the Kriz–Nordentoft density theorem, a positive proportion of the characters η attain the minimal valuation m simultaneously on all the non-trivial conjugates. An analytic count of the fields of $\mathcal{K}_p(X, \mathcal{L})$ ordered by conductor then converts this into a positive proportion of fields.

We now turn to the case where F is an arbitrary totally real field with the notation as in the first paragraph of §1.2.

Theorem 2b. *Let F be any totally real field. Then the conclusion of Theorem 2a holds with $e := v_p(\#\,\mathrm{Cl}(H)_{\chi_0}) + f(p-1)m$.*

The proof of this theorem uses the fact that S and T satisfy the conditions (C1) and (C2) in §4, which automatically hold in the setting of Theorem 2a.

Outline of the paper

Section 2 collects the algebraic and analytic preliminaries: Fitting ideals and their behavior under base change, quadratic presentations, and the density theorem of Kriz and Nordentoft. In Section 3 we recall the Ritter–Weiss module and the theorem of Dasgupta–Kakde that computes its Fitting ideal. Section 4 constructs the horizontal tower together with the algebra Λ, the module ∇_∞, and the L-function Θ_∞, and establishes the norm relations for the Stickelberger elements and the compatibilities among the Ritter–Weiss modules that make these limits well defined. This section also computes the Chebotarev density δ of the set of orderly primes. The section concludes with a proof of Theorem 1. Section 5 deduces the arithmetic-statistical applications (Theorems 2a and 2b) through an explicit size formula and a Selberg–Delange count.

2 Preliminaries

In this section we collect the algebraic and analytic background that will be used in later sections. The algebraic concepts include Fitting ideals, quadratic presentation of modules, behavior of projective limits and topological Nakayama lemma. We also discuss Kriz–Nordentoft's theory of horizontal Iwasawa algebras and state their density theorem on character zeros.

2.1 Fitting ideals

Let R be a commutative ring and let M be a finitely generated R-module. Choose a presentation

$$R^m \xrightarrow{A} R^n \longrightarrow M \longrightarrow 0,$$

where n is a positive integer.

Definition 2.1.1. For an integer $i \geq 0$, the i-th Fitting ideal of M, denoted $\mathrm{Fitt}^i_R(M)$, is the ideal of R generated by the determinants of all the (possibly infinitely many) $(n-i)\times(n-i)$ minors of the matrix A. The 0-th Fitting ideal of M is defined as

$$\mathrm{Fitt}_R(M) := \mathrm{Fitt}^0_R(M).$$

We recall some basic properties of the Fitting ideal in the following proposition.

Proposition 2.1.2. *Let R be a commutative ring and let M be a finitely generated R-module.*

1. *The ideals $Fitt^i_R(M)$ are independent of the choice of presentation of M.*
2. *For any map of rings $R \to S$,*
$$Fitt^i_S(M \otimes_R S) = Fitt^i_R(M)\cdot S$$

See [Eis13, §20.2] for a proof.

2.2 Quadratic presentation

Definition 2.2.1. A finitely presented R-module M is said to admit a quadratic presentation if there exists an exact sequence of R-modules

$$R^d \xrightarrow{A} R^d \longrightarrow M \longrightarrow 0$$

for some integer $d \geq 0$.

In other words, M admits a presentation in which the number of generators equals the number of relations. Quadratic presentations impose rigid constraints on the structure of Fitting ideals.

Proposition 2.2.2. *Let R be a commutative ring and let M be a finitely presented R-module admitting a quadratic presentation*

$$R^d \xrightarrow{A} R^d \longrightarrow M \longrightarrow 0.$$

Then the 0-th Fitting ideal of M is principal and satisfies

$$\mathrm{Fitt}_R(M) = (\det A).$$

This observation is very useful in Iwasawa theory, where the relevant modules are often expected to admit quadratic presentations and their Fitting ideals are conjecturally principally generated by p-adic L-functions.

2.3 Exactness of Projective Limits

Definition 2.3.1. Let R be a commutative ring with identity, and let $\{M_i, \phi_{ji}\}$ be an inverse system of R-modules with transition maps $\phi_{ji} : M_j \to M_i$ for non-negative integers $j \geq i$. This inverse system is said to satisfy the Mittag-Leffler condition if for any i, the family $\{\phi_{ji}(M_j)\}_{j \geq i}$ of submodules of M_i are eventually stationary.

The Mittag-Leffler condition is automatically satisfied if either all morphisms ϕ_{ji} are surjective or all modules M_i are of finite length. We have the following proposition about projective systems which satisfy the Mittag-Leffler condition:

Proposition 2.3.2. *Let $\{A_i\}_i, \{B_i\}_i,$ and $\{C_i\}_i$ be projective systems of R-modules, and let*

$$0 \longrightarrow A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \longrightarrow 0 \tag{3}$$

be an exact sequence of the projective systems. Suppose $\{A_i\}_i$ satisfies the Mittag-Leffler condition. Then the sequence,

$$0 \longrightarrow \varprojlim A_i \xrightarrow{f} \varprojlim B_i \xrightarrow{g} \varprojlim C_i \longrightarrow 0 \tag{4}$$

is exact. Here, the maps f and g are obtained as the projective limits of $\{f_i\}$ and $\{g_i\}$.

We will use the following topological version of Nakayama's lemma in later sections. For a proof, refer to Lemma 1.10 in [PY26].

Proposition 2.3.3. *(Topological Nakayama lemma) Let R be a commutative, topological ring which is Hausdorff and compact. Let J be a closed ideal of R contained in the Jacobson radical. Let M be a topological R-module which is a limit $\varprojlim M_i$ of finitely generated, compact, Hausdorff R-modules. Suppose the transition maps in the projective system $\{M_i\}_{i \geq 1}$ are all surjective and the topology on M is the projective limit topology. If $M/\overline{JM}$ is finitely generated over R/J, then M is finitely generated over R.*

Finally, we record a fact about local profinite rings which we will use to infer principality of an ideal from the principality of its closure

Lemma 2.3.4. *Let R be a local profinite ring and I be an R-ideal whose closure is principal. Then I is a closed principal ideal.*

Proof. Let $\bar{I} = (a)$ and $I = aJ$ where $J = \{r \in R : ar \in I\}$. Note that J is an R-ideal. Suppose I is not closed. This means J is a proper ideal of R. The multiplication by a map $m_a : R \to R$ is continuous and sends the closed maximal ideal $\mathfrak{m}$ to $a\mathfrak{m}$, and hence $a\mathfrak{m}$ is closed. We have $I \subset a\mathfrak{m}$ and therefore $\bar{I} \subset a\mathfrak{m}$. Thus we get $(a) \subset a\mathfrak{m}$, which means $a = a \cdot r_m$ for some $r_m \in \mathfrak{m}$. This gives $a(1 - r_m) = 0$ and since $1 - r_m$ is a unit, we get $a = 0$ which is a contradiction. □

2.4 Horizontal Iwasawa algebras

We follow [KN24] and recall the notion of horizontal Iwasawa algebras. Let p be an odd prime and $(G_n)_{n\geq 1}$ be a sequence of finite abelian groups and R be a comutative ring. We can form the following product group

$$G_{\mathbb{N}} := \varprojlim_{\substack{A\subset\mathbb{N}\\ \text{finite}}} \prod_{i\in A} G_i$$

where the transition maps are given by canonical projections

$$\pi_{A,A'} : \prod_{i\in A} G_i \to \prod_{i\in A'} G_i \tag{5}$$

if $A' \subset A \subset \mathbb{N}$. For a finite set $A \subset \mathbb{N}$, the group $\prod_{i\in A} G_i$ is endowed with the discrete topology. This makes $G_{\mathbb{N}}$ a profinite group with the product topology. We define the *horizontal Iwasawa algebra* as follows

$$\Lambda^{\mathrm{hor}} := R[\![G_{\mathbb{N}}]\!] = \varprojlim_{\substack{A\subset\mathbb{N}\\ \text{finite}}} R\left[\prod_{i\in A} G_i\right] \tag{6}$$

We say that Λ^{hor} is pro-p if $G_{\mathbb{N}}$ is a pro-p abelian group and R is a p-adically complete subring of $\mathcal{O}_{\mathbb{C}_p}$. We refer to elements of Λ^{hor} as *horizontal measures*. The classical Iwasawa algebra $R[\![G]\!]$ is obtained by taking the projective limit over open subgroups $H \leq G$ of $R[G/H]$ where G is a profinite abelian group, such as $\mathbb{Z}_p$. The horizontal Iwasawa algebra is not as well behaved as the classical Iwasawa algebra. For instance, Λ^{hor} is an integral domain only when the G_n are all trivial.

We will use the following key result on the p-adic valuation of horizontal measures on continuous characters in §5 to prove Theorem 2a.

Theorem 2.4.1 ([KN24, Thm. 2.12])**.** *Let G be an abelian pro-p group with finite exponent $e(G) < \infty$. Let $\nu \in \mathbb{Z}_p[\![G]\!]$ be a non-zero measure. There exists a finite set $M_\nu \leq \widehat{G}$ such that the following holds: for any continuous character $\chi \in \widehat{G}$ there exists $\chi_0 \in M_\nu$ such that*

$$v_p(\nu(\chi\chi_0)) = v_p(\nu) := \inf_{\chi'\in\widehat{G}} v_p(\nu(\chi')) \tag{7}$$

Since M_ν is finite and independent of χ, Theorem 2.4.1 yields that a *positive proportion* of characters χ (in a sense made precise in §5) satisfy $v_p(\nu(\chi)) = v_p(\nu)$. This fact will translate via Theorem 1, into a statement about the p-part of a positive proportion of class numbers in the family of degree-p cyclic extensions.

3 Ritter–Weiss modules and the result of Dasgupta–Kakde

In this section, we recall Ritter–Weiss modules, which were first constructed by Ritter and Weiss in their work on Stark units [RW96]. They were further studied and used extensively by Dasgupta and Kakde in their work on the Brumer–Stark conjecture [DK23]. We also state the main theorem of Dasgupta–Kakde about the Fitting ideal of these modules. This will be the starting point in our construction of the horizontal Iwasawa module.

3.1 Ritter–Weiss modules

Let H/F be a finite abelian CM extension of number fields, with F totally real, and let

$$G := \mathrm{Gal}(H/F).$$

Fix two non-empty finite disjoint sets of places S and T of F, and an odd prime p. We assume the following conditions on S and T following [DK23].

(A1) $S \cup T$ contains $S_{\mathrm{ram}} \cup S_\infty$.

