Sharp Identification Regions
in Models with Convex Moment Predictions

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First draft: February 15, 2007. Current draft: March 2, 2011

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Outline

This Supplement includes four appendices. Appendix B establishes that the methodology of Andrews and Shi (2009) can be applied in our context, to obtain confidence sets that uniformly cover each element of the sharp identification region with a prespecified asymptotic probability. Appendix C shows that our approach easily applies also to finite games of incomplete information, and characterizes Θ_I through a finite number of moment inequalities. Appendix D specializes our results in the context of complete information games, to the case that players are restricted to use pure strategies only and Nash equilibrium is the solution concept. In this case, Θ_I is characterized through a finite number of moment inequalities, and further insights are provided on how to reduce the number of inequalities to be checked in order to compute it. Appendix E shows that our methodology is applicable to static simultaneous move finite games regardless of the solution concept used.[1] Appendix F applies the results in Section 2 of Beresteanu, Molchanov, and Molinari (2010, BMM henceforth) to the analysis of individual decision making, looking at random utility models of multinomial choice in the presence of interval regressors data.

B   Applicability of Andrews and Shi's (2009) Generalized Moment Selection Procedure[2]

B.1   Finite Games of Complete and Incomplete Information

Andrews and Shi (2009, Section 9; AS henceforth) consider conditional moment inequality problems of the form $\mathbf{E}\left(m_d\left(y,\underline{x},\theta,u\right)|\underline{x}\right)\geq 0$ for all $u\in B$ $\underline{x}-a.s.$, $d=1,...,D$. They show that the conditional moment inequalities can be transformed into equivalent unconditional moment inequalities, by choosing appropriate weighting functions (instruments) $g\in\mathcal{G}$, with $\mathcal{G}$ a collection of instruments and g that depend on $\underline{x}$. This yields $\mathbf{E}\left(m_d\left(y,\underline{x},\theta,g,u\right)\right)\geq 0$, for all $u\in B$, $g=\left[g_1,...,g_D\right]'\in\mathcal{G}$ and $d=1,...,D$, where $m_d\left(y,\underline{x},\theta,g,u\right)=m_d\left(y,\underline{x},\theta,u\right)g\left(\underline{x}\right)$. In the models that we analyze in BMM Section 3 and in Appendix C below, the conditional moment inequalities are of the " $\leq$ " type, and

$$
\begin{aligned}
m\left(y,\underline{x},\theta,u\right) &= u'\left[1\left(y=t^k\right),\ k=1,...,\kappa_{\mathcal{Y}}\right]-\mathbf{E}\left[h\left(Q_\theta,u\right)|\underline{x}\right], \\
m\left(y,\underline{x},\theta,g,u\right) &= \left(u'\left[1\left(y=t^k\right),\ k=1,...,\kappa_{\mathcal{Y}}\right]-\mathbf{E}\left[h\left(Q_\theta,u\right)|\underline{x}\right]\right)g\left(\underline{x}\right).
\end{aligned}
$$

Notice that $\mathbf{E}\left[h\left(Q_\theta, u\right)|\underline{x}\right]$ is a known (or simulated) function of θ, u and $\underline{x}$, and that for each $u \in B$, we have only one inequality. Notice also that by the positive homogeneity of the support function, our moment inequalities can be written equivalently as $\mathbf{E}\left(m\left(y, \underline{x}, \theta, g, u\right)\right) \leq 0$, for all $g \in \mathcal{G}$ and $u \in S \equiv \{u \in \Re^{\kappa_{\mathcal{Y}}} : \|u\| = 1\}$. Hence, they are invariant to rescaling of the moment function, which is important for finite sample inference (see, e.g., Andrews and Soares (2010)).

In all that follows, to simplify the exposition, we abstract from the choice of $\mathcal{G}$. Once we establish that our problem fits into the general framework of AS, one can choose instruments g as detailed in Section 3 of AS. To avoid ambiguity, in this Section we denote $F\left(y|\underline{x}\right) \equiv \left[\mathbf{P}\left(y = t^k|\underline{x}\right), \; k = 1, ..., \kappa_{\mathcal{Y}}\right]$. We first establish that Θ_I can be equivalently defined using only the first $\kappa_{\mathcal{Y}} - 1$ entries of $\mathcal{Y}$, thereby avoiding the problems for inference associated with linear dependence among the entries of $F\left(y|\underline{x}\right)$ and also lowering the dimension over which the maximization is performed. Let $\widetilde{F}\left(y|\underline{x}\right)$ denote the first $\kappa_{\mathcal{Y}} - 1$ rows of $F\left(y|\underline{x}\right)$, $B^{\kappa_{\mathcal{Y}}-1} = \{u \in \Re^{\kappa_{\mathcal{Y}}-1} : \|u\| \leq 1\}$, $S^{\kappa_{\mathcal{Y}}-1} = \{u \in \Re^{\kappa_{\mathcal{Y}}-1} : \|u\| = 1\}$, and

$$\tilde{Q}_\theta = \left\{\tilde{q} = \left[\left[q\left(\sigma\right)\right]_k, \; k = 1, ..., \kappa_{\mathcal{Y}} - 1\right], \; \sigma \in \text{Sel}(S_\theta)\right\}.$$

Theorem B.1 *Let Assumptions 3.1 in BMM (or C.1 below) and 3.2 in BMM hold. Then*

$$
\begin{aligned}
\tilde{\Theta}_I &\equiv \left\{\theta \in \Theta : \max_{u \in B^{\kappa_{\mathcal{Y}}-1}} \left(u'\widetilde{F}\left(y|\underline{x}\right) - \mathbf{E}\left[h\left(\tilde{Q}_\theta, u\right)|\underline{x}\right]\right) = 0 \;\; \underline{x} - a.s.\right\} \\
&= \left\{\theta \in \Theta : \left[\max_{u \in S^{\kappa_{\mathcal{Y}}-1}} \left(u'\widetilde{F}\left(y|\underline{x}\right) - \mathbf{E}\left[h\left(\tilde{Q}_\theta, u\right)|\underline{x}\right]\right)\right]_+ = 0 \;\; \underline{x} - a.s.\right\} \\
&= \Theta_I
\end{aligned}
$$

Proof. The equality between the two representations above follows by standard arguments, see, e.g., Beresteanu and Molinari (2008, Lemma A.1). To establish that $\tilde{\Theta}_I = \Theta_I$, observe that $\theta \in \tilde{\Theta}_I$ if and only if $\widetilde{F}\left(y|\underline{x}\right) \in \mathbb{E}\left(\tilde{Q}_\theta|\underline{x}\right)$. Pick $\theta \in \Theta_I$. Then $F\left(y|\underline{x}\right) = \mathbf{E}\left(q|\underline{x}\right)$ for some $q \in \text{Sel}\left(Q_\theta\right)$. Notice that this implies $\widetilde{F}\left(y|\underline{x}\right) = \mathbf{E}\left(\tilde{q}|\underline{x}\right)$ for $\tilde{q} \in \text{Sel}\left(\tilde{Q}_\theta\right)$, hence, $\theta \in \tilde{\Theta}_I$. Conversely, pick $\theta \in \tilde{\Theta}_I$. Then $\widetilde{F}\left(y|\underline{x}\right) = \mathbf{E}\left(\tilde{q}|\underline{x}\right)$ for some $\tilde{q} \in \text{Sel}\left(\tilde{Q}_\theta\right)$, which in turn implies that $q = \left[\tilde{q}; 1 - \sum_{k=1}^{\kappa_{\mathcal{Y}}-1} \tilde{q}\right] \in \text{Sel}(Q_\theta)$ and $F\left(y|\underline{x}\right) = \mathbf{E}\left(q|\underline{x}\right)$. Hence, $\theta \in \Theta_I$. $\blacksquare$

