Locally Nilpotent Derivations and the Cancellation Theorem

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Let k be an algebraically closed field of characteristic 0. The $n = 1$ case of the Cancellation problem asserts the following:
If R is a k-algebra such that $k[x, y] \cong R[t]$ as k-algebras then $R \cong k[z]$.
In this report, we develop some basic theory of Locally Nilpotent Derivations and prove the above theorem.

Locally Nilpotent Derivations

Let B be an integral domain containing a field k of characteristic 0. Let B^* denote the group of units of B and frac(B) denote the field of fractions of B.

Definition 1 *A derivation $D : B \mapsto B$ is a function which satisfies the following properties for all $a, b \in B$:*
$D(a + b) = D(a) + D(b)$
$D(ab) = aD(b) + bD(a)$

The set of derivations on B is denoted by $Der(B)$. If A is a subring of B, then $Der_A(B)$ denotes the subset of derivations D such that $D(A) = 0$.
In particular, a derivation $D \in Der(B)$ is said to be a locally nilpotent derivation (lnd) if for every $b \in B$ there exists a positive integer n such that $D^n(b) = 0$. Here, $D^n(b)$ is D composed n times on b. The set of all locally nilpotent derivations on B is denoted by $LND(B)$.

Definition 2 *A subring A of B is said to be factorially closed in B if for all non-zero $f, g \in B$, the product $fg \in A \implies f, g \in A$*

Proposition 1 *Let $D \in LND(B)$. The subring $ker(D)$ of B is factorially closed in B*

Proof: Let $a, b \in B$ such that $ab \in ker(D)$. Since $D \in LND(B)$, there exists positive integers m, n such that $D^m(a) = D^n(b) = 0$ but $D^{m-1}(a)$ and $D^{n-1}(b) \neq 0$.
Suppose $a \notin ker(D)$ so that $m > 1$.
Look at

$$D^{m+n-2}(ab) = \sum_{i+j=m+n-2} \binom{m+n-2}{i} D^i(a) D^j(b)$$

LHS is 0 and the only term contributing to the RHS is when $i = m-1$ and $j = n-1$.

$$\implies D^{m-1}(a) D^{n-1}(b) = 0$$

Contradicting $D^{n-1}(b) \neq 0$.
Therefore we get the proposition.

Definition 3 *Given $D \in LND(B)$ with a local slice r (i.e $Dr \neq 0$ and $D^2 r = 0$), the Dixmier map induced by r is $\pi_r : B \to B_{Dr}$, where:*

$$\pi_r(f) = \sum_{i \geq 0} \frac{(-1)^i}{i!} D^i f \frac{r^i}{(Dr)^i}$$

and the exponential map determined by D is $exp(D) : B \to B$, where:

$$exp(D)(f) = \sum_{i \geq 0} \frac{1}{i!} D^i f$$

Proposition 2 *Let $D \in LND(B)$ be given, $D \neq 0$, and set $A = ker(D)$. Choose a local slice $r \in B$ of D, and let $\pi_r : B \to B_{Dr}$ denote the Dixmier map defined by r.*
a) $\pi_r(B) \subset A_{Dr}$
b) π_r is a k-algebra homomorphism.
c) $ker \pi_r = rB_{Dr} \cap B$
d) $B_{Dr} = A_{Dr}[r]$
e) The transcendence degree of B over A is 1.

Proof: Consider first the case $Ds = 1$ for some $s \in B$.
For a), we have:

$$\pi_s(h) = \sum_{i \geq 0} \frac{(-1)^i}{i!} D^i h \frac{s^i}{(Ds)^i}$$

Therefore, it is easy to see that $D(\pi_s(h)) = 0$ for all $h \in B$. Therefore, $\pi_s(B) \subset A = A_{Ds}$
For b), let t be transcendental over B, and extend D to $B[t]$ by setting $Dt = 0$. Let $\imath : B \to B[t]$ be the inclusion, and let $\epsilon : B[t] \to B$ be the evaluation map $\epsilon(t) = s$. Then $exp(-tD)$ is an automorphism of $B[t]$. In addition, $\pi_s = \epsilon \circ exp(-tD) \circ \imath$. Therefore, π_s is a homomorphism.
For c), note that $\pi_s(s) = s - (Ds)s = 0$. Therefore, $\pi(sB) = 0$. Conversely, if $\pi_s(f) = 0$, then since $\pi_s(f) = f + sb$ for some $b \in B$, we conclude that $f \in sB$. Therefore, $ker(\pi_s) = sB$ when $Ds = 1$.
Next, since the kernel of D on $B[t]$ equals $A[t]$, π_s extends to a homomorphism $\pi_s : B[t] \to A[t]$. Define the homomorphism $\phi : B \to A[s]$ by $\phi = \epsilon \circ \pi_s \circ exp(tD) \circ \imath$. Specifically, ϕ is defined by:

$$\phi(g) = \sum_{n \geq 0} \frac{1}{n!} \pi_s(D^n g) s^n$$

Then ϕ is a surjection, since $\phi(a) = a$ for all $a \in A$ and $\phi(s) = s$. Also, if $\phi(g) = 0$, then since s is transcendental over A, it follows that each coefficient of $\phi(g) \in A[s]$ is zero. If $g \neq 0$, then the highest degree coefficient of $\phi(g)$ equals $\frac{1}{n!}\pi_s(D^n g)$, where $n = deg_D(g) \geq 0$. Thus, $D^n g \in ker(\pi_s) = sB$, and since also $D^n g \in A - 0$, we conclude that $s \in A$ (since A is factorially closed). But $s \notin A$, so it must be the case that $g = 0$. Therefore, ϕ is an isomorphism which proves d).
For the general case when there need not exist an s such that $Ds = 1$, suppose that, for the local slice r, $Dr = f \in A$. Let D_f denote the extension of D to B_f. Then $s := \frac{r}{f}$ is a slice of D_f. Since π_r is the restriction to B of the homomorphism $\pi_s : B_f \to B_f$, it follows that π_r is a homomorphism. The kernel is $sB_f \cap B = rB_f \cap B$, and $B_f = A_f[s] = A_f[r]$. Therefore, results a)-d) hold in the general case. Part e) follows from d).

Proposition 3 *If B is a UFD and A is a factorially closed subring of B, then A is a UFD.*

Proof: Suppose that $a \in A$ is a prime in B. If $fg \in aA$ for $f, g \in A$, then either $f \in aB \cap A = aA$ or $g \in aB \cap A = aA$. So a is prime in A as well.
Given a nonzero $h \in A - A*$, write $h = b_1 b_2 \dots b_n$, where b_i are primes in B. Since A is factorially closed, $b_i \in A$ which implies that b_i are primes in A. Since this factorization is unique in B, it is also unique in A. Therefore, A is a UFD.

Lemma 1 *Let A be a factorially closed subring of B. Then A is algebraically closed in B.*

Proof: Let $b \in B$ satisfy a polynomial with coefficients in A.

$$\sum_{i=0}^{n} a_i b^i = 0$$

where $a_i \in A$. Then

$$\sum_{i=1}^{n} a_i b^i = -a_0 \in A$$

Therefore, we get $b(\sum_{i=1}^{n} a_i b^{i-1}) \in A$.
A is factorially closed, therefore, we get $b \in A$.
Therefore, we get A is algebraically closed in B.

Theorem 1 *(Cancellation theorem n=1) Let k be an algebraically closed field of characteristic 0 and let R be a k-algebra such that $k[x, y] \cong R[t]$ as k- algebras. Then $R \cong k[z]$ as k-algebras*

Proof: Let ϕ be the isomorphism from $k[x, y]$ to $R[t]$. Let $d = \frac{d}{dt}$ be the usual derivation on $R[t]$. This induces a locally nilpotent derivation D on $k[x, y]$ defined by

$$D(p) := \phi^{-1}(\frac{d}{dt}(\phi(p)))$$

Now since ϕ is an isomorphism, we get $R = ker(d) \cong ker(D) =: A$.
Therefore it is enough to show that A is one-generated. If $A = k[x]$ or $k[y]$ then we are done. Otherwise, pick some irreducible $f \in A$ with the smallest x-degree. We know that $k[x, y]$ has transcendence degree 1 over A by part e) of Proposition 2 and $k[f]$ has transcendence degree 1 over k. Therefore we get that A is algebraic over $k[f]$. We show that $A = k[f]$. We do this by showing that $k[f]$ is algebraically closed in A, which would imply that $A = k[f]$ since A is algebraic over $k[f]$. But by Lemma 1, it is enough to show that $k[f]$ is factorially closed in A.
Suppose $0 \neq g, h \in A$ such that $gh \in k[f]$. Since k is algebraically closed, we can factorize

$$gh = \prod_{i=1}^{n} (f - \lambda_i)$$

where $\lambda_i \in k$
We have $f - \lambda_i \in A$ and since f is irreducible, each $f - \lambda_i$ is also irreducible (if $f - \lambda_i = f_1 f_2 \in A$ then A being factorially closed would imply that both $f_1, f_2 \in A$ contradicting the fact that f is the element of smallest x-degree).
By Proposition 3, A is a UFD and therefore $gh = \prod_{i=1}^{n}(f - \lambda_i)$ is the unique factorization into irreducibles. Therefore,

$$g = \prod_{i \in F}(f - \lambda_i)$$

and

$$h = \prod_{i \in \{1,\ldots,n\} - F}(f - \lambda_i)$$

for some proper subset $F \subset \{1, 2, \ldots, n\}$.
Therefore, both $g, h \in k[f]$ and hence $k[f]$ is factorially closed in A. By Lemma 1, we get that $k[f]$ is algebraically closed in A, which completes the proof of the theorem.

References

1) *Algebraic Theory of Locally Nilpotent Derivations* by Gene Freudenburg