(A2) The torsion subgroup of

$$\mathcal{O}^*_{H,S_\infty,T} = \{u \in \mathcal{O}_H^\times : u \equiv 1 \pmod{\mathfrak{q}} \text{ for all primes } \mathfrak{q} \text{ above primes in } T\}$$

is trivial

(A3) $S \supset \{v \in S_{\mathrm{ram}} : v \mid p\}$

(A4) $T \supset \{v \in S_{\mathrm{ram}} : v \nmid p\}$

Let S_H and T_H denote the sets of primes of H above S and T, respectively. Let $\mathrm{Cl}^T(H)$ be the ray class group of H with conductor equal to the product of primes in T_H. This group is defines as

$$\mathrm{Cl}^T(H) := I_T(H)/P_T(H)$$

where $I_T(H)$ is the group of fractional ideals of H relatively primes to the primes in T_H, and $P_T(H)$ is its subgroup generated by principal ideals (α) where $\alpha \in \mathcal{O}_H$ satisfies $\alpha \equiv 1 \pmod{\mathfrak{p}}$ for all $\mathfrak{p} \in T_H$. Define $\mathrm{Cl}^T_S(H)$ to be the quotient of $\mathrm{Cl}^T(H)$ by the subgroup generated by classes of primes in S_H. Let

$$\mathrm{Div}_S(H) := \bigoplus_{w \in S_H} \mathbb{Z} \cdot w$$

be the free abelian group generated by primes of H above S, and let $\mathrm{Div}^0_S(H)$ be its subgroup of degree-zero divisors. All of these groups are naturally $\mathbb{Z}[G]$-modules via the Galois action. The *Ritter–Weiss module* associated to $(H/F, S, T)$ is the $\mathbb{Z}[G]$-module $\nabla^T_S(H)$, which sits in the fundamental exact sequence:

$$0 \longrightarrow \mathrm{Cl}^T_S(H) \longrightarrow \nabla^T_S(H) \longrightarrow \mathrm{Div}^0_S(H) \longrightarrow 0. \tag{8}$$

For a detailed definition of $\nabla^T_S(H)$, refer to [DKSW23, §4]. The module $\nabla^T_S(H)$ is finitely generated over $\mathbb{Z}[G]$. Tensoring with $\mathbb{Z}_p$ yields a finitely generated $\mathbb{Z}_p[G]$-module

$$\nabla^T_S(H)_p := \nabla^T_S(H) \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

Denote by $c \in G$ the complex conjugation. We define $\mathbb{Z}[G]^-$ by

$$\mathbb{Z}[G]^- := \mathbb{Z}[G]/(1+c)$$

For a $\mathbb{Z}[G]$-module M, we write M^- to mean $M \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]^- \simeq M/(1+c)M$.

3.2 Stickelberger elements

Recall for a character χ of G and finite sets S and T of places of F as before, we have L-functions

$$L_S(\chi, s) = \prod_{\mathfrak{q} \notin S} \left(1 - \chi(\mathfrak{q})\,\mathrm{N}\,\mathfrak{q}^{-s}\right)^{-1}, \qquad L^T_S(\chi, s) = L_S(\chi, s) \prod_{\mathfrak{q} \in T} \left(1 - \chi(\mathfrak{q})\,\mathrm{N}\,\mathfrak{q}^{\,1-s}\right),$$

We recall here a fact about the order of vanishing at $s = 0$ of $L_S(\chi, s)$. Let $G_v \subset G$ be the decomposition group at a place $v \in S$. For a non-trivial $\chi \in \widehat{G}$, we have

$$\mathrm{ord}_{s=0} L_S(\chi, s) = \#\{v \in S : \chi(G_v) = 1\}. \tag{9}$$

If χ is the trivial character, then the order of vanishing is given by $\#S - 1$.

Definition 3.2.1. The Stickelberger element $\Theta^T_S(H)$ is the unique element in $\mathbb{C}[G]$ defined by

$$\chi\big(\Theta^T_S(H)\big) = L^T_S(\chi^{-1}, 0)$$

for all $\chi \in \widehat{G}$.

Since T satisfies (A1)–(A4), a theorem of Deligne–Ribet [DR80] says that we have $\Theta^T_S(H) \in \mathbb{Z}[G]$.

3.2.1 Dasgupta–Kakde's result

In [DK23], Dasgupta and Kakde prove an unconditional result about the Fitting ideal of these Ritter–Weiss modules from which they derive the Brumer–Stark conjecture away from 2 and it is extended to include the prime 2 in [DKSW23]. The result is as follows.

Theorem 3.2.2 ([DKSW23, Thm. 1.2])**.** *Let p be a prime. Assume S and T satisfy conditions (A1)-(A4) defined above. Then $\nabla_S^T(H/F)_p^-$ is quadratically presented as a $\mathbb{Z}_p[G]^-$-module and there is an equality of the following ideals of $\mathbb{Z}_p[G]^-$*

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(\nabla_S^T(H/F)_p^-) = (\Theta_S^T(H)/2^t)$$

where $t = \#(S_\infty \cap T)$.

Remark 3.2.3. If we take p to be an odd prime, then 2 is invertible in $\mathbb{Z}_p[G]$ and hence

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(\nabla_S^T(H/F)_p^-) = (\Theta_S^T(H))$$

We work with an odd prime p throughout this paper, and thus suppress the 2^t in the above theorem.

One can regard Theorem 3.2.2 as a finite level specialization of an Iwasawa main conjecture, in which the Fitting ideal of an arithmetic module is generated by an analytic element built from L-values. As we climb the horizontal tower in §4, the identities at each layer should patch together into a single statement over the horizontal Iwasawa algebra, relating a limit module to a horizontal p-adic L-function. Making this precise is the content of §4.

We now discuss *character group rings* following [DKSW23]. Let $\mathcal{O}$ denote the ring of integers of a finite extension of $\mathbb{Q}_p$ containing all the values of all characters $\chi \in \hat{G}$. In §4.2 we work with a specific choice of $\mathcal{O}$. We have a $\mathbb{Z}_p$-algebra injection

$$\mathbb{Z}_p[G] \longrightarrow \prod_{\chi \in \hat{G}} \mathcal{O}, \qquad x \mapsto (\chi(x)).$$

For every $\mathrm{Gal}(\mathbb{Q}_p/Q)$-stable set of characters $\Psi \subset \hat{G}$, define the character group ring R_Ψ by

$$R_\Psi = \mathrm{Image}\big(\mathbb{Z}_p[G] \to \prod_{\psi \in \Psi} \mathcal{O}\big)$$

One reason to define these rings is to view the Stickelberger element $\Theta_S^T(H)$ as a nonzerodivisor. This property is crucial to apply the conclusion of the following size lemma.

Lemma 3.2.4 ([DKSW23, Lemma 2.6])**.** *Let M be a quadratically presented module over a character group ring R_Ψ, and let $\mathrm{Fitt}_{R_\Psi}(M) = (x)$. Then M is finite if and only if x is a nonzerodivisor, and in this case we have*

$$\#M = \#\big(\mathbb{Z}_p/\prod_{\psi \in \Psi} \psi(x)\big)$$

We apply the above lemma to the Ritter–Weiss module $\nabla_S^T(H/F)_p^-$ and its character components in §5 in order to deduce Theorems 2a and 2b.

4 Construction of the horizontal Iwasawa module and the horizontal L-function

In this section, we derive the compatibility of a component of the Ritter–Weiss modules and the corresponding Stickelberger elements in a horizontal tower. We thereby construct the horizontal Iwasawa Ritter–Weiss module $\nabla_S^T(H_\infty)$ and the horizontal p-adic L-function $\Theta_S^T(H_\infty)$, by taking suitable projective limits. We conclude the section with a proof of Theorem 1.

Recall that we have fixed an odd prime p and a finite CM abelian extension H/F of a totally real field F with Galois group $G = \mathrm{Gal}(H/F)$, whose degree is prime to p. We also fix an odd character $\chi_0 \in \mathrm{Hom}(G, \bar{\mathbb{F}}_p^\times)$

and orderly primes $\{\mathfrak{l}_i\}_{i\geq 1}$ of F which have a positive density δ (computed in §4.1) among the set of all primes of F. Let F_n be the maximal p-power degree sub-extension of the ray class field of F of conductor $\mathfrak{l}_1\mathfrak{l}_2\cdots\mathfrak{l}_n$ and set

$$H_n = H\cdot F_n, \quad G_n = \mathrm{Gal}(H_n/F), \quad \Delta_n = \mathrm{Gal}(H_n/H), \quad \mathcal{G} = \mathrm{Gal}(H_\infty/F).$$

Each $F(\mathfrak{l}_n)$ is totally real and hence each H_n is CM with the complex conjugation c as H. We have also fixed finite disjoint non-empty sets of places S and T of F, and $T_n = T\sqcup\{\mathfrak{l}_1,\mathfrak{l}_2,\ldots,\mathfrak{l}_n\}$. These finite sets of places satisfy (A1)–(A4) (3.1), and each $\mathfrak{l}_i$ $(i\leq n)$ is ramified in H_n and is therefore forced into T_n by (A4).

For each $n\geq 1$, we have the Ritter–Weiss module $\nabla_S^{T_n}(H_n)_p^-$ and the Stickelberger element $\Theta_S^{T_n}(H_n)$ of §3, and due to Theorem 3.2.2 we have the equality of the following ideals of $\mathbb{Z}_p[G_n]^-$

$$\mathrm{Fitt}_{\mathbb{Z}_p[G_n]^-}(\nabla_S^{T_n}(H_n)_p^-) = (\Theta_S^{T_n}(H_n)). \tag{10}$$

4.1 Density of the orderly primes

Recall the orderly primes of Definition 1.2.1. Conditions (O2) and (O3) are Frobenius conditions in the finite abelian extension

$$M := F(\mu_p)\cdot H_{\chi_0}, \qquad H_{\chi_0} = H^{\ker\chi_0},$$

namely $\mathrm{Frob}_\mathfrak{l}$ trivial on $F(\mu_p)$ (i.e. $\mathrm{N}\mathfrak{l}\equiv 1 \bmod p$) and $\mathrm{Frob}_\mathfrak{l}$ non-trivial on H_{χ_0} (i.e. $\chi_0(\mathrm{Frob}_\mathfrak{l})\neq 1$). Excluding the finitely many primes failing (O1), the Chebotarev density theorem gives

$$\delta \;=\; \frac{\#\{\sigma\in\mathrm{Gal}(M/F) : \sigma|_{F(\mu_p)} = \mathrm{id},\ \sigma|_{H_{\chi_0}}\neq \mathrm{id}\}}{[M:F]} \;>\; 0.$$

For $F=\mathbb{Q}$, H imaginary quadratic with $H\not\subseteq\mathbb{Q}(\mu_p)$, and $\chi_0=\chi_H$: here $H_{\chi_0}=H$, and we have that $\mathbb{Q}(\mu_p)$ and H are linearly disjoint over $\mathbb{Q}$, so $\mathrm{Gal}(M/\mathbb{Q})\cong(\mathbb{Z}/p)^\times\times\mathrm{Gal}(H/\mathbb{Q})$. Hence the density is given by

$$\delta = \frac{1}{p-1}\cdot\frac{1}{2} = \frac{1}{2(p-1)}.$$

Over an arbitrary totally real field F, we get

$$\delta = \frac{1}{[F(\mu_p):F]}\cdot\left(1-\frac{1}{[H_{\chi_0}:F]}\right) \;>\; 0.$$

as long as $F(\mu_p)$ and H_{χ_0} are linearly disjoint. The density is positive even when they are not linearly disjoint.