AS propose a confidence set with nominal value $1 - \alpha$ for the true parameter vector, as follows:

$$CS_n = \left\{\theta \in \Theta : T_n\left(\theta\right) \leq c_{n,1-\alpha}\left(\theta\right)\right\},$$

where $T_n\left(\theta\right)$ is a test statistic and $c_{n,1-\alpha}\left(\theta\right)$ is a corresponding critical value for a test with nominal significance level α. AS establish that, under certain assumptions, this confidence set has correct

45

uniform asymptotic size.[3] In order to apply the construction in AS, we maintain the following:

Assumption B.1 *The researcher observes an i.i.d. sequence of equilibrium outcomes and observable payoff shifters* $\{y_i, \underline{x}_i\}_{i=1}^n$. *Define* $\tilde{\Sigma}_{\underline{x}} = \mathrm{diag}\left(\widetilde{F}(y|\underline{x})\right) - \widetilde{F}(y|\underline{x})\,\widetilde{F}(y|\underline{x})'$, *and let* $\tilde{\Sigma}_{\underline{x}}$ *be non-singular with* $a < \left\|\tilde{\Sigma}_{\underline{x}}\right\| < b\ \underline{x} - a.s.$ *for some constants* $0 < a < b < \infty$, *where* $\left\|\tilde{\Sigma}_{\underline{x}}\right\|$ *is a matrix norm for* $\tilde{\Sigma}_{\underline{x}}$ *compatible with the Euclidean norm.*

AS propose various criterion functions T_n, some of the Cramér-von Mises type, some of the Kolmogorov-Smirnov type. Here, we work with a mix of Cramér-von Mises and Kolmogorov-Smirnov statistic, using a modification of the function S_1 on page 10 of AS. Specifically, we use

$$(\text{B.1}) \qquad T_n(\theta) = \int \left(\max_{u \in B^{\kappa_{\mathcal{Y}}-1}} \sqrt{n}\bar{m}_n(\theta, g, u)\right)^2 d\Gamma$$

$$= \int \left(\max_{u \in S^{\kappa_{\mathcal{Y}}-1}} \sqrt{n}\bar{m}_n(\theta, g, u)\right)_+^2 d\Gamma = \int \max_{u \in S^{\kappa_{\mathcal{Y}}-1}} \left(\sqrt{n}\bar{m}_n(\theta, g, u)\right)_+^2 d\Gamma,$$

where Γ denotes a probability measure on $\mathcal{G}$ whose support is $\mathcal{G}$ as detailed in Section 3 of AS, the second equality follows from the proof of Theorem B.1, and

$$\bar{m}_n(\theta, g, u) = \frac{1}{n}\sum_{i=1}^n \left(u'w(y_i) - f(\underline{x}_i, \theta, u)\right) g(\underline{x}_i),$$

$$f(\underline{x}_i, \theta, u) = \mathbf{E}\left[h\left(\tilde{Q}_\theta, u\right)|\underline{x}_i\right],$$

$$w(y_i) = \left[\mathbf{1}\left(y_i = t^k\right),\ k = 1, ..., \kappa_{\mathcal{Y}} - 1\right],$$

so that $\bar{m}_n(\theta, g, u)$ is the sample analog of a version of $\mathbf{E}(m(y, \underline{x}, \theta, g, u))$ which is based on the first $\kappa_{\mathcal{Y}} - 1$ entries of $\mathcal{Y}$ and on $\tilde{Q}_\theta$. Note that by the same argument which follows, our problem specified as in equation (3.6) in BMM corresponds to the Cramér-von Mises test statistic of AS, with modified function S_1.

Below we show that our modified function S_1 satisfies Assumptions S1-S4 of AS, and that Assumption M2 of AS is also satisfied. This establishes that their generalized moment selection procedure with infinitely many conditional moment inequalities is applicable. We note that one

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[3]Imbens and Manski (2004) discuss the difference between confidence sets that uniformly cover the true parameter vector with a prespecified asymptotic probability, and confidence sets that uniformly cover Θ_I (see also Stoye (2009)). Providing methodologies to obtain asymptotically valid confidence sets of either type when the conditioning variables have a continuous distribution is a developing area of research, to which the method of AS belongs. In certain empirically relevant models (see for example Appendix C and Appendix D) the characterization in Theorem 2.1 of BMM yields a finite number of (conditional) moment inequalities. In such cases, the methods of Chernozhukov, Hong, and Tamer (2007) and Romano and Shaikh (2010) can be applied after discretizing the conditioning variables, to obtain confidence sets which cover Θ_I with a prespecified asymptotic probability, uniformly in the case of Romano and Shaikh (2010). Ciliberto and Tamer (2009) verify the required regularity conditions for finite games of complete information.

can take the confidence set CS_n applied with confidence level $1/2$ to obtain half-median-unbiased estimated sets, see AS and Chernozhukov, Lee, and Rosen (2009). Finally, one can also take the criterion function in Theorem B.1, replace there $\widetilde{F}(y|\underline{x})$ with its sample analog, and construct an Hausdorff-consistent estimator of Θ_I using the methodology proposed by Chernozhukov, Hong, and Tamer (2007, equation 3.2 and Theorem 3.1). To see that their results are applicable, recall that the payoff functions are assumed to be continuous in (x_j, ε_j). Hence, the Nash equilibrium correspondence has a closed graph, see Fudenberg and Tirole (1991, Section 1.3.2). This implies that Q_θ has a closed graph, and therefore the same is true for $\mathbb{E}(Q_\theta|\underline{x})$, see Aumann (1965, Corollary 5.2). In turn, this yields $\limsup_{\theta_n \to \theta} \mathbb{E}(Q_{\theta_n}|\underline{x}) \subseteq \mathbb{E}(Q_\theta|\underline{x})$. Observe that

$$\max_{u \in B^{\kappa_{\mathcal{Y}}-1}} \left(u' \widetilde{F}\left(y|\underline{x}\right) - \mathbb{E}\left[h\left(\tilde{Q}_\theta, u\right) |\underline{x}\right] \right) = d_H \left(\widetilde{F}\left(y|\underline{x}\right), \mathbb{E}\left(\tilde{Q}_\theta|\underline{x}\right) \right).$$

The criterion function $s(\theta) \equiv \int d_H \left(\widetilde{F}(y|\underline{x}), \mathbb{E}\left(\tilde{Q}_\theta|\underline{x}\right) \right) dF_{\underline{x}}$, with $F_{\underline{x}}$ the probability distribution of $\underline{x}$ (or a probability measure which dominates it), is therefore lower semicontinuous in θ, because

$$
\begin{aligned}
\liminf_{\theta_n \to \theta} s(\theta_n) &\geq \int \liminf_{\theta_n \to \theta} d_H \left(\widetilde{F}(y|\underline{x}), \mathbb{E}\left(\tilde{Q}_{\theta_n}|\underline{x}\right) \right) dF_{\underline{x}} \geq \int d_H \left(\widetilde{F}(y|\underline{x}), \limsup \mathbb{E}\left(\tilde{Q}_{\theta_n}|\underline{x}\right) \right) dF_{\underline{x}} \\
&\geq \int d_H \left(\widetilde{F}(y|\underline{x}), \mathbb{E}\left(\tilde{Q}_\theta|\underline{x}\right) \right) dF_{\underline{x}} = s(\theta).
\end{aligned}
$$

Conditions (c-e) in Assumption C1 of Chernozhukov, Hong, and Tamer (2007) are verified by standard arguments.

