Grothendieck-Lefschetz fixed point theorem

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Outline

- Lefschetz FPT
- Examples and applications
- Varieties over $\mathbb{F}_p$ and the Frobenius map
- Grothendieck-Lefschetz FPT
- Weil conjectures

Fixed Point Theorems

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Natural to ask for generalizations

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Theorem (Lefschetz)

Let X be a finite simplicial complex and $f : X \to X$ be a continuous map. Define the Lefschetz number of f by

$$\Lambda_f = \sum_{i \geq 0} (-1)^i tr(f_*|_{H_i(X,\mathbb{Q})})$$

If $\Lambda_f \neq 0$ then f has a fixed point.

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In fact, if we work with H^* instead of H_*, then Λ_f is the intersection number of Δ and Γ_f in $X \times X$.

Lefschetz applications

Lefschetz $\implies$ Brouwer

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Let $f : D^2 \to D^2$ be a continuous map
Only the 0^{th} homology of D^2 is non-zero (isomorphic to $\mathbb{Z}$)
The induced map on 0^{th} homology has non-zero trace and therefore by Lefschetz FPT, f has a fixed point

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Why study them?

They are of interest in number theory (among many other fields). For instance, the points over $\mathbb{Q}$ of $X^2 + Y^2 - 1$ give Pythagorean triples.

Varieties over $\mathbb{F}_p$ and the Frobenius map

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We are interested in counting $\#V(\mathbb{F}_p) = \#Fix(Fr_p)$
We will be able to do this if we have an analogue of the Lefschetz fixed point theorem in the setting of varieties over finite fields

New cohomology theory

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For X a variety over $\mathbb{F}_p$, the cohomology theory should be such that Fr_p acts on $H^i(X, K)$ where K is a field of characteristic 0

Let k and K be fields with $char(K) = 0$

A Weil cohomology theory is a contravariant functor from the category of smooth projective varieties over k to the category of graded $K-$algebras, satisfying some axioms

The axioms include Poincare duality and Kunneth formula

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Theorem (Artin Comparison)

Let X be a smooth proper variety over $\mathbb{C}$. For any $i \geq 0$, we have an isomorphism

$$H^i_{et}(X, \mathbb{Q}_l) \cong H^i_{sing}(X(\mathbb{C}), \mathbb{Q}_l)$$

Theorem

Let X be a smooth, proper variety over $\mathbb{F}_p$. Then

$$X(\mathbb{F}_p) = \sum_i (-1)^i tr(Fr_p | H^i(\bar{X}, \mathbb{Q}_l))$$

Weil Conjectures

These are conjectures about the generating function involving the above point counts

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The Grothendieck-Lefschetz FPT gives a formula for N_m in terms of traces, which is the main step in proving that $Z(X,T)$ is rational

References/Sources

Thanks to Eugene Eyeson for useful discussions and references

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