4.2 The character component

We discuss here what passing to the χ_0-component of the module means. We have $\chi_0\in\mathrm{Hom}(G,\bar{\mathbb{F}}_p^\times)$, which has a unique Teichmüller lift to $\widetilde{\chi_0}\in\mathrm{Hom}(G,\bar{\mathbb{Q}}_p^\times)$. Let

$$\mathcal{O} := \mathbb{Z}_p[\chi_0] = \mathbb{Z}_p[\widetilde{\chi_0}(g) : g\in G].$$

The prime to p part of G_n is independent of n, and the character χ_0 can be extended to a character of G_n for all n by making it trivial on the p-Sylow subgroup Δ_n of G_n. Define the idempotent $e_{\chi_0}\in\mathcal{O}[G]$ by

$$e_{\chi_0} = \frac{1}{|G|}\sum_{g\in G}\widetilde{\chi_0}(g)^{-1}\cdot g.$$

The corresponding idempotent in $\mathcal{O}[G_n]$ is also denoted e_{χ_0} and it is the image of e_{χ_0} under the map $\mathcal{O}[G]\to\mathcal{O}[G_n]$ sending $g\mapsto(g,1)$. Now for a $\mathbb{Z}[G_n]$-module M, we define its χ_0-component by

$$M_{\chi_0} := e_{\chi_0}(M\otimes_{\mathbb{Z}}\mathcal{O}). \tag{11}$$

This is a module for the ring $\Lambda_n := \mathcal{O}[\Delta_n]$, and by passing to the χ_0-component of module in equation (10), we get

$$\mathrm{Fitt}_{\mathcal{O}[\Delta_n]}(\nabla_S^{T_n}(H_n)_{\chi_0}) = (\Theta_S^{T_n}(H_n)_{\chi_0}). \tag{12}$$

Note that since p is odd, 2 is invertible in $\mathcal{O}[G_n]$ and we can define M^+ and M^- as the images of $e^{\pm} := \frac{1\pm c}{2}$. By our hypothesis, χ_0 is an odd character and thus $e_{\chi_0}e^- = e_{\chi_0}$. Therefore the χ_0-component is already contained in the minus part of the module.

4.2.1 Norm relations and units

In this section, we show that the elements $\theta_n := \Theta_S^{T_n}(H_n)_{\chi_0}$ are compatible under the maps $\pi_{n+1,n}$ up to an explicit Euler factor. For each n, the prime $\mathfrak{l} = \mathfrak{l}_{n+1}$ is unramified in H_n, so its Frobenius $\sigma_{\mathfrak{l}} := \mathrm{Frob}_{\mathfrak{l}} \in G_n$ is well defined. We let

$$u_{n+1} := e_{\chi_0}\big(1 - \sigma_{\mathfrak{l}_{n+1}}^{-1}\,\mathrm{N}\,\mathfrak{l}_{n+1}\big) \;\in\; \Lambda_n.$$

Lemma 4.2.1. *The element u_{n+1} is a unit in Λ_n for all $n \geq 1$.*

Proof. Recall that the prime $\mathfrak{l}$ satisfy $\chi_0(\mathfrak{l}) \neq 1$ and $\mathrm{N}\,\mathfrak{l} \equiv 1 \pmod{p}$ by Definition 1.2.1. Hence

$$u_{n+1} \equiv 1 - \chi_0(\mathfrak{l})^{-1}\,\mathrm{N}\,\mathfrak{l} \equiv 1 - \chi_0(\mathfrak{l})^{-1} \not\equiv 0 \pmod{p}$$

Therefore, $u_{n+1} \in \Lambda_n^{\times}$. □

The following proposition shows that the Stickelberger elements θ_n are compatible up to these units u_n. This lets us define the horizontal p-adic L-function as the projective limit of θ_n, after normalization.

Proposition 4.2.2. *For every $n \geq 1$, we have*

$$\pi_{n+1,n}(\theta_{n+1}) = u_{n+1}\,\theta_n$$

Proof. We first show that

$$\pi_{n+1,n}\big(\Theta_S^{T_{n+1}}(H_{n+1})\big) = \Theta_S^{T_{n+1}}(H_n) = \big(1 - \sigma_{\mathfrak{l}}^{-1}\,\mathrm{N}\,\mathfrak{l}\big)\,\Theta_S^{T_n}(H_n). \tag{13}$$

For the first equality let $\chi \in \widehat{G_n}$ and let $\widetilde{\chi}$ be its inflation to G_{n+1}. Since $\widetilde{\chi}$ is trivial on $\mathrm{Gal}(H_{n+1}/H_n)$ it is unramified at $\mathfrak{l}$, and Artin L-functions are inflation-invariant, so $L_S^{T_{n+1}}(\widetilde{\chi}^{-1}, s) = L_S^{T_{n+1}}(\chi^{-1}, s)$; evaluating at $s = 0$,

$$\chi\big(\pi_{n+1,n}\Theta_S^{T_{n+1}}(H_{n+1})\big) = \widetilde{\chi}\big(\Theta_S^{T_{n+1}}(H_{n+1})\big) = L_S^{T_{n+1}}(\chi^{-1}, 0) = \chi\big(\Theta_S^{T_{n+1}}(H_n)\big),$$

and equality on all $\chi \in \widehat{G_n}$ gives the first equality of (13). For the second equality, every $\chi \in \widehat{G_n}$ is unramified at $\mathfrak{l}$, and

$$L_S^{T_{n+1}}(\chi^{-1}, 0) = \big(1 - \chi^{-1}(\mathfrak{l})\,\mathrm{N}\,\mathfrak{l}\big)L_S^{T_n}(\chi^{-1}, 0) = \chi\big((1 - \sigma_{\mathfrak{l}}^{-1}\,\mathrm{N}\,\mathfrak{l})\Theta_S^{T_n}(H_n)\big).$$

Applying e_{χ_0} to (13) and using $\pi_{n+1,n} \circ e_{\chi_0} = e_{\chi_0} \circ \pi_{n+1,n}$ gives

$$\pi_{n+1,n}(\theta_{n+1}) = e_{\chi_0}(1 - \sigma_{\mathfrak{l}}^{-1}\,\mathrm{N}\,\mathfrak{l})\,\theta_n = u_{n+1}\theta_n.$$

which proves the proposition □

We now normalize the Stickelberger elements by these units u_i and define

$$\hat{\theta}_n := \left(\prod_{i=1}^{n} u_i\right)^{-1} \theta_n \quad \in\; \Lambda_n$$

so that $\pi_{n+1,n}(\hat{\theta}_{n+1}) = \hat{\theta}_n$ for all $n \geq 1$. Also, since u_i is a unit for all $i \geq 1$, we get that the ideals generated by $\hat{\theta}_n$ and θ_n are the same. Thus we have,

$$\mathrm{Fitt}_{\Lambda_n}(\nabla_n) = (\theta_n) = (\hat{\theta}_n). \tag{14}$$

Definition 4.2.3. The horizontal p-adic L-function is defined as

$$\Theta_\infty := \varprojlim_n \widehat{\theta}_n \;\in\; \Lambda = \varprojlim_n \Lambda_n.$$

The projective limit in the above definition makes sense because the elemenets $\{\hat{\theta}_n\}_{n\geq 1}$ form a compatible system under the transition maps $\pi_{n+1,n}$.

4.3 Compatibility of the Ritter–Weiss modules

We now turn to the compatibility of the Ritter–Weiss modules in our horizontal tower.

Fix $n \geq 1$ and put $\mathfrak{l} = \mathfrak{l}_{n+1}$ to simplify notation in this section. Define the $\mathbb{Z}[G_n]$-module

$$V_n := \bigoplus_{\lambda \mid \mathfrak{l}} (\mathcal{O}_{H_n}/\lambda)^\times$$

where the sum is over all primes of H_n lying above $\mathfrak{l}$. Define the (S, T_n)-unit group $\mathcal{O}^\times_{H_n,S,T_n}$ by

$$\mathcal{O}^\times_{H_n,S,T_n} := \{u \in \mathcal{O}^\times_{H_n,S_{H_n}} : v_{\mathfrak{p}}(u) = 0 \text{ for all finite } \mathfrak{p} \notin S_{H_n}, \text{ and } u \equiv 1 \pmod{\mathfrak{q}} \text{ for all } \mathfrak{q} \in T_n\}.$$

Lemma 4.3.1. *There is an exact sequence of $\mathbb{Z}[G_n]$-modules*

$$\mathcal{O}^\times_{H_n,S,T_n} \xrightarrow{\mathrm{red}_{\mathfrak{l}}} V_n \xrightarrow{\psi} \mathrm{Cl}_S^{T_{n+1}}(H_n) \xrightarrow{\alpha_n} \mathrm{Cl}_S^{T_n}(H_n) \longrightarrow 0, \tag{15}$$

where $\mathrm{red}_{\mathfrak{l}}$ *is reduction of (S, T_n)-units modulo the primes above $\mathfrak{l}$ and α_n is the natural surjection.*

Proof. The map $\mathrm{red}_{\mathfrak{l}}$ is well defined because $\mathfrak{l} \notin S$ and each S-unit is also a λ-unit. The map ψ is defined as follows: For an element $(u_\lambda)_\lambda \in V_n$, choose $\gamma \in \mathcal{O}_{H_n}$ such that $\gamma \equiv u_\lambda \pmod{\lambda}$ for each $\lambda \mid \mathfrak{l}$, and $\gamma \equiv 1 \pmod{\mathfrak{q}}$ for all $\mathfrak{q} \in T_n$. One can do this because of Chinese Remainder Theorem. Define $\psi((u_\lambda)_\lambda) := [(\gamma)]$. This is well defined because if γ' is another such element, then $\frac{\gamma}{\gamma'} \equiv 1 \pmod{T_{n+1}}$ and hence $[(\gamma)] = [(\gamma')]$ in $\mathrm{Cl}_S^{T_{n+1}}(H_n)$.

Exactness at $\mathrm{Cl}_S^{T_{n+1}}(H_n)$: Since $(\gamma) \in P_{H_n}(T_n)$, we have $\alpha_n\psi = 0$. Conversely, suppose $[\mathfrak{a}] \in \ker(\alpha_n)$. Then $[\mathfrak{a}]$ is trivial in $\mathrm{Cl}_S^{T_n}(H_n)$. Thus there exists $\beta \in H_n^\times$ and an ideal $\mathfrak{s}$ supported on S such that $\mathfrak{a} = (\beta)\mathfrak{s}$ and $\beta \equiv 1 \pmod{T_n}$. Since S is disjoint from T_{n+1}, we have that (β) is coprime to T_{n+1}. Thus $\psi(\mathrm{red}_{\mathfrak{l}}(\beta)) = [(\beta)] = \mathfrak{a}$. This proves exactness at $\mathrm{Cl}_S^{T_{n+1}}(H_n)$.

Exactness at V_n: Suppose $\mathrm{red}_{\mathfrak{l}}(v) = (u_\lambda)_\lambda$. Since $v \in \mathcal{O}^\times_{H_n,S,T_n}$, we can use v itself as the element γ in the construction of ψ. Thus $\psi(\mathrm{red}_{\mathfrak{l}}(v)) = [(v)]$. But (v) is trivial in $\mathrm{Cl}_S^{T_{n+1}}(H_n)$ since v is an S-unit. Converesely, suppose $\psi((u_\lambda)_\lambda) = 0$. Choose a γ as in the construction of ψ, so we get that (γ) is trivial in $\mathrm{Cl}_S^{T_{n+1}}(H_n)$. Thus there exists $\beta \in H_n^\times$ such that $(\gamma) = (\beta)\mathfrak{s}$, where $\mathfrak{s}$ is supported only on S and $\beta \equiv 1 \pmod{T_{n+1}}$. The element $w := \frac{\gamma}{\beta}$ generates the ideal $\mathfrak{s}$, and is an S-unit satisfying $w \equiv 1 \pmod{T_n}$. Thus $w \in \mathcal{O}^\times_{H_n,S,T_n}$ and $\mathrm{red}_{\mathfrak{l}}(w) = (u_\lambda)_\lambda$. This proves exactness at V_n. □

Now we show that by passing to the χ_0-component, the module V_n vanishes.