We now verify AS's assumptions.

Theorem B.2 *Let Assumption B.1 hold. Then Assumptions S1-S4 and M2 of AS are satisfied.*

Proof. Assumption S1-a follows because the moment inequalities are defined for $u \in S^{\kappa_{\mathcal{Y}}-1}$, hence any rescaling of the moment function is absorbed by a corresponding rescaling in u. The rest of Assumption S1 and Assumptions S2-S4 are verified by AS. To verify Assumption M2, observe that

$$\widetilde{m}\left(y, \underline{x}, \theta, u\right) \equiv u' w\left(y\right) - f\left(\underline{x}, \theta, u\right)$$

is given by the sum of a linear function of u and a Lipschitz function of u, with Lipschitz constant equal to 1. It is immediate that the processes $\{u' w\left(y_{in}\right), \ u \in S^{\kappa_{\mathcal{Y}}-1}, \ i \leq n, \ n \geq 1\}$ satisfy Assumption M2. We now show that the same holds for the processes $\{f\left(\underline{x}_{in}, \theta_n, u\right), \ u \in S^{\kappa_{\mathcal{Y}}-1},$

$i \leq n$, $n \geq 1\}$. Assumption M2-(a) holds because for all $u \in S^{\kappa_{\mathcal{Y}}-1}$,

$$\left| \frac{f(\underline{x}, \theta, u)}{Var(\widetilde{m}(y, \underline{x}, \theta, u))} \right| \leq \left| \frac{f(\underline{x}, \theta, u)}{\mathbf{E}(u'\tilde{\Sigma}_{\underline{x}}u)} \right| \leq c \left| \mathbf{E}\left[h\left(\tilde{Q}_\theta, u \right) |\underline{x} \right] \right| \leq c\mathbf{E}\left(\left\| \tilde{Q}_\theta \right\|_H |\underline{x} \right) \leq c \quad \underline{x} - a.s.,$$

where the first inequality follows from the variance decomposition formula, c is a constant that depends on a and b from Assumption B.1, and the last inequality follows recalling that $\tilde{Q}_\theta$ takes its realizations in the unit simplex which is a subset of the unit ball. Assumption M2-(b) follows immediately because the envelope function is a constant. Assumption M2-(c) is verified observing that $f(\underline{x}, \theta, u)$ is Lipschitz in u, with Lipschitz constant equal to 1. By Lemma 2.13 in Pakes and Pollard (1989), the class of functions $\{f(\cdot, u), u \in S^{\kappa_{\mathcal{Y}}-1}\}$ is Euclidean with envelope equal to a constant, and therefore manageable. Assumption M2 for the processes $\{(u'w(y_{in}) - f(\underline{x}_{in}, \theta_n, u)),$ $u \in S^{\kappa_{\mathcal{Y}}-1}$, $i \leq n$, $n \geq 1\}$ then follows by Lemma E1 of AS. $\blacksquare$

B.2 BLP with Interval Outcome and Covariate Data

We maintain the following:

Assumption B.2 *The researcher observes an i.i.d. sequence of tuples $\{y_{iL}, y_{iU}, x_{iL}, x_{iU}\}_{i=1}^n$. $\mathbf{E}\left(|y_i|^2 \right)$, $\mathbf{E}\left(|x_j|^2 \right)$, $\mathbf{E}\left(|y_i x_j|^2 \right)$, and $\mathbf{E}\left(x_j^4 \right)$ are all finite, for each $i, j = L, U$.*

Let $Q_{\theta i}$ be the mapping defined as in equation (5.1) in BMM using $(y_{iL}, y_{iU}, x_{iL}, x_{iU})$. Beresteanu and Molinari (2008, Lemmas A.4, A.5 and proof of Theorem 4.2) establish that $\{Q_{\theta i}\}_{i=1}^n$ is a sequence of i.i.d. random closed sets, such that $\mathbf{E}\left(\|Q_{\theta i}\|_H^2 \right) < \infty$. Define $T_n(\theta)$ similarly to the previous Section:

$$
\begin{aligned}
T_n(\theta) &= \left(\max_{u \in B} \left(-\sqrt{n}\bar{m}_n(\theta, u) \right) \right)^2 = \left(\max_{u \in S} -\sqrt{n}\bar{m}_n(\theta, u) \right)_+^2 = \max_{u \in S} \left(-\sqrt{n}\bar{m}_n(\theta, u) \right)_+^2, \\
\bar{m}_n(\theta, u) &= \frac{1}{n} \sum_{i=1}^n h(Q_{\theta i}, u),
\end{aligned}
$$

where, again, the fact that $u \in S$ guarantees that the above test statistic is invariant to rescaling of the moment function. This preserves concavity of the objective function. We then have the following result:

Theorem B.3 *Let Assumption 5.1 in BMM and Assumption B.2 hold. Then Assumption EP on page 37 of AS is satisfied.*

Proof. Let $m\left(y_{iL}, y_{iU}, x_{iL}, x_{iU}, \theta, u\right) = h\left(Q_{\theta i}, u\right).$ Following AS notation, define

$$
\begin{aligned}
\sqrt{n}\bar{m}_n\left(\theta, u\right) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} h\left(Q_{\theta i}, u\right), \\
\gamma_{1,n}\left(\theta, u\right) &= \sqrt{n}\mathbf{E}\left[h\left(Q_{\theta i}, u\right)\right], \\
\gamma_2\left(\theta, u, u^*\right) &= \mathbf{E}\left[h\left(Q_{\theta i}, u\right) h\left(Q_{\theta i}, u^*\right)\right] - \mathbf{E}\left[h\left(Q_{\theta i}, u\right)\right]\mathbf{E}\left[h\left(Q_{\theta i}, u^*\right)\right], \\
\nu_n\left(\theta, u\right) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[h\left(Q_{\theta i}, u\right) - \mathbf{E}\left(h\left(Q_{\theta i}, u\right)\right)\right].
\end{aligned}
$$

Given the above definitions, we have

$$
\sqrt{n}\bar{m}_n\left(\theta, u\right) = \nu_n\left(\theta, u\right) + \gamma_{1,n}\left(\theta, u\right).
$$

By the Central Limit Theorem for i.i.d. sequences of random sets (Molchanov (2005, Theorem 2.2.1))

$$
\nu_n\left(\theta, \cdot\right) \Longrightarrow \nu_{\gamma_2(\theta)}\left(\cdot\right),
$$

a Gaussian process with mean zero, covariance kernel $\gamma_2\left(\theta, u, u^*\right)$, and continuous sample paths. It follows from the Strong Law of Large Numbers in Banach spaces of Mourier (1955) that the sample analog estimator $\hat{\gamma}_{2,n}\left(\theta, u, u^*\right)$ which replaces population moments with sample averages, satisfies $\hat{\gamma}_{2,n}\left(\theta, \cdot, \cdot\right) \overset{a.s.}{\to} \gamma_2\left(\theta, \cdot, \cdot\right)$, uniformly in u, u^*. $\blacksquare$

C   Entry Games of Incomplete Information

We now consider the case that players have incomplete information (see, e.g. Aradillas-López (2010), Brock and Durlauf (2001, 2007), Seim (2006), Sweeting (2009)). We retain the notation introduced in BMM, but we substitute Assumption 3.1 there with the following one, which is fairly standard in the literature. We continue to maintain Assumption 3.2.