Lemma 4.3.2. *The χ_0-component of V_n vanishes (i.e. $(V_n)_{\chi_0} = 0$.)*

Proof. The prime $\mathfrak{l}$ is unramified in H_n and hence $\mathrm{Frob}_{\mathfrak{l}}$ is a well defined element of G_n. Its action on $\mathcal{O}_{H_n}/\lambda$ is given by $x \to x^{N\mathfrak{l}}$. Thus, the operator $(\mathrm{Frob}_{\mathfrak{l}} - N\mathfrak{l})$ acts as 0 on V_n. Passing to the χ_0-component, we get that $e_{\chi_0}(\mathrm{Frob}_{\mathfrak{l}} - N\mathfrak{l})$ annihilates $(V_n)_{\chi_0}$. But by the definition of orderly primes, we have $\chi_0(\mathfrak{l}) \neq 1$ and $N\mathfrak{l} \equiv 1 \pmod{p}$. Thus $\chi_0(\mathfrak{l}) - N\mathfrak{l} \not\equiv 0 \pmod{p}$ and hence is a unit in $\Lambda_n := e_{\chi_0}\mathcal{O}[G_n]$. The module $(V_n)_{\chi_0}$ is thus annihilated by a unit and therefore vanishes. □

We construct a map from $\nabla_S^{T_{n+1}}(H_{n+1})$ to $\nabla_S^{T_n}(H_n)$, as the composition of two maps. The first one is the norm map from $\nabla_S^{T_{n+1}}(H_{n+1})$ to $\nabla_S^{T_{n+1}}(H_n)$, while the second one β_n from $\nabla_S^{T_{n+1}}(H_n)$ to $\nabla_S^{T_n}(H_n)$ forgets ramification at $\mathfrak{l}_{n+1}$. This is illustrated in the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Cl}_S^{T_{n+1}}(H_{n+1}) & \longrightarrow & \nabla_S^{T_{n+1}}(H_{n+1}) & \xrightarrow{\partial_{n+1}} & X_{S,H_{n+1}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Nm}_n} & & \downarrow{\scriptstyle \mathrm{Nm}_n^\nabla} & & \downarrow{\scriptstyle \mathrm{Nm}} & & \\
0 & \longrightarrow & \mathrm{Cl}_S^{T_{n+1}}(H_n) & \longrightarrow & \nabla_S^{T_{n+1}}(H_n) & \xrightarrow{\partial} & X_{S,H_n} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \beta_n} & & \| {\scriptstyle id} & & \\
0 & \longrightarrow & \mathrm{Cl}_S^{T_n}(H_n) & \longrightarrow & \nabla_S^{T_n}(H_n) & \xrightarrow{\partial_n} & X_{S,H_n} & \longrightarrow & 0.
\end{array} \tag{16}$$

We now compare the Ritter–Weiss modules of the bottom two rows

Lemma 4.3.3. *The map β_n induces an isomorphism $\beta_{n,\chi_0} : \nabla_S^{T_{n+1}}(H_n)_{\chi_0} \xrightarrow{\sim} \nabla_S^{T_n}(H_n)_{\chi_0}$*

Proof. Since $(V_n)_{\chi_0} = 0$, applying e_{χ_0} to equation (15), we see that α_n induces an isomorphism

$$\alpha_{n,\chi_0} : \mathrm{Cl}_S^{T_{n+1}}(H_n)_{\chi_0} \xrightarrow{\sim} \mathrm{Cl}_S^{T_n}(H_n)_{\chi_0}.$$

Applying the five lemma to the bottom two rows, we get that the middle vertical map β_{n,χ_0} is also an isomorphism. □

We are now ready to prove the main result of this section, which is the compatibility of the χ_0-components of the Ritter–Weiss modules.

Proposition 4.3.4. *For every $n \geq 1$, there is a natural isomorphism of Λ_n-modules*

$$\nabla_S^{T_{n+1}}(H_{n+1})_{\chi_0} \otimes_{\Lambda_{n+1}} \Lambda_n \xrightarrow{\sim} \nabla_S^{T_n}(H_n)_{\chi_0}. \tag{17}$$

and the composite transition map $t_n : \nabla_S^{T_{n+1}}(H_{n+1})_{\chi_0} \to \nabla_S^{T_n}(H_n)_{\chi_0}$ is surjective.

Proof. By Dasgupta–Kakde–Silliman–Wang [DKSW23, Lemma 4.6], we have an isomorphism

$$\nabla_S^{T_{n+1}}(H_{n+1}) \otimes_{\mathbb{Z}[G_{n+1}]} \mathbb{Z}[G_n] \xrightarrow{\sim} \nabla_S^{T_{n+1}}(H_n).$$

Passing to the χ_0-component, we get $\nabla_S^{T_{n+1}}(H_{n+1})_{\chi_0} \otimes_{\Lambda_{n+1}} \Lambda_n \xrightarrow{\sim} \nabla_S^{T_{n+1}}(H_n)_{\chi_0}$. Composing this with the isomorphism β_{n,χ_0} from Lemma (4.3.3), we get

$$\nabla_S^{T_{n+1}}(H_{n+1})_{\chi_0} \otimes_{\Lambda_{n+1}} \Lambda_n \xrightarrow{\sim} \nabla_S^{T_{n+1}}(H_n)_{\chi_0} \xrightarrow{\beta_{n,\chi_0}} \nabla_S^{T_n}(H_n)_{\chi_0}.$$

The norm map Nm_n is surjective on the χ_0-component and composing it with β_{n,χ_0} makes the transition map t_n surjective. □

4.4 Construction of ∇_∞

Let $\Lambda := \varprojlim \Lambda_n = e_{\chi_0}\mathcal{O}[\![\mathcal{G}]\!]$ be the χ_0-component of the horizontal Iwasawa algebra. This is a non-Noetherian local profinite ring with maximal ideal $\mathfrak{m}_\Lambda = \varprojlim \mathfrak{m}_n$. We can now define the horizontal Ritter–Weiss module over Λ because of the surjective transition maps from Proposition 4.3.4.

Definition 4.4.1. The horizontal Ritter–Weiss module is defined as

$$\nabla_\infty := \varprojlim_n \nabla_S^{T_n}(H_n)_{\chi_0}. \tag{18}$$

Since Fitting ideals are defined only for finitely generated modules, we need the following proposition to talk about $\mathrm{Fitt}_\Lambda(\nabla_\infty)$.

Proposition 4.4.2. *The module ∇_∞ is finitely generated over Λ, and we have for all $n \geq 1$,*

$$\nabla_\infty \otimes_\Lambda \Lambda_n \simeq \nabla_S^{T_n}(H_n)_{\chi_0}.$$

Proof. The ring Λ_n is local with maximal ideal $\mathfrak{m}_n$. The limit ring $\Lambda = \varprojlim \Lambda_n$ is local, compact, Hausdorff with respect to the $\mathfrak{m}_\infty$-adic topology where $\mathfrak{m}_\infty = \varprojlim \mathfrak{m}_n$ is the maximal ideal of Λ. We have

$$\Lambda/\mathfrak{m}_\infty\Lambda = \mathbb{F}_p[\chi_0],$$

and

$$\nabla_\infty/\mathfrak{I}_\infty\nabla_\infty = M_0 := \nabla_S^T(H)_{\chi_0}.$$

where $\mathfrak{I}$ is the augmentation ideal of Λ. The module M_0 is finitely generated over $\Lambda/\mathfrak{m}_\infty\Lambda = \mathbb{F}_p[\chi_0]$, and the ideal $\mathfrak{m}_\infty$ is closed in Λ. Therefore, by the topological Nakayama lemma (2.3.3), we get that ∇_∞ is finitely generated over Λ. □

4.5 The horizontal Iwasawa main conjecture

The horizontal Iwasawa Ritter–Weiss module ∇_∞ is finitely generated over Λ by Proposition 4.4.2, and thus we can define the zeroth Fitting ideal of ∇_∞ as an ideal of Λ.
By (14), we have

$$\mathrm{Fitt}_{\Lambda_n}(\nabla_n) = (\hat{\theta}_n).$$

Together with Proposition 2.1.2, we obtain

$$\mathrm{Fitt}_{\Lambda}(\nabla_\infty) \cdot \Lambda_n = \mathrm{Fitt}_{\Lambda_n}(\nabla_n) = (\hat{\theta}_n).$$

On the other hand, for each $n \geq 1$, we have

$$(\Theta_\infty) \cdot \Lambda_n = (\Theta_n) = (\widehat{\theta}_n) = \mathrm{Fitt}_{\Lambda}(\nabla_\infty) \cdot \Lambda_n.$$

Since the ideal (Θ_∞) is closed in Λ under the inverse limit topology, we get

$$\overline{\mathrm{Fitt}_{\Lambda}(\nabla_\infty)} = (\Theta_\infty)$$

due to the following lemma.

Lemma 4.5.1. *Let $A_\infty = \varprojlim A_n$ be the inverse limit of discrete topological rings A_n with surjective transition maps $A_{n+1} \to A_n$ for all $n \geq 1$. Let I and J be ideals of A_∞ such that for all $n \geq 1$, we have*

$$IA_n = JA_n.$$

Then $\overline{I} = \overline{J}$, where the closure is taken in the inverse limit topology.

Proof. Let $\pi_n : A_\infty \to A_n$ be the natural surjection. By definition of the inverse limit topology on A_∞, a fundamental system of open neighbourhoods around 0 is given by $\ker(\pi_n)$. We have

$$\pi_n^{-1}(IA_n) = I + \ker(\pi_n).$$

Since $IA_n = JA_n$ for all $n \geq 1$, we get for all $n \geq 1$

$$I + \ker(\pi_n) = J + \ker(\pi_n).$$

Taking the intersection over all $n \geq 1$, we get $\overline{I} = \overline{J}$. □

In fact, since Λ is a local profinite ring, Lemma 2.3.4 gives

$$\mathrm{Fitt}_{\Lambda}(\nabla_\infty) = (\Theta_\infty),$$

which proves Theorem 1.

In the following section, we turn towards proving the arithmetic-statistical consequence of Theorem 1.

5 A size formula and the arithmetic statistical application

In this section we use the horizontal Iwasawa main conjecture and using convert it to an exact size formula for the class group. We then use Kriz–Nordentoft's density theorem (Theorem 2.4.1) to prove the arithmetic-statistics consequences: for a positive proportion of cyclic degree-p extensions K/F in our family, the χ_0-component of the class group of $H_K := HK$ has a fixed, smallest possible size (Theorems 2a and 2b). We keep the notation of §4, and let

$$\mathcal{K}_p(\mathcal{L}) := \big\{K/F : \ K \text{ cyclic of degree } p, \ \mathfrak{f}(K) \text{ a squarefree product of primes in } \mathcal{L}\big\},$$
$$\mathcal{K}_p(X, \mathcal{L}) := \big\{K \in \mathcal{K}_p(\mathcal{L}) : \ \mathrm{N}\,\mathfrak{f}(K) \leq X\big\}.$$

We now impose two further conditions on the sets of places S and T, namely

(C1) (Depletion): for every finite place $v \in S$ we have $\chi_0(G_v) \neq 1$, where $G_v \subseteq G$ is the decomposition group at v.