Assumption C.1 *(i) The set of outcomes of the game $\mathcal{Y}$ is finite. The observed outcome of the game results from simultaneous move, pure strategy Bayesian Nash play.*
(ii) All players and the researcher observe payoff shifters x_j, $j = 1, ..., J$. The payoff shifter ε_j is private information to player $j = 1, ..., J$, and unobservable to the researcher. Conditional on $\{x_j, j = 1, ..., J\}$, ε_j is independent of $\{\varepsilon_i\}_{i \neq j}$. Players have correct common prior $F_\theta\left(\varepsilon | \underline{x}\right).$
(iii) The payoffs are additively separable in ε : $\pi_j\left(y_j, y_{-j}, x_j, \varepsilon_j; \theta\right) = \tilde{\pi}_j\left(y_j, y_{-j}, x_j; \theta\right) + \varepsilon_j$. Assumption 3.1-(iii) in BMM holds.

The independence condition in Assumption C.1-(iii) substantially simplifies the task of calculating the set of BNE. Conceptually, however, our methodology applies also when players' types are correlated. The resulting difficulties associated with calculating the set of BNE are to be faced with any methodology for inference in this class of games. The correct-common-prior condition in Assumption C.1-(iii) can be relaxed, but we maintain it here for simplicity.

For the sake of brevity, we restrict attention to two player entry games. However, this restriction is not necessary. Our results easily extend, with appropriate modifications to the notation and the definition of the set of pure strategy Bayesian Nash Equilibria (BNE), to the case of $J \geq 2$ players each with $2 \leq \kappa_{\mathcal{Y}_j} < \infty$ strategies. In what follows, we characterize the set of BNE of the game, borrowing from the treatment in Grieco (2009, Section 4), and then apply our methodology to this set.[4] To conserve space, we do not explicitly verify Assumptions 2.1-2.5 in BMM. Assumptions 2.1-2.3 follow by similar arguments as in Section 3 of BMM. Assumptions 2.4-2.5 follow by the same construction that we provide at the end of Section 3 of BMM, replacing equation (3.7) there with equation (8) in Grieco (2009, Theorem 4).

With incomplete information, players' strategies are decision rules $y_j : \mathcal{E} \rightarrow \{0, 1\}$, with $\mathcal{E}$ the support of ε. The set of outcomes of the game is $\mathcal{Y} = \{(0,0),(1,0),(0,1),(1,1)\}$. Given $\theta \in \Theta$ and a realization of $\underline{x}$ and ε_j, player j enters the market if and only if his expected payoff is non-negative. Therefore, equilibrium mappings (decision rules) are step functions determined by a threshold: $y_j(\varepsilon_j) = 1(\varepsilon_j \geq t_j)$, $j = 1, 2$. As a result, player j's beliefs about player $-j$'s probability of entry under the common prior assumption is $\int y_{-j}(\varepsilon_{-j}) \, dF_\theta(\varepsilon_{-j}|\underline{x}) = 1 - F_\theta(t_{-j}|\underline{x})$, and therefore player j's best response cutoff is[5]

$$t_j^b(t_{-j}, \underline{x}; \theta) = -\tilde{\pi}_j(1, 0, x_j; \theta) F_\theta(t_{-j}|\underline{x}) - \tilde{\pi}_j(1, 1, x_j; \theta)(1 - F_\theta(t_{-j}|\underline{x})).$$

Hence, the set of equilibria can be defined as the set of cutoff rules:

$$T_\theta(\underline{x}) = \left\{ (t_1, t_2) : \ t_j = t_j^b(t_{-j}, \underline{x}; \theta) \ \forall \ j = 1, 2 \right\}.$$

Note that the equilibrium thresholds are functions of $\underline{x}$ only. The set $T_\theta(\underline{x})$ might contain a finite number of equilibria (e.g., if the common prior is the Normal distribution), or a continuum of equilibria. For ease of notation we write the set $T_\theta(\underline{x})$ and its realizations, respectively, as T_θ and $T_\theta(\omega) \equiv T_\theta(\underline{x}(\omega))$, $\omega \in \Omega$.

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[4]We refer to Grieco (2009) for a thorough discussion of the related literature and of identification problems in games of incomplete information with multiple BNE. See also Berry and Tamer (2007, Section 3).

[5]For example, with payoffs linear in $\underline{x}$ and given by $\pi(y_j, y_{-j}, \underline{x}, \varepsilon_j; \theta) = y_j(y_{-j}\theta_{1j} + x_j\theta_{2j} + \varepsilon_j)$, we have that player 1 enters if and only if $(\varepsilon_1 + x_1\theta_{21}) F_\theta(t_2|\underline{x}) + (\varepsilon_1 + x_1\theta_{21} + \theta_{11})(1 - F_\theta(t_2|\underline{x})) \geq 0$. Therefore the cutoff is $t_j^b(t_{-j}, \underline{x}; \theta) = -x_1\theta_{21}F_\theta(t_2|\underline{x}) - (x_1\theta_{21} + \theta_{11})(1 - F_\theta(t_2|\underline{x})) = -x_1\theta_{21} - \theta_{11}(1 - F_\theta(t_2|\underline{x}))$.

For given realization of the random variables characterizing the model, i.e., for given $\omega \in \Omega$, we need to map the set of equilibrium decision rules of each player, into outcomes of the game. Consider the realization $t(\omega)$ of $t \in \text{Sel}(T_\theta)$. Through the threshold decision rule, such realization implies the following action profile:

$$
\text{(C.1)} \qquad q(t(\omega)) = \left[
\begin{array}{c}
1\left(\varepsilon_1(\omega) \leq t_1(\omega), \varepsilon_2(\omega) \leq t_2(\omega)\right) \\
1\left(\varepsilon_1(\omega) \geq t_1(\omega), \varepsilon_2(\omega) \leq t_2(\omega)\right) \\
1\left(\varepsilon_1(\omega) \leq t_1(\omega), \varepsilon_2(\omega) \geq t_2(\omega)\right) \\
1\left(\varepsilon_1(\omega) \geq t_1(\omega), \varepsilon_2(\omega) \geq t_2(\omega)\right)
\end{array}
\right] \in \Delta^3,
$$

with Δ^3 the simplex in $\Re^4$. The vector $q(t(\omega))$ indicates which of the four possible pairs of actions is played with probability 1, when the realization of $(\underline{x}, \varepsilon)$ is $(\underline{x}(\omega), \varepsilon(\omega))$ and the equilibrium threshold is $t(\omega) \in T_\theta(\underline{x}(\omega))$. Applying this construction to all measurable selections of T_θ, we construct a random closed set in Δ^3:

$$
Q_\theta = \{q(t) : t \in \text{Sel}(T_\theta)\}.
$$

For given $\underline{x}$ and $\theta \in \Theta$, define the conditional Aumann expectation

$$
\mathbb{E}(Q_\theta | \underline{x}) = \{\mathbf{E}(q(t)| \underline{x}) : t \in \text{Sel}(T_\theta)\}.
$$

Notice that for a specific selection $t \in \text{Sel}(T_\theta)$, given the independence assumption on $\varepsilon_1, \varepsilon_2$, the first entry of the vector $\mathbf{E}(q(t)| \underline{x})$ is

$$
\mathbf{E}\left(1\left(\varepsilon_1 \leq t_1, \varepsilon_2 \leq t_2\right)| \underline{x}\right) = \left(1 - F_\theta(t_1|\underline{x})\right)\left(1 - F_\theta(t_2|\underline{x})\right),
$$

and similarly for other entries of $\mathbf{E}(q(t)| \underline{x})$. This yields the multinomial distribution over outcome profiles determined by equilibrium threshold $t \in \text{Sel}(T_\theta)$. By the same logic as in Section 3 of BMM, $\mathbb{E}(Q_\theta|\underline{x})$ is the set of probability distributions over action profiles conditional on $\underline{x}$ which are consistent with the maintained modeling assumptions, i.e., with *all* the model's implications. By the same results that we applied in BMM, the set $\mathbb{E}(Q_\theta|\underline{x})$ is closed and convex.