(C2) (Smoothing): for every $\mathfrak{q} \in T$ we have $\mathrm{N}\,\mathfrak{q} \not\equiv \chi_0(\mathfrak{q}) \pmod{\mathfrak{m}_{\mathcal{O}}}$.

The condition (C1) ensures that the divisor part of the short exact sequence (8) vanishes on the χ_0-component, and (C2) ensures that the conclusion of Theorem 2b holds for the ideal class group instead of the T-ray class group.

5.1 The size formula

Fix $K \in \mathcal{K}_p(\mathcal{L})$, a degree p cyclic extension of F with conductor a squarefree product of orderly primes. Thus $K \subseteq F_n$ for some n, and K is cut out by an order-p character

$$\eta_K \colon \Delta_\infty \longrightarrow \mu_p,$$

well defined up to the action $\eta_K \mapsto \eta_K^a$ of $(\mathbb{Z}/p)^\times$. The $p-1$ characters $\eta_K, \eta_K^2, \dots, \eta_K^{p-1}$ all cut out the same field K. We write $H_K = H \cdot K$, a CM field with $\mathrm{Gal}(H_K/H) \cong \mathrm{Gal}(K/F) \cong \mathbb{Z}/p$.

Proposition 5.1.1. *For $K \in \mathcal{K}_p(\mathcal{L})$ cut out by a character $\eta := \eta_K$, we have*

$$v_p\big(\#\,\mathrm{Cl}(H_K)_{\chi_0}\big) \;=\; f\sum_{j=0}^{p-1} v_p\big(\eta^j(\Theta_\infty)\big) \;=\; v_p\big(\#\,\mathrm{Cl}(H)_{\chi_0}\big) + f\,(p-1)\,v_p\big(\eta(\Theta_\infty)\big).$$

Proof. Let $T_K = T \cup \{\mathfrak{l}_i : \mathfrak{l}_i \mid \mathfrak{f}(K)\}$, $G_K = \mathrm{Gal}(H_K/F)$ and $\Lambda_K = e_{\chi_0}\mathcal{O}[G_K]$. Then $\Lambda_K \cong \mathcal{O}[\mathrm{Gal}(K/F)] \cong \mathcal{O}[\mathbb{Z}/p]$ is a character group ring. By the theorem of Dasgupta–Kakde (3.2.2), applied to H_K and passed to the χ_0-component, the module $\nabla_S^{T_K}(H_K)_{\chi_0}$ has a quadratic presentation over Λ_K, and

$$\mathrm{Fitt}_{\Lambda_K}\big(\nabla_S^{T_K}(H_K)_{\chi_0}\big) = (\theta_K), \qquad \theta_K := \Theta_S^{T_K}(H_K)_{\chi_0}.$$

Fix a character η^j of $\mathrm{Gal}(K/F)$. We have

$$\eta^j(\theta_K) = (\chi_0\eta^j)\big(\Theta_S^{T_K}(H_K)\big) = L_S^{T_K}\big((\chi_0\eta^j)^{-1}, 0\big)$$

Since θ_K is a non-zerodivisor, we can apply Lemma 3.2.4 to get

$$\#\nabla_S^{T_K}(H_K)_{\chi_0} = \prod_{j=0}^{p-1} \#\big(\mathcal{O}_\eta/\eta^j(\theta_K)\big).$$

We now apply $(-)_{\chi_0\eta^j}$ to the exact sequence $0 \to \mathrm{Cl}_S^{T_K}(H_K) \to \nabla_S^{T_K}(H_K) \to \mathrm{Div}_S^0(H_K) \to 0$. The divisor term vanishes on each $\chi_0\eta^j$-component by the depletion condition (C1). Therefore we get,

$$\#\,\mathrm{Cl}_S^{T_K}(H_K)_{\chi_0} = \prod_{j=0}^{p-1} \#\big(\mathcal{O}_\eta/\eta^j(\theta_K)\big).$$

Since $\eta^j(\theta_K)$ and $\eta^j(\Theta_\infty)$ differ only by the unit Euler factors $\eta^j(u_i)$ of §4.2.1, the sizes $\#\big(\mathcal{O}_{\eta^j}/\eta^j(\theta_K)\big)$ and $\#\big(\mathcal{O}_{\eta^j}/\eta^j(\Theta_\infty)\big)$ agree. Together with the depletion and smoothing conditions (C1) and (C2) on S and T ensure that the factors at the places of S and T are p-adic units, and this gives

$$\#\,\mathrm{Cl}(H_K)_{\chi_0} = \prod_{j=0}^{p-1} \#\big(\mathcal{O}_\eta/\eta^j(\theta_K)\big).$$

The first equality of the proposition follows from this. For the second equality, the $j=0$ term is $v_p\big(\#(\mathcal{O}/\Theta_\infty(1))\big) = v_p(\#\,\mathrm{Cl}(H)_{\chi_0})$, which is the case $K = F$. For $j = 1, \dots, p-1$ the characters $\eta, \dots, \eta^{p-1}$ form a single Galois orbit and thus their valuations all have the same value $v_p(\eta(\Theta_\infty))$. □

Definition 5.1.2. Set

$$m := \min_\eta v_p\big(\eta(\Theta_\infty)\big), \qquad e := v_p\big(\#\,\mathrm{Cl}(H)_{\chi_0}\big) + f\,(p-1)\,m,$$

the minimum taken over order-p characters η of Δ_∞.

Therefore for every $K \in \mathcal{K}_p(\mathcal{L})$, we have $v_p(\#\,\mathrm{Cl}(H_K)_{\chi_0}) \geq e$, where equality holds if and only if $v_p(\eta_K(\Theta_\infty)) = m$. Our arithmetic-statistics results (Theorems 2a and 2b) say that equality holds for a positive proportion of fields in $\mathcal{K}_p(\mathcal{L})$, ordered by the norm of the conductor.

5.2 Model count over $\mathbb{Q}$

In this subsection, we let $F = \mathbb{Q}$ and H be an imaginary quadratic with $H \not\subseteq \mathbb{Q}(\mu_p)$. Let $\chi_0 = \chi_H$ be the quadratic character, so that $\mathcal{O} = \mathbb{Z}_p$, $f = 1$, and $\mathrm{Cl}(H_K)_{\chi_0}$ is the minus part of the p-part of $\mathrm{Cl}(H_K)$. Our goal in this subsection is to prove Theorem 2a: when ordered by conductor, a positive proportion of fields K in $\mathcal{K}_p(\mathcal{L})$ satisfy $v_p(\#\,\mathrm{Cl}(H_K)_{\chi_0}) = e$.

The argument has two parts. The first part is combinatorial: the density theorem of Kriz–Nordentoft produces a large "stable" set of characters, and the pigeonhole principle turns it into a lower bound for the proportion. This part uses nothing special about $\mathbb{Q}$. The second part is analytic: we use the Selberg–Delange method to count the the relevant families and show that the lower bound obtained in the first part converges to a positive constant. Over a general base field F, we only need to modify the analytic part of the argument, and this is done in §5.3. We start with the combinatorial argument.

5.2.1 Applying Kriz–Nordentoft's density result

The horizontal p-adic L-function Θ_∞ is nonzero and so is its push forward to the exponent p quotient $\bar{\Delta}_\infty$ of Δ_∞. Theorem 2.4.1 applied to this nonzero measure gives a finite set $M = \{\eta_1, \eta_2, \dots, \eta_t\}$ of characters of $\bar{\Delta}_\infty$ with the following property: for every order p character η of $\bar{\Delta}_\infty$, some $\eta\eta_j$ attains the minimum valuation m.

Recall that we order the orderly primes of F by the absolute norm. Such an ordering need not be unique, but we fix one $\mathcal{L} = \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \dots\}$ and use it throughout. Choose an integer k large enough so that every prime dividing any of the conductors $\mathfrak{f}(\eta_1), \mathfrak{f}(\eta_2), \dots, \mathfrak{f}(\eta_t)$ lies in $\{\mathfrak{l}_1, \mathfrak{l}_2, \dots, \mathfrak{l}_k\}$. Such a choice exists because M is finite. For a finite set $A \subset \{k+1, k+2, \dots\}$ with $N(\mathfrak{m}_k \prod_{j\in A} \mathfrak{l}_j) \le X$, define

$$S_A(X) := \Big\{\text{order-}p \text{ characters } \eta \text{ of } \bar{\Delta}_\infty : \prod_{j\in A} \mathfrak{l}_j \mid \mathfrak{f}(\eta) \mid \mathfrak{m}_k \prod_{j\in A} \mathfrak{l}_j\Big\}$$

and set $\mathcal{X}_{\mathrm{stable}}(X) := \bigsqcup_A S_A(X)$. Also define the following sets of characters

$$\mathcal{X}_p(X) := \{\text{order-}p \text{ characters } \eta : \mathrm{N}\,\mathfrak{f}(\eta) \le X\}, \qquad \mathcal{X}_{\min}(X) := \{\eta \in \mathcal{X}_p(X) : v_p(\eta(\Theta_\infty)) = m\}.$$

Since every field in $\mathcal{K}_p(X, \mathcal{L})$ is cut out by exactly $p-1$ order-p characters, we have

$$\#\mathcal{X}_p(X) = (p-1)\#\mathcal{K}_p(X, \mathcal{L}),$$

and the proportion in Theorem 2a equals $\#\mathcal{X}_{\min}(X)/\#\mathcal{X}_p(X)$. We start by showing that $\mathcal{X}_{\mathrm{stable}}$ is stable under multiplication by characters in the finite set $M = \{\eta_1, \eta_2, \dots, \eta_t\}$.

Lemma 5.2.1. *If $\eta \in \mathcal{X}_{\mathrm{stable}}(X)$ then $\eta\eta_i \in \mathcal{X}_{\mathrm{stable}}(X)$ for every $1 \le i \le t$.*

Proof. Suppose $\eta \in S_A(X)$ for some finite set A. Each η_i is ramified only at primes of $\{\mathfrak{l}_1, \dots, \mathfrak{l}_k\}$, so multiplying by η_i changes only the $\mathfrak{m}_k$-part of the conductor. The part away from $\mathfrak{m}_k$ stays $\prod_{j\in A} \mathfrak{l}_j$ and the $\mathfrak{m}_k$-part still divides $\mathfrak{m}_k$. Hence $\eta\eta_i \in S_A(X) \subseteq \mathcal{X}_{\mathrm{stable}}(X)$. □

Lemma 5.2.2. *For every $X > 0$, we have*

$$\mathcal{X}_{\mathrm{stable}}(X) \;\subseteq\; \mathcal{X}_p(X) \;\subseteq\; \mathcal{X}_{\mathrm{stable}}(\mathrm{N}\,\mathfrak{m}_k \cdot X).$$

Proof. Write $\mathfrak{f}(\eta) = \mathfrak{f}_k\,\mathfrak{f}'$ with $\mathfrak{f}_k \mid \mathfrak{m}_k$ and $\mathfrak{f}'$ coprime to $\mathfrak{m}_k$. If $\eta \in S_A(X)$ then $\mathrm{N}(\mathfrak{m}_k\mathfrak{f}') \le X$, and $\mathfrak{f}(\eta) \mid \mathfrak{m}_k\mathfrak{f}'$ gives $\mathrm{N}\,\mathfrak{f}(\eta) \le X$, so $\eta \in \mathcal{X}_p(X)$. This proves the first inclusion. For the second inclusion, if $\eta \in \mathcal{X}_p(X)$ then $\mathrm{N}\,\mathfrak{f}' \le \mathrm{N}\,\mathfrak{f}(\eta) \le X$, so $\mathrm{N}(\mathfrak{m}_k\mathfrak{f}') = \mathrm{N}\,\mathfrak{m}_k \cdot \mathrm{N}\,\mathfrak{f}' \le \mathrm{N}\,\mathfrak{m}_k \cdot X$. Thus $\eta \in S_A(\mathrm{N}\,\mathfrak{m}_k X)$ where A is the finite set of indices $\{j > k\}$ such that $\mathfrak{l}_j \mid \mathfrak{f}'$. □

By Lemma 5.2.1 and the pigeon-hole principle, we get $\#\mathcal{X}_{\min}(X) \ge \frac{1}{t}\#\mathcal{X}_{\mathrm{stable}}(X)$. Together with Lemma 5.2.2, this gives the following proposition.

Proposition 5.2.3. *For every X,*

$$\frac{\#\mathcal{X}_{\min}(X)}{\#\mathcal{X}_p(X)} \;\ge\; \frac{1}{t} \cdot \frac{\#\mathcal{X}_{\mathrm{stable}}(X)}{\#\mathcal{X}_{\mathrm{stable}}(\mathrm{N}\,\mathfrak{m}_k\,X)}.$$

Everything is now reduced to the asymptotics of $\#\mathcal{X}_{\mathrm{stable}}(X)$, which we treat analytically in the next subsection.

5.2.2 Counting via the Selberg–Delange method

By Proposition 5.2.3, the counting problem is reduced to understanding the asymptotics of $\#\mathcal{X}_{\mathrm{stable}}(X)$ as $X \to \infty$. We first find the asymptotics of $\#\mathcal{X}_p(X)$ as $X \to \infty$ using the Selberg–Delange method, and use it to study $\#\mathcal{X}_{\mathrm{stable}}(X)$.

We have $\bar{\Delta} \simeq \prod_{i=1}^{\infty} \mathbb{Z}/p$, and thus an order-$p$ character of squarefree conductor exactly $\mathfrak{f}$ is a free choice of a non-trivial character at each prime dividing $\mathfrak{f}$. If $\mathfrak{f}$ is a squarefree product of primes in $\mathcal{L}$, define

$$a(\mathfrak{f}) := (p-1)^{\omega(\mathfrak{f})},$$

where $\omega(\mathfrak{f})$ is the number of prime factors of $\mathfrak{f}$. For other $\mathfrak{f}$, define $a(\mathfrak{f}) = 0$. Thus we have

$$\#\mathcal{X}_p(X) = \sum_{\mathrm{N}(\mathfrak{f}) \le X} a(\mathfrak{f})$$

where the sum is taken over squarefree $\mathfrak{f}$ supported on $\mathcal{L}$. Let $\alpha := p-1$ and consider the Dirichlet series

$$F(s) = \sum_{\mathfrak{f}} \frac{a(\mathfrak{f})}{\mathrm{N}\,\mathfrak{f}^s}.$$

Since the sum is only over squarefree $\mathfrak{f}$, we get the following Euler product

$$F(s) = \prod_{\mathfrak{l} \in \mathcal{L}} \Big(1 + \frac{\alpha}{\mathrm{N}\,\mathfrak{l}^s}\Big) = \zeta_{\mathcal{L}}(s)^{\alpha} H(s),$$

where $\zeta_{\mathcal{L}}(s) = \prod_{\mathfrak{l}\in\mathcal{L}} (1 - \mathrm{N}\,\mathfrak{l}^{-s})^{-1}$ and $H(s) = \prod_{\mathfrak{l}\in\mathcal{L}} \big(1 + \alpha\,\mathrm{N}\,\mathfrak{l}^{-s}\big)(1 - \mathrm{N}\,\mathfrak{l}^{-s})^{\alpha}$.

Proposition 5.2.4. *The function $H(s)$ converges at $s = 1$, with $H(1) \neq 0$.*

Proof. The logarithm of the $\mathfrak{l}$-th factor of $H(s)$ is $\log(1 + \alpha\,\mathrm{N}\,\mathfrak{l}^{-s}) + \alpha \log(1 - \mathrm{N}\,\mathfrak{l}^{-s}) = -\frac{\alpha^2+\alpha}{2}\,\mathrm{N}\,\mathfrak{l}^{-2s} + O(\mathrm{N}\,\mathfrak{l}^{-3\Re s})$, and $\sum_{\mathfrak{l}} \mathrm{N}\,\mathfrak{l}^{-2}$ converges. Thus $H(s)$ converges at $s = 1$ and $H(1) \neq 0$. □

In order to use the Selberg–Delange method, we need to know the behavior of $\zeta_{\mathcal{L}}(s)$ near $s = 1$ and the following proposition tells us that.

Proposition 5.2.5. *As $s \to 1^+$, we have $\zeta_{\mathcal{L}}(s) \sim C(s-1)^{-\delta}$ for some $C > 0$.*

Proof. Near $s = 1$, $\log \zeta_{\mathcal{L}}(s) = \sum_{\mathfrak{l}\in\mathcal{L}} \mathrm{N}\,\mathfrak{l}^{-s} + O(1)$. Since $\mathcal{L}$ has density $\delta > 0$ and $\sum_{\mathfrak{l}} \mathrm{N}\,\mathfrak{l}^{-s} = \log \frac{1}{s-1} + O(1)$ by the prime ideal theorem, partial summation gives $\sum_{\mathfrak{l}\in\mathcal{L}} \mathrm{N}\,\mathfrak{l}^{-s} = \delta \log \frac{1}{s-1} + O(1)$. □

Hence $F(s) = \zeta_{\mathcal{L}}(s)^{\alpha} H(s)$ has a singularity of type $(s-1)^{-(p-1)\delta}$ at $s = 1$, and the Selberg–Delange method ([Ten15, Thm. 5.2]; see also [PTBZ25]) gives as $X \to \infty$

$$\sum_{\mathrm{N}\,\mathfrak{f} \le X} a(\mathfrak{f}) \;\sim\; \frac{H(1)}{\Gamma\big((p-1)\delta\big)}\, X (\log X)^{(p-1)\delta - 1}. \tag{19}$$

Therefore, we get the following asymptotics for $\#\mathcal{X}_p(X, \mathcal{L})$ as $X \to \infty$.

Proposition 5.2.6. *As $X \to \infty$, we have*

$$\#\mathcal{X}_p(X, \mathcal{L}) \sim \frac{H(1)}{\Gamma((p-1)\delta)}\, X (\log X)^{(p-1)\delta - 1}.$$

For the stable family $\mathcal{X}_{\mathrm{stable}}(X, \mathcal{L})$, we need the same count with the Euler factors at the primes dividing $\mathfrak{m}_k$ removed. We record it as a proposition below.

Proposition 5.2.7. *Let*

$$P(Y) := \sum_{\substack{(\mathfrak{f}',\mathfrak{m}_k)=1 \\ \mathrm{N}\,\mathfrak{f}'\le Y}} a(\mathfrak{f}'),$$

where the sum is taken over squarefree $\mathfrak{f}'$ supported on $\mathcal{L}$ which are coprime to $\mathfrak{m}_k$. Then as $Y\to\infty$

$$P(Y) \;\sim\; c_1\, Y(\log Y)^{(p-1)\delta-1},$$

where $c_1 = \frac{H(1)}{\Gamma((p-1)\delta)}\prod_{\mathfrak{l}\mid\mathfrak{m}_k}\left(1+\frac{p-1}{\mathrm{N}\,\mathfrak{l}}\right)^{-1} > 0$.

Proof. The Dirichlet series of P is obtained from $F(s)$ by deleting the finitely many Euler factors at the primes dividing $\mathfrak{m}_k$:

$$\sum_{(\mathfrak{f}',\mathfrak{m}_k)=1}\frac{a(\mathfrak{f}')}{\mathrm{N}\,\mathfrak{f}'^{\,s}} = \prod_{\substack{\mathfrak{l}\in\mathcal{L}\\ \mathfrak{l}\nmid\mathfrak{m}_k}}\left(1+\frac{\alpha}{\mathrm{N}\,\mathfrak{l}^{\,s}}\right) = F(s)\cdot\prod_{\mathfrak{l}\mid\mathfrak{m}_k}\left(1+\frac{\alpha}{\mathrm{N}\,\mathfrak{l}^{\,s}}\right)^{-1}.$$

Each removed factor $\left(1+\alpha\,\mathrm{N}\,\mathfrak{l}^{-s}\right)^{-1}$ is holomorphic and non-zero at $s=1$ (as $1+\alpha\,\mathrm{N}\,\mathfrak{l}^{-1}>0$). Consequently, near $s=1$, the series is $(s-1)^{-(p-1)\delta}$ times a function holomorphic and non-zero at $s=1$, with the same leading constant as $F(s)$ multiplied by $\prod_{\mathfrak{l}\mid\mathfrak{m}_k}(1+\alpha\,\mathrm{N}\,\mathfrak{l}^{-1})^{-1}$. The Selberg–Delange method then gives the stated asymptotic. □

We can now count the stable family. Fix A and write $\mathfrak{f}'_A=\prod_{j\in A}\mathfrak{l}_j$. A character lies in $S_A(X)$ when it is non-trivial on each factor from A, and arbitrary on the $\mathfrak{m}_k$-factors. This gives

$$\#S_A(X) = p^{\omega(\mathfrak{m}_k)}(p-1)^{|A|}. \tag{20}$$

Summing over A, and using the bijection $A\leftrightarrow\mathfrak{f}'_A$ between finite subsets of $\{k+1,k+2,\dots\}$ and squarefree $\mathfrak{f}'$ coprime to $\mathfrak{m}_k$, we get

$$\#\mathcal{X}_{\mathrm{stable}}(X) = p^{\omega(\mathfrak{m}_k)}\sum_{\substack{(\mathfrak{f}',\mathfrak{m}_k)=1\\ \mathrm{N}\,\mathfrak{f}'\le X/\mathrm{N}\,\mathfrak{m}_k}}(p-1)^{\omega(\mathfrak{f}')} = p^{\omega(\mathfrak{m}_k)}P\!\left(\tfrac{X}{\mathrm{N}\,\mathfrak{m}_k}\right). \tag{21}$$

By Proposition 5.2.7 the ratio in Proposition 5.2.3 converges:

$$\frac{\#\mathcal{X}_{\mathrm{stable}}(X)}{\#\mathcal{X}_{\mathrm{stable}}(\mathrm{N}\,\mathfrak{m}_kX)} = \frac{p^{\omega(\mathfrak{m}_k)}P(X/\mathrm{N}\,\mathfrak{m}_k)}{p^{\omega(\mathfrak{m}_k)}P(X)} \xrightarrow{X\to\infty} \frac{1}{\mathrm{N}\,\mathfrak{m}_k} > 0,$$

since $P(X/\mathrm{N}\,\mathfrak{m}_k)/P(X)\to(\mathrm{N}\,\mathfrak{m}_k)^{-1}$ as $X\to\infty$. Combining this with Proposition 5.2.3, we get

$$\liminf_{X\to\infty}\frac{\#\{K\in\mathcal{K}_p(X,\mathcal{L}) : v_p(\#\,\mathrm{Cl}(H_K)_{\chi_0})=e\}}{\#\mathcal{K}_p(X,\mathcal{L})} \;=\; \liminf_{X\to\infty}\frac{\#\mathcal{X}_{\min}(X)}{\#\mathcal{X}_p(X)} \;\ge\; \frac{1}{t\,\mathrm{N}\,\mathfrak{m}_k} \;>\; 0,$$

which proves Theorem 2a.