Observe that regardless of whether T_θ contains a finite number of equilibria or a continuum, Q_θ can take on only a finite number of realizations corresponding to each of the vertices of Δ^3, because the vectors $q(t)$ in equation C.1 collect threshold decision rules.[6] As we show in the proof of Theorem C.1, this implies that $\mathbb{E}(Q_\theta|\underline{x})$ is a closed convex polytope $\underline{x}-a.s.$, fully characterized by a finite number of supporting hyperplanes. In turn, this allows us to characterize Θ_I through a finite number of moment inequalities, and to compute it using efficient algorithms in linear programming.

---

[6]Hence, the set Q_θ is a "simple" random closed set in Δ^3, in the sense that there exists a finite measurable partition $\Omega_1, \ldots, \Omega_m$ of Ω and sets $K_1, ..., K_m \in \mathcal{F}$ such that $Q_\theta(\omega) = K_i$ for all $\omega \in \Omega_i$, $1 \leq i \leq m$.

Theorem C.1 *Let Assumption C.1 and Assumption 3.2 of BMM hold. Then*

$$
\begin{aligned}
\Theta_I &= \left\{ \theta \in \Theta : \max_{u \in B} \left(u' \mathbf{P} \left(y | \underline{x} \right) - \mathbf{E} \left[h \left(Q_\theta, u \right) | \underline{x} \right] \right) = 0 \; \underline{x} - a.s. \right\} \\
&= \left\{ \theta \in \Theta : u' \mathbf{P} \left(y | \underline{x} \right) \le \mathbf{E} \left[h \left(Q_\theta, u \right) | \underline{x} \right] \; \forall \, u \in D, \; \underline{x} - a.s. \right\},
\end{aligned}
$$

where $D = \left\{ u = \begin{bmatrix} u_1 & ... & u_{\kappa_{\mathcal{Y}}} \end{bmatrix}' : u_i \in \{0, 1\}, \; i = 1, ..., \kappa_{\mathcal{Y}} \right\}.$

Proof. By the same argument as in the proof of Theorem 2.1 in BMM,

$$
\begin{aligned}
\Theta_I &= \left\{ \theta \in \Theta : \mathbf{P} \left(y | \underline{x} \right) \in \mathbb{E} \left(Q_\theta | \underline{x} \right), \; \underline{x} - a.s. \right\} \\
&= \left\{ \theta \in \Theta : \max_{u \in B} \left(u' \mathbf{P} \left(y | \underline{x} \right) - \mathbf{E} \left[h \left(Q_\theta, u \right) | \underline{x} \right] \right) = 0 \; \underline{x} - a.s. \right\} \\
&= \left\{ \theta \in \Theta : u' \mathbf{P} \left(y | \underline{x} \right) \le \mathbf{E} \left[h \left(Q_\theta, u \right) | \underline{x} \right] \; \forall \, u \in B, \; \underline{x} - a.s. \right\}.
\end{aligned}
$$

It remains to show equivalence of the conditions

$$
\begin{aligned}
(i) \quad u' \mathbf{P} \left(y | \underline{x} \right) &\le \mathbf{E} \left[h \left(Q_\theta, u \right) | \underline{x} \right] \; \forall u \in B, \\
(ii) \quad u' \mathbf{P} \left(y | \underline{x} \right) &\le \mathbf{E} \left[h \left(Q_\theta, u \right) | \underline{x} \right] \; \forall u \in D.
\end{aligned}
$$

By the positive homogeneity of the support function, condition (i) is equivalent to $u' \mathbf{P} \left(y | \underline{x} \right) \le \mathbf{E} \left[h \left(Q_\theta, u \right) | \underline{x} \right] \; \forall u \in \Re^{\kappa_{\mathcal{Y}}}$. It is obvious that this condition implies condition (ii). To see why condition (ii) implies condition (i), observe that because the set Q_θ and the set $\mathrm{co} \left[Q_\theta \right]$ are simple, one can find a finite measurable partition $\Omega_1, \ldots, \Omega_m$ of Ω and convex sets $K_1, ..., K_m \in \Delta^{\kappa_{\mathcal{Y}} - 1}$, such that by Theorem 2.1.21 in Molchanov (2005)

$$
\mathbb{E} \left(Q_\theta | \underline{x} \right) = K_1 \mathbf{P} \left(\Omega_1 | \underline{x} \right) \oplus K_2 \mathbf{P} \left(\Omega_2 | \underline{x} \right) \oplus ... \oplus K_m \mathbf{P} \left(\Omega_m | \underline{x} \right),
$$

with K_i the value that $\mathrm{co} \left[Q_\theta \left(\omega \right) \right]$ takes for $\omega \in \Omega_i$, $i = 1, ..., m$ (see Molchanov (2005, Definition 1.2.8)). By the properties of the support function, see Schneider (1993, Theorem 1.7.5),

$$
h \left(\mathbb{E} \left(Q_\theta | \underline{x} \right), u \right) = \sum_{i=1}^{m} \mathbf{P} \left(\Omega_i | \underline{x} \right) h \left(K_i, u \right).
$$

Finally, for each $i = 1, ..., m$, the vertices of K_i are a subset of the vertices of $\Delta^{\kappa_{\mathcal{Y}} - 1}$. Hence the supporting hyperplanes of K_i, $i = 1, ..., m$, are a subset of the supporting hyperplanes of the simplex $\Delta^{\kappa_{\mathcal{Y}} - 1}$, which in turn are obtained through its support function evaluated in directions $u \in D$. Therefore the supporting hyperplanes of $\mathbb{E} \left(Q_\theta | \underline{x} \right)$ are a subset of the supporting hyperplanes of $\Delta^{\kappa_{\mathcal{Y}} - 1}$. ∎

Remark 1 Grieco (2009) introduces an important model, where each player has a vector of payoff shifters unobservable by the researcher. Some of the elements of this vector are private information to the player, while the others are known to all players. Our results in Section 2 of BMM apply to this set-up as well, by the same arguments as in Section 3 of BMM and in this Appendix.

Remark 2 Appendix B verifies the regularity conditions required by AS for models satisfying Assumptions C.1 and 3.2 in BMM under the additional assumption that the researcher observes an i.i.d. sequence of equilibrium outcomes and observable payoff shifters $\{y_i, \underline{x}_i\}_{i=1}^n$.

D   Pure Strategies Only: Further Simplifications

We now assume that players in each market do not randomize across their actions. In a finite game, when restricting attention to pure strategies, one necessarily contends with the issue of possible non-existence of an equilibrium for certain parameter values $\theta \in \Theta$ and realizations of $(\underline{x}, \varepsilon)$. To deal with this problem, one can impose Assumption D.1 below:

Assumption D.1 *One of the following holds:*
(i) For a subset of values of $\theta \in \Theta$ which includes the values of θ that have generated the observed outcomes y, a pure strategy Nash equilibrium exists $(\underline{x}, \varepsilon) - a.s.$
(ii) For each $\theta \in \Theta$ and realizations of $\underline{x}, \varepsilon$ such that a pure strategy Nash equilibrium does not exist, $S_\theta(\underline{x}, \varepsilon) = \text{vert}(\Sigma(\mathcal{Y}))$, with $\text{vert}(\cdot)$ the vertices of the set in parenthesis.

Assumption D.1-(i) requires an equilibrium always to exist for the values of θ that have generated the observed outcomes y. If the model is correctly specified and players in fact follow pure strategy Nash behavior, then this assumption is satisfied. However, the assumption implicitly imposes strong restrictions on the parameter vector θ, the payoff functions, and the payoff shifters $\underline{x}, \varepsilon$. On the other hand, Assumption D.1-(ii) posits that if the model does not have an equilibrium for a given $\theta \in \Theta$ and realization of $(\underline{x}, \varepsilon)$, then the model has no prediction on what should be the action taken by the players, and "anything can happen." In this respect, one may argue that Assumption D.1-(ii) is more conservative than Assumption D.1-(i). We do not take a stand here on which solution to the existence problem the applied researcher should follow. Either way, the approach that we propose delivers the sharp identification region Θ_I, although the set Θ_I will differ depending on whether Assumption D.1-(i) or D.1-(ii) is imposed. Moreover, one may choose not to impose Assumption D.1 at all, and use a different solution concept. In that case as well, as we illustrate in Appendix E, our approach can be applied to characterize the sharp identification region.

When players play only pure strategies, the set S_θ takes its realizations as subsets of the vertices of $\Sigma(\mathcal{Y})$, because each pure strategy Nash equilibrium is equivalent to a degenerate mixed strategy Nash equilibrium placing probability one on a specific pure strategy profile. Hence, the realizations of the set Q_θ lie in the subsets of the vertices of $\Delta^{\kappa_\mathcal{Y}-1}$.

Example 1 *Consider a simple two player entry game similar to the one in Tamer (2003), omit the covariates, assume that players' payoffs are given by $\pi_j = y_j(y_{-j}\theta_j + \varepsilon_j)$, where $y_j \in \{0,1\}$ and $\theta_j < 0$, $j = 1,2$. Assume that players do not randomize across their actions, so that each σ_j, $j = 1,2$, can take only values 0 and 1. Figure 1 plots the set S_θ resulting from the possible realizations of $\varepsilon_1, \varepsilon_2$. In this case, S_θ assumes only five values:*

$$
S_\theta(\varepsilon) = \left\{
\begin{array}{ll}
\{(0,0)\} & \text{if } \varepsilon \in \mathcal{E}_\theta^{(0,0)} \equiv (-\infty,0] \times (-\infty,0], \\
\{(1,0)\} & \text{if } \varepsilon \in \mathcal{E}_\theta^{(1,0)} \equiv [-\theta_1,+\infty) \times (-\infty,-\theta_2] \cup [0,-\theta_1] \times (-\infty,0], \\
\{(0,1)\} & \text{if } \varepsilon \in \mathcal{E}_\theta^{(0,1)} \equiv (-\infty,0] \times [0,+\infty) \cup [0,-\theta_1] \times [-\theta_2,+\infty), \\
\{(1,1)\} & \text{if } \varepsilon \in \mathcal{E}_\theta^{(1,1)} \equiv [-\theta_1,+\infty) \times [-\theta_2,+\infty), \\
\{(0,1),(1,0)\} & \text{if } \varepsilon \in \mathcal{E}_\theta^M \equiv [0,-\theta_1] \times [0,-\theta_2],
\end{array}
\right.
$$

where in the above expressions $\mathcal{E}_\theta^{(\cdot,\cdot)}$ denotes a region of values for ε such that the game admits the pair in the superscript as a unique equilibrium, and $\mathcal{E}_\theta^M$ denotes the region of values for ε such that the game has multiple equilibria. Consequently, also the set Q_θ assumes only five values, equal respectively to $\{[1 \ 0 \ 0 \ 0]'\}$, $\{[0 \ 1 \ 0 \ 0]'\}$, $\{[0 \ 0 \ 1 \ 0]'\}$, $\{[0 \ 0 \ 0 \ 1]'\}$, and $\{[0 \ 1 \ 0 \ 0]', [0 \ 0 \ 1 \ 0]'\}$. □

Hence, the sets S_θ and Q_θ are "simple" random closed sets in $\Sigma(\mathcal{Y})$ and $\Delta^{\kappa_\mathcal{Y}-1}$, respectively. Because the probability space is non-atomic and Q_θ is simple, $\mathbb{E}(Q_\theta|\underline{x})$ is a closed convex polytope, fully characterized by a finite number of supporting hyperplanes.

Example 1 (Cont.) *Consider again the simple two player entry game with pure strategies only in Example 1. Then for $\varepsilon \in \mathcal{E}_\theta^M$ the set Q_θ contains only two points, $[0 \ 1 \ 0 \ 0]'$ and $[0 \ 0 \ 1 \ 0]'$, and for $\varepsilon \notin \mathcal{E}_\theta^M$ it is a singleton. Therefore, the expectations of the selections of Q_θ are given by*

$$
\begin{aligned}
\mathbf{E}(q) = \ & \left[\mathbf{P}\left(\varepsilon \in \mathcal{E}_\theta^{(0,0)}\right) \quad \mathbf{P}\left(\varepsilon \in \mathcal{E}_\theta^{(1,0)}\right) \quad \mathbf{P}\left(\varepsilon \in \mathcal{E}_\theta^{(0,1)}\right) \quad \mathbf{P}\left(\varepsilon \in \mathcal{E}_\theta^{(1,1)}\right)\right]' \\
& + [0 \quad p_1 \quad 1-p_1 \quad 0]' \mathbf{P}\left(\varepsilon \in \mathcal{E}_\theta^M\right),
\end{aligned}
$$

where $p_1 = \mathbf{P}\left(\Omega_1^M \middle| \omega : \varepsilon(\omega) \in \mathcal{E}_\theta^M\right)$, for all measurable $\Omega_1^M \subset \{\omega : \varepsilon(\omega) \in \mathcal{E}_\theta^M\}$, $i = 1,2$. If the probability space has no atoms, then the possible values for p_1 fill in the whole $[0,1]$ segment. Hence, $\mathbb{E}(Q_\theta)$ is a segment in Δ^3. □

54

Hence, checking whether $\mathbf{P}\left(y|\,\underline{x}\right) \in \mathbb{E}\left(Q_\theta|\,\underline{x}\right)$ amounts to checking whether a point belongs to a polytope, i.e. whether a finite number of moment inequalities hold $\underline{x} - a.s.$ In Theorem D.1 we show that these inequalities are obtained by checking inequality $u'\mathbf{P}\left(y|\,\underline{x}\right) \leq \mathbf{E}\left[h\left(Q_\theta, u\right)|\,\underline{x}\right]$ for the $2^{\kappa_{\mathcal{Y}}}$ possible u vectors whose entries are either equal to zero or to one.