5.3 Count over an arbitrary totally real field F

We now turn to proving Theorem 2b, which repalces the base field $\mathbb{Q}$ with an arbitrary totally real field F. Our goal is to prove: when ordered by the norm of the conductor, a positive proportion of fields $K\in\mathcal{K}_p(\mathcal{L})$ satisfy $v_p(\#\,\mathrm{Cl}(H_K)_{\chi_0})=e$, now with $e=v_p(\#\,\mathrm{Cl}(H)_{\chi_0})+f(p-1)m$. We keep the notation of the previous section.

The combinatorial half of argument (i.e. §5.2.1) over $\mathbb{Q}$ carries over to F. Kriz–Nordentoft's density theorem, Proposition 5.1.1, Lemmas 5.2.1 and 5.2.2, and Proposition 5.2.3 all hold over an arbitrary totally real field F. Thus we have as before,

$$\frac{\#\{K\in\mathcal{K}_p(X,\mathcal{L}) : v_p(\#\,\mathrm{Cl}(H_K)_{\chi_0})=e\}}{\#\mathcal{K}_p(X,\mathcal{L})} \;\ge\; \frac{1}{t}\cdot\frac{\#\mathcal{X}_{\mathrm{stable}}(X)}{\#\mathcal{X}_{\mathrm{stable}}(\mathrm{N}\,\mathfrak{m}_kX)}. \tag{22}$$

The only step that used $F = \mathbb{Q}$ was the count (20)–(21) of $\#\mathcal{X}_{\text{stable}}(X)$, where we used that the number of order-p characters ramified exactly at $\mathfrak{f}$ is given by $(p-1)^{\omega(\mathfrak{f})}$. This followed from the fact that over $\mathbb{Q}$, the Galois group $\mathrm{Gal}(H_n/H) = \Delta_n$ was isomorphic to $\prod_{i=1}^{n} \mathbb{Z}/p$. This is no longer true over F due to the influence of global units $\mathcal{O}_F^\times$ and the p-part $\mathrm{Cl}(F)_{(p)}$ of the class group of F. However, we argue in this section that this finite influence causes only the constants to change in the asymptotics. In other words, we show that beyond a finite set of orderly primes, the growth of the Galois groups Δ_n is similar to the $F = \mathbb{Q}$ case. In the following subsection, we explain the finite defects coming from F.

5.3.1 The ray class group and its finite defects

For a squarefree product $\mathfrak{f}$ of orderly primes, let $\mathrm{Cl}_{\mathfrak{f}}(F)$ be the p-part of the ray class group of F of conductor $\mathfrak{f}$. Thus if $\mathfrak{f} = \mathfrak{l}_1\mathfrak{l}_2\cdots\mathfrak{l}_n$, then $\mathrm{Cl}_{\mathfrak{f}}(F) = \mathrm{Gal}(H_n/H) = \Delta_n$. In general, $\mathrm{Cl}_{\mathfrak{f}}(F)$ is a finite abelian p-group, and the order-p characters of Δ_∞ unramified outside $\mathfrak{f}$ are precisely the non-trivial homomorphisms $\mathrm{Cl}_{\mathfrak{f}}(F) \to \mu_p$. Denote their number by

$$N_F(\mathfrak{f}) := \#\mathrm{Hom}(\mathrm{Cl}_{\mathfrak{f}}(F), \mathbb{Z}/p) = p^{\mu(\mathfrak{f})}, \qquad \mu(\mathfrak{f}) := \dim_{\mathbb{F}_p} \mathrm{Cl}_{\mathfrak{f}}(F)/p\,\mathrm{Cl}_{\mathfrak{f}}(F),$$

where $\mu(\mathfrak{f})$ is the minimal number of generators of $\mathrm{Cl}_{\mathfrak{f}}(F)$. For comparison, when $F = \mathbb{Q}$ we have $N_{\mathbb{Q}}(\mathfrak{f}) = p^{\omega(\mathfrak{f})}$. We have the following exact sequence from class field theory

$$\mathcal{O}_F^\times \xrightarrow{\lambda_{\mathfrak{f}}} \bigoplus_{\mathfrak{l}\mid\mathfrak{f}} V_{\mathfrak{l}} \xrightarrow{\iota_{\mathfrak{f}}} \mathrm{Cl}_{\mathfrak{f}}(F) \longrightarrow \mathrm{Cl}(F)_{(p)} \longrightarrow 0, \tag{23}$$

where $V_{\mathfrak{l}} := \left((\mathcal{O}_F/\mathfrak{l})^\times\right)_{(p)}$ is the p-part of the residue field's unit group. By Dirichlet's unit theorem $\mathcal{O}_F^\times$ has free $\mathbb{Z}$-rank $r := [F:\mathbb{Q}] - 1$ (as F is totally real), so the image of $\lambda_{\mathfrak{f}}$ is generated by at most r elements. Denote the p-rank of $\mathrm{Cl}(F)_{(p)}$ by $\rho := \mathrm{rk}_p\,\mathrm{Cl}(F)$.

Over $F = \mathbb{Q}$ the two defects vanish: $\mathcal{O}_{\mathbb{Q}}^\times = \{\pm 1\}$ has trivial p-part (as p is odd), and $\mathrm{Cl}(\mathbb{Q})$ is trivial, so $r = \rho = 0$. The short exact sequence (23) gives rise to an isomorphism $\mathcal{R}_{\mathfrak{f}} \cong \bigoplus_{\mathfrak{l}\mid\mathfrak{f}} V_{\mathfrak{l}}$, which is why we get $a(\mathfrak{f}) = (p-1)^{\omega(\mathfrak{f})}$. For a general F the defects are present but bounded, and the next subsection shows that they perturb the count by only a constant.

5.3.2 Generator ranks and bounded defects

For a finite abelian p-group G, denote the minimal number of generators by $\mu(G) = \dim_{\mathbb{F}_p} G/pG$. We use the following well-known fact about $\mu(G)$:

Fact: If $0 \to H \to G \to Q \to 0$ is a short exact sequence of finite abelian p-groups, then

$$\mu(Q) \;\le\; \mu(G) \;\le\; \mu(Q) + \mu(H).$$

The following lemma uses this fact to compare $\mu(\mathfrak{f}) := \mu(\mathrm{Cl}_{\mathfrak{f}}(F))$ and $\omega(\mathfrak{f})$.

Lemma 5.3.1. *Let $r = [F:\mathbb{Q}] - 1$ and $\rho = \mathrm{rk}_p\,\mathrm{Cl}(F)$. For every squarefree product $\mathfrak{f}$ of orderly primes, we have*

$$\omega(\mathfrak{f}) - r \;\le\; \mu(\mathfrak{f}) \;\le\; \omega(\mathfrak{f}) + \rho.$$

Proof. The sum $\bigoplus_{\mathfrak{l}\mid\mathfrak{f}} V_{\mathfrak{l}}$ of cyclic groups has $\mu = \omega(\mathfrak{f})$, and since r is the unit rank of F, we have $\mu(\mathrm{im}(\lambda_f)) \le \omega(\mathfrak{f})$ by (23). Applying the above fact to the short exact sequence $0 \to \mathrm{im}(\lambda_{\mathfrak{f}}) \to \bigoplus_{\mathfrak{l}\mid\mathfrak{f}} V_{\mathfrak{l}} \to \mathrm{im}(\iota_{\mathfrak{f}}) \to 0$, we get

$$\omega(\mathfrak{f}) - r \;\le\; \mu(\mathrm{im}(\lambda_{\mathfrak{f}})) \;\le\; \omega(\mathfrak{f}).$$

Since $\mathrm{im}(\lambda_{\mathfrak{f}}))$ is a subgroup of $\mathrm{Cl}_{\mathfrak{f}}(F)$, this proves the left inequality of the lemma. For the other inequality, we apply the above fact to

$$0 \to \mathrm{im}(\lambda_{\mathfrak{f}}) \to \mathrm{Cl}_{\mathfrak{f}}(F) \to \mathrm{Cl}(F)_{(p)} \to 0$$

and obtain

$$\mu(\mathfrak{f}) \le\; \mu(\mathrm{im}(\lambda_{\mathfrak{f}})) + \mu(\mathrm{Cl}(F)_p) \;\le\; \omega(\mathfrak{f}) + \rho.$$

□

The above lemma thus shows that the quantities $\mu(\mathfrak{f})$ and $\omega(\mathfrak{f})$ can differ at most by comstants r and ρ coming from F. The lemma that follows is the key result of this section, and it that shows that $\mu(\mathfrak{f})$ and $\omega(\mathfrak{f})$ behave the same way beyond a finite defect B.

Lemma 5.3.2. *There is a finite set B of orderly primes such that, for every squarefree product $\mathfrak{f}$ of orderly primes with $B \subset supp(\mathfrak{f})$ and every orderly prime $\mathfrak{l}' \nmid \mathfrak{f}$, we have*

$$\mu(\mathfrak{f}\mathfrak{l}') = \mu(\mathfrak{f}) + 1.$$

Consequently, we have $N_F(\mathfrak{f}\mathfrak{l}') = p \cdot N_F(\mathfrak{f})$.