Theorem D.1 *Assume that players use only pure strategies, that Assumptions 3.1 and 3.2 in BMM and Assumption D.1 are satisfied. Then for $\underline{x} - a.s.$ these two conditions are equivalent:*

1. $u'\mathbf{P}\left(y|\,\underline{x}\right) \leq \mathbf{E}\left[h\left(Q_\theta, u\right)|\,\underline{x}\right] \ \forall \ u \in \Re^{\kappa_{\mathcal{Y}}},$

2. $u'\mathbf{P}\left(y|\,\underline{x}\right) \leq \mathbf{E}\left[h\left(Q_\theta, u\right)|\,\underline{x}\right] \ \forall \ u \in D = \left\{u = \begin{bmatrix} u_1 & ... & u_{\kappa_{\mathcal{Y}}} \end{bmatrix}' : u_i \in \{0,1\}, \ i = 1,...,\kappa_{\mathcal{Y}}\right\}.$

Proof. Same argument as in the proof of Theorem C.1. ∎

In Appendix D.2 we connect this result to a related notion in the theory of random sets, that of a *capacity functional* (the "probability distribution" of a random closed set), and we provide an equivalent characterization of the sharpness result which gives further insights into our approach. In Appendix D.2 we provide results that significantly reduce the number of inequalities to be checked, by showing that depending on the model under consideration, many of the $2^{\kappa_{\mathcal{Y}}}$ inequalities in Theorem D.1 are redundant.

To conclude this Appendix, it is important to discuss why the sharp identification region cannot in general be obtained through a finite number of moment inequalities. When players are not allowed to randomize over their actions, the family of possible equilibria is finite. Hence, the range of values that ε takes can be partitioned into areas in which the set of equilibria remains constant, that is, does not depend on ε any longer. However, when players randomize across their actions, in equilibrium they must be indifferent among the actions over which they place positive probability. This implies that there exist regions in the sample space where the equilibrium mixed strategy profiles are a function of ε directly.[7] When the distribution of ε is continuous, Q_θ may take a continuum of values as a function of ε, and $\mathbb{E}\left(Q_\theta|\,\underline{x}\right)$ may have infinitely many extreme points. Therefore, one needs an infinite number of moment inequalities to determine whether $\mathbf{P}\left(y|\,\underline{x}\right)$ belongs to it. In this case, the most practical approach to obtain the sharp identification region is by solving the maximization problem in Theorem 3.2 of BMM.

---

[7] For example, in the two player entry game in Example 1 of BMM, for $\varepsilon \in \mathcal{E}_M^\theta$, $S_\theta = \left\{(0,1), \left(\frac{\varepsilon_2}{-\theta_2}, \frac{\varepsilon_1}{-\theta_1}\right), (1,0)\right\}$. However, if one restricts players to use pure strategies, then for $\varepsilon \in \mathcal{E}_M^\theta$, $S_\theta = \{(0,1), (1,0)\}$, with no additional dependence of the equilibria on ε.

D.1 Example: Two Type, Four Player Entry Game with Pure Strategies Only

Consider a game where in each market there are four potential entrants, two of each type. The two types differ from each other by their payoff function. This model is an extension of the seminal papers by Bresnahan and Reiss (1990, 1991). An empirical application of a version of this model appears in Ciliberto and Tamer (2009, CT henceforth). We adopt the version of this model described in Berry and Tamer (2007, pages 84-85), and for illustration purposes we simplify it by omitting the observable payoff shifters $\underline{x}$ and by setting to zero the constant in the payoff function.

Let $a_{jm} \in \{0,1\}$ be the strategy of firm $j = 1, 2$ of type $m = 1, 2$. Entry is denoted by $a_{jm} = 1$, with $a_{jm} = 0$ denoting staying out. Players $j = 1, 2$ of type 1 and type 2 have respectively the following payoff functions:

$$\text{(D.1)} \qquad \pi_{j1}\left(a_{j1}, a_{-j1}, a_{12}, a_{22}, \varepsilon_1\right) = y_{j1}\left(\theta_{11}\left(a_{-j1} + a_{12} + a_{22}\right) - \varepsilon_1\right),$$

$$\text{(D.2)} \qquad \pi_{j2}\left(a_{j2}, a_{-j2}, a_{11}, a_{21}, \varepsilon_2\right) = a_{j2}\left(\theta_{21}\left(a_{11} + a_{21}\right) + \theta_{22}a_{-j2} - \varepsilon_2\right).$$

We assume that θ_{11}, θ_{21} and θ_{22} are strictly negative and that $\theta_{22} > \theta_{21}$. This means that a type 2 firm is worried more about rivals of type 1 than of rivals of its own type. Since firms of a given type are indistinguishable to the econometrician, the observable outcome is the number of firms of each type which enter the market. Let $y_1 = a_{11} + a_{21}$ denote the number of entrants of type 1 and $y_2 = a_{12} + a_{22}$ the number of entrants of type 2 that a firm faces, so that $y_m \in \{0,1,2\}$, $m = 1, 2$. Then there are 9 possible outcomes to this game, ordered as follows: $\mathcal{Y} = \{(0,0), (0,1), (1,0), (1,1), (2,0), (0,2), (1,2), (2,1), (2,2)\}$. Notice that here players' actions and observable outcomes of the game differ. Figure 2 plots the outcomes of the game against the realizations of $\varepsilon_1, \varepsilon_2$. In this case, Q_θ takes its realizations in the vertices of Δ^8. For example, for $\omega : \varepsilon_1(\omega) \geq \theta_{11}$, $\varepsilon_2(\omega) \geq \theta_{22}$, the game has a unique equilibrium outcome, $y = (0,0)$, and $Q_\theta(\omega) = \{[1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]'\}$; for $\omega : 2\theta_{11} \leq \varepsilon_1(\omega) \leq \theta_{11}$, $2\theta_{22} \leq \varepsilon_2(\omega) \leq \theta_{22}$, the game has two equilibrium outcomes, $y = (0,1)$ and $y = (1,0)$, and $Q_\theta(\omega) = \{[0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0]', [0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]'\}$; etc.

Because the set $\mathcal{Y}$ has cardinality 9, in principle there are $2^9 = 512$ inequality restrictions to consider, corresponding to each binary vector of length 9. However, the number of inequalities to be checked is significantly smaller. Because we are allowing only pure strategy equilibria, the realizations of any $\sigma \in S_\theta$ are vectors of zeros and ones. Hence, for all $\omega \in \Omega$, $[q(\sigma(\omega))]_k = 1$ if $\prod_{j=1}^J \sigma_j\left(\omega, t_j^k\right) = 1$, and zero otherwise. Consider two equilibria $t^k, t^l \in \mathcal{Y}$, $1 \leq k \neq l \leq \kappa_{\mathcal{Y}}$, such that

$$\text{(D.3)} \qquad \left\{\omega : \prod_{j=1}^J \sigma_j\left(\omega, t_j^k\right) = 1 \bigg| \underline{x}\right\} \cap \left\{\omega : \prod_{j=1}^J \sigma_j\left(\omega, t_j^l\right) = 1 \bigg| \underline{x}\right\} = \emptyset,$$

that is, the set of ω for which S_θ admits both t^k and t^l as equilibria has probability zero. Let u^k be a vector with each entry equal to zero, and entry k equal to 1, and similarly for u^l. Then the inequality $\left(u^k + u^l\right)' \mathbf{P}\left(y|\underline{x}\right) \leq \mathbf{E}\left[h\left(Q\left(S_\theta\right), u^k + u^l\right)|\underline{x}\right]$ does not add any information beyond that provided by the inequalities $u'\mathbf{P}\left(y|\underline{x}\right) \leq \mathbf{E}\left[h\left(Q\left(S_\theta\right), u\right)|\underline{x}\right]$ for $u = u^k$ and for $u = u^l$. The same reasoning can be extended to tuples of pure strategy equilibria of size up to $\kappa_{\mathcal{Y}}$. Applying this simple reasoning, the sharp identification region that we give in this example is based on 26 inequalities, whereas Θ_O^{ABJ} and Θ_O^{CT} are based, respectively, on 9 and 18 inequalities. Hence, the computational burden is essentially equivalent.