Proof. Let $\mathfrak{f}$ be a squarefree product of orderly primes, and $\mathfrak{l}'$ be an orderly prime coprime to $\mathfrak{f}$. Since $\mu(\bigoplus_{\mathfrak{l}|\mathfrak{f}\mathfrak{l}'} V_\mathfrak{l}) = \mu(\bigoplus_{\mathfrak{l}|\mathfrak{f}} V_\mathfrak{l}) + 1$, we have

$$\mu(\mathfrak{f}) \;\le\; \mu(\mathfrak{f}\mathfrak{l}') \;\le\; \mu(\mathfrak{f}) + 1.$$

In other words, the minimal number of generators of $\mathrm{Cl}_{\mathfrak{f}\mathfrak{l}'}(F)$ can grow by at most 1. Define the defect

$$D(\mathfrak{f}) := \mu(\mathfrak{f}) - \omega(\mathfrak{f}) + r$$

By Lemma 5.3.1, we have

$$0 \;\le\; D(\mathfrak{f}) \;\le\; \rho + r.$$

Since $D(\mathfrak{f}\mathfrak{l}') \le D(\mathfrak{f})$, we get that the function D is integer-valued, bounded, and non-increasing in $\omega(\mathfrak{f})$. Thus $D(\mathfrak{f})$ stabilizes to its minimum. In other words, there exists a finite set B of orderly primes, such that for any $\mathfrak{f}$ with $B \subset \mathrm{supp}(\mathfrak{f})$ and any $\mathfrak{l}' \nmid \mathfrak{f}$, we have $D(\mathfrak{f}\mathfrak{l}') = D(\mathfrak{f})$. Equivalently, $\mu(\mathfrak{f}\mathfrak{l}') = \mu(\mathfrak{f}) + 1$. □

5.3.3 Counting via the Selberg–Delange method over F

We now fix a finite set B as in the above lemma, and enlarge k so that $\{\mathfrak{l}_1, \mathfrak{l}_2, \dots, \mathfrak{l}_k\}$ contains B as well as all the prime factors of $\mathfrak{f}(\eta_1), \mathfrak{f}(\eta_2), \dots, \mathfrak{f}(\eta_t)$, where $M = \{\eta_1, \eta_2, \dots, \eta_t\}$ is as in §5.2.1. We use the same notation as before: for a finite set $A \subset \{k+1, k+2, \dots\}$, let

$$\mathfrak{f}_A := \prod_{j \in A} \mathfrak{l}_j, \qquad \mathfrak{m}_k = \prod_{i=1}^{k} \mathfrak{l}_i,$$

and

$$S_A := \Big\{\text{order-}p \text{ characters } \eta \text{ of } \bar{\Delta}_\infty : \mathfrak{f}_A \mid \mathfrak{f}(\eta) \mid \mathfrak{m}_k\mathfrak{f}_A\Big\}.$$

Let $\mathfrak{f}$ be a squarefree product of orderly primes. For an orderly prime $\mathfrak{l}' \nmid \mathfrak{f}$, allowing extra ramification at $\mathfrak{l}'$ enlarges the ray class field, giving a surjection

$$\mathrm{Cl}_{\mathfrak{f}\mathfrak{l}'}(F) \;\twoheadrightarrow\; \mathrm{Cl}_{\mathfrak{f}}(F), \tag{24}$$

whose kernel is the image of the inertia $V_{\mathfrak{l}'}$ at the new prime, and is therefore cyclic. Using this fact from class field theory, we prove the following lemma about the kernel of $\mathrm{Cl}_{\mathfrak{f}\mathfrak{f}_A}(F)/p \twoheadrightarrow \mathrm{Cl}_{\mathfrak{f}}(F)/p$ as an $\mathbb{F}_p$-vector space.

Lemma 5.3.3. *Let $A \subset \{k+1, k+2, \dots\}$ be a finite set and let $\overline{\sigma}_{\mathfrak{l}_j} \in \mathrm{Cl}_{\mathfrak{m}_k\mathfrak{f}_A}(F)/p$ denote the class of a generator of the inertia subgroup $\iota_{\mathfrak{m}_k\mathfrak{f}_A}(V_{\mathfrak{l}_j})$ for each $j \in A$. Then the reduction map $\pi : \mathrm{Cl}_{\mathfrak{m}_k\mathfrak{f}_A}(F)/p \to \mathrm{Cl}_{\mathfrak{m}_k}(F)/p$ is surjective, its kernel is spanned by $\{\overline{\sigma}_{\mathfrak{l}_j} : j \in A\}$, and these $\omega(\mathfrak{f}_A) = |A|$ classes form an $\mathbb{F}_p$-basis of $\ker\pi$. In particular, they are linearly independent in $\mathrm{Cl}_{\mathfrak{m}_k\mathfrak{f}_A}(F)/p$, and there is a short exact sequence of $\mathbb{F}_p$-vector spaces*

$$0 \;\longrightarrow\; \bigoplus_{j \in A} \mathbb{F}_p\, \overline{\sigma}_{\mathfrak{l}_j} \;\longrightarrow\; \mathrm{Cl}_{\mathfrak{m}_k\mathfrak{f}_A}(F)/p \xrightarrow{\;\pi\;} \mathrm{Cl}_{\mathfrak{m}_k}(F)/p \;\longrightarrow\; 0. \tag{25}$$

Proof. Repeatedly applying the surjection (24) one prime at a time gives

$$0 \longrightarrow N \longrightarrow \mathrm{Cl}_{\mathfrak{m}_k\mathfrak{f}_A}(F) \xrightarrow{\pi} \mathrm{Cl}_{\mathfrak{m}_k}(F) \longrightarrow 0. \tag{26}$$

where the kernel N is generated by inertia at the primes in A. This is a short exact sequence of finite abelian p-groups, and it need not split. Since $\otimes_{\mathbb{Z}} \mathbb{Z}/p$ is right exact, applying it to the above sequence gives

$$N/pN \longrightarrow \mathrm{Cl}_{\mathfrak{m}_k\mathfrak{f}_A}(F)/p \xrightarrow{\pi} \mathrm{Cl}_{\mathfrak{m}_k}(F)/p \longrightarrow 0. \tag{27}$$

Since N is generated by the cyclic inertia subgroups $\iota_{\mathfrak{m}_k\mathfrak{f}_A}(V_\mathfrak{l})$ for $\mathfrak{l} \in A$, the image of π is spanned by the classes $\overline{\sigma}_\mathfrak{l}$, with $\mathfrak{l} \in A$. We that these classes are also linearly independent over $\mathbb{F}_p$ by a dimension count.

$$\dim_{\mathbb{F}_p}(\ker(\pi)) \;=\; \dim_{\mathbb{F}_p}(\mathrm{Cl}_{\mathfrak{m}_k\mathfrak{f}_A}(F)/p) - \dim_{\mathbb{F}_p}(\mathrm{Cl}_{\mathfrak{m}_k}(F)/p) \;=\; \mu(\mathfrak{m}_k\mathfrak{f}_A) - \mu(\mathfrak{m}_k)$$

By Lemma 5.3.2 we have $\mu(\mathfrak{m}_k\mathfrak{f}_A) - \mu(\mathfrak{m}_k) = \omega(\mathfrak{f}_A) = |A|$. Therefore, the set of classes $\{\overline{\sigma}_\mathfrak{l} : \mathfrak{l} \in A\}$ is an $\mathbb{F}_p$-basis of N/pN which gives the short exact sequence of $\mathbb{F}_p$-vector spaces in 25. □

For the A as in the above lemma, we can extend the $\mathbb{F}_p$-basis of $\ker\pi$ to a basis of $\mathrm{Cl}_{\mathfrak{m}_k\mathfrak{f}_A}(F)/p$ by adding $\mu(\mathfrak{m}_k\mathfrak{f}_A) - \mu(\mathfrak{m}_k) = |A|$ many vectors. A character $\eta : \mathrm{Cl}_{\mathfrak{m}_k\mathfrak{f}_A}/p \to \mu_p$ is characterized by its values on such a basis of $\mathrm{Cl}_{\mathfrak{m}_k\mathfrak{f}_A}(F)/p$, and η is ramified at $\mathfrak{f}_A$ if and only if $\eta(\overline{\sigma}_{\mathfrak{l}_j}) \neq 0$ for all $j \in A$. We therefore get the following count

$$\#S_A(X) = p^{\mu(\mathfrak{m}_k)}\ (p-1)^{|A|}.$$

Note that the only difference from the counts over $F = \mathbb{Q}$ is that we have $p^{\mu(\mathfrak{m}_k)}$ replacing $p^{\omega(\mathfrak{m}_k)}$. Therefore, we get the analogue of equation (21) over F

$$\#\mathcal{X}_{\mathrm{stable}}(X) = p^{\mu(\mathfrak{m}_k)} \sum_{\substack{(\mathfrak{f}',\mathfrak{m}_k)=1 \\ \mathrm{N}\,\mathfrak{f}' \le X/\mathrm{N}\,\mathfrak{m}_k}} (p-1)^{\omega(\mathfrak{f}')} = p^{\mu(\mathfrak{m}_k)} P\Big(\tfrac{X}{\mathrm{N}\,\mathfrak{m}_k}\Big).$$

It follows that

$$\frac{\#\mathcal{X}_{\mathrm{stable}}(X)}{\#\mathcal{X}_{\mathrm{stable}}(\mathrm{N}\,\mathfrak{m}_k X)} = \frac{p^{\mu(\mathfrak{m}_k)}P(X/\mathrm{N}\,\mathfrak{m}_k)}{p^{\mu(\mathfrak{m}_k)}P(X)} \xrightarrow{X\to\infty} \frac{1}{\mathrm{N}\,\mathfrak{m}_k} > 0,$$

which gives

$$\liminf_{X\to\infty} \frac{\#\{K \in \mathcal{K}_p(X,\mathcal{L}) : v_p(\#\,\mathrm{Cl}(H_K)_{\chi_0}) = e\}}{\#\mathcal{K}_p(X,\mathcal{L})} \;=\; \liminf_{X\to\infty} \frac{\#\mathcal{X}_{\min}(X)}{\#\mathcal{X}_p(X)} \;\ge\; \frac{1}{t\,\mathrm{N}\,\mathfrak{m}_k} \;>\; 0.$$

References

[DK23] Samit Dasgupta and Mahesh Kakde. On the Brumer-Stark conjecture. *Ann. of Math. (2)*, 197(1):289–388, 2023.

[DKSW23] Samit Dasgupta, Mahesh Kakde, Jesse Silliman, and Jiuya Wang. The Brumer–Stark Conjecture over Z. Preprint, arXiv:2310.16399 [math.NT] (2023), 2023.

[DR80] Pierre Deligne and Kenneth A. Ribet. Values of abelian l-functions at negative integers over totally real fields. *Inventiones Mathematicae*, 59(3):227–286, 1980.

[Eis13] David Eisenbud. *Commutative algebra: with a view toward algebraic geometry*. Springer Science & Business Media, 2013.

[GP25] Rusiru Gambheera and Cristian D. Popescu. Equivariant iwasawa theory for ritter-weiss modules and applications, 2025.

[KN24] Daniel Kriz and Asbjørn Christian Nordentoft. Horizontal p-adic l-functions, 2024.

[KW20] Jürgen Klüners and Jiuya Wang. ℓ-torsion bounds for the class group of number fields with an ℓ-group as galois group, 2020.

[PTBZ25] Lillian B. Pierce, Caroline L. Turnage-Butterbaugh, and Asif Zaman. A guide to tauberian theorems for arithmetic applications, 2025.

[PY26] Cristian D. Popescu and Wei Yin. Fitting ideals of projective limits of modules over non-noetherian iwasawa algebras, 2026.

[RW96] Jürgen Ritter and Alfred Weiss. A Tate sequence for global units. *Compositio Math.*, 102(2):147–178, 1996.

[Smi23] Alexander Smith. The distribution of ℓ^∞-selmer groups in degree ℓ twist families i, 2023.

[Ten15] Gérald Tenenbaum. *Introduction to analytic and probabilistic number theory*, volume 163 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, third edition, 2015.

[TW95] Richard Taylor and Andrew Wiles. Ring-theoretic properties of certain hecke algebras. *Annals of Mathematics*, 141(3):553–572, 1995.