Figure 3 and Table 1 report Θ_I, Θ_O^{CT}– the outer region proposed by CT, and Θ_O^{ABJ}– the outer region proposed by Andrews, Berry, and Jia (2004, ABJ henceforth), in a simple example with $(\varepsilon_1, \varepsilon_2) \overset{iid}{\sim} N\left(0, 1\right)$ and $\Theta = [-5, 0]^3$. In the figure, Θ_O^{ABJ} is given by the union of the yellow, red and black segments, and Θ_O^{CT} by the union of the red and black segments. Θ_I is the black segment. Notice that the identification regions are segments because the outcomes $(0, 0)$ and $(2, 2)$ can only occur as unique equilibrium outcomes, and therefore imply two moment equalities which make θ_{21} and θ_{22} a function of θ_{11}. While, strictly speaking, the approach in ABJ does not take into account this fact, as it uses only upper bounds on the probabilities that each outcome occurs, it is clear (and indicated in their paper) that one can incorporate equalities into their method. Hence, we use the equalities on $\mathbf{P}\left(y = (0, 0)\right)$ and $\mathbf{P}\left(y = (2, 2)\right)$ also when calculating Θ_O^{ABJ}. We generate the data with $\theta_{11}^\star = -0.15$, $\theta_{21}^\star = -0.20$, and $\theta_{22}^\star = -0.10$ and use a selection mechanism to choose the equilibrium played in the many regions of multiplicity. The resulting observed distribution is $\mathbf{P}\left(y\right) = [0.3021\ 0.0335\ 0.0231\ 0.0019\ 0.2601\ 0.2779\ 0.0104\ 0.0158\ 0.0752]'$. Our results clearly show that Θ_I is substantially smaller than Θ_O^{CT} and Θ_O^{ABJ}. The width of the bounds on each parameter vector obtained using our method is about 46% of the width obtained using ABJ's method, and about 63% of the width obtained using CT's method.

In order to further illustrate the computational advantages of our characterization of Θ_I in Theorem 3.2 of BMM, we also recalculated the sharp identification region for this example *without* taking advantage of our knowledge of the structure of the game which reduces the number of inequalities to be checked to 26, but by simply solving for each candidate $\theta \in \Theta$ the problem $\max_{u \in B}\left(u'\mathbf{P}\left(y|\underline{x}\right) - \mathbf{E}\left[h\left(Q_\theta, u\right)|\underline{x}\right]\right)$. We modified the simple Nelder-Mead algorithm described in Section 3.4 of BMM to apply to a minimization in $\Re^9$, wrote it as a program in Fortran 90, and compiled and ran it on a Unix machine with a single processor of 3.2 GHz. Our recalculation of Θ_I yielded exactly the same result as described above, and checking 10^6 candidate values for $\theta \in \Theta$ took less than one minute.

D.2 Dual Characterization of the Sharpness Result in the Pure Strategies Case

For a given realization of $(\underline{x}, \varepsilon)$ and value of $\theta \in \Theta$, the set of outcomes generated by pure strategy Nash equilibria[8] is

(D.4) $\qquad Y_\theta(\underline{x}, \varepsilon) = \{y \in \mathcal{Y} : \pi_j(y_j, y_{-j}, x_j, \varepsilon_j, \theta) \geq \pi_j(\tilde{y}_j, y_{-j}, x_j, \varepsilon_j, \theta) \ \forall \tilde{y}_j \in \mathcal{Y}_j \ \forall j\}.$

As we did for S_θ, we omit the explicit reference to this set's dependence on $\underline{x}$ and ε. Given Assumption 3.1 of BMM, one can easily show that Y_θ is a random closed set in $\mathcal{Y}$ (see Definition A.1 of BMM). Because the realizations of Y_θ are subsets of the finite set $\mathcal{Y}$, it immediately follows that Y_θ is a random closed set in $\mathcal{Y}$ without any requirement on the payoff functions.

The researcher observes the tuple $(y, \underline{x})$, and the random set Y_θ is a function of $\underline{x}$ (and of course ε). Under Assumptions 3.1 and 3.2 of BMM and Assumption D.1, and given the covariates $\underline{x}$, the observed outcomes y are consistent with the model if and only if there exists at least one $\theta \in \Theta$ such that $y(\omega) \in Y_\theta(\omega) \ \underline{x} - a.s.$ (i.e., y is a selection of $Y_\theta \ \underline{x} - a.s.$, see Definition A.3 in BMM). A necessary and sufficient condition which guarantees that a random vector $(y, \underline{x})$ is a selection of $(Y_\theta, \underline{x})$ is given by the results of Artstein (1983), Norberg (1992) and Molchanov (2005, Theorem 1.2.20 and Section 1.4.8), and amounts to the following:[9]

$$\mathbf{P}\{(y, \underline{x}) \in K \times L\} \leq \mathbf{P}\{(Y_\theta, \underline{x}) \cap K \times L \neq \emptyset\} \ \forall K \subset \mathcal{Y}, \ \forall \text{ compact sets } L \subset \mathcal{X}.$$

This inequality can be written as $\mathbf{P}\{y \in K | \underline{x} \in L\} \mathbf{P}\{\underline{x} \in L\} \leq \mathbf{P}\{Y_\theta \cap K \neq \emptyset | \underline{x} \in L\} \mathbf{P}\{\underline{x} \in L\}$ for all $K \subset \mathcal{Y}$ and compact sets $L \subset \mathcal{X}$ such that $\mathbf{P}\{\underline{x} \in L\} > 0$, and it is satisfied if and only if

(D.5) $\qquad \mathbf{P}\{y \in K | \underline{x}\} \leq \mathbf{P}\{Y_\theta \cap K \neq \emptyset | \underline{x}\} \ \forall K \subset \mathcal{Y} \ \underline{x} - a.s.$

Because $\mathcal{Y}$ is finite, all its subsets are compact. The functional $\mathbf{P}\{Y_\theta \cap K \neq \emptyset | \underline{x}\}$ on the right-hand side of (D.5) is called the capacity functional of Y_θ given $\underline{x}$. The following definitions formally introduce the unconditional version of this functional and a few related ones:

---

[8]Restrict the set S_θ to be a set of pure strategy Nash equilibria. Then when players actions and outcomes of the game coincide, Y_θ coincides with S_θ. However, under the more general assumption that $y = g(a)$, where $a \in \mathcal{A}$ is a strategy profile and g is an outcome rule, these two sets differ, and

$$Y_\theta(\underline{x}, \varepsilon) = \{y \in \mathcal{Y} : y = g(a), \ a \in \mathcal{A} \text{ and } \pi_j(a_j, a_{-j}, x_j, \varepsilon_j, \theta) \geq \pi_j(\tilde{a}_j, a_{-j}, x_j, \varepsilon_j, \theta) \ \forall \tilde{a}_j \in \mathcal{A}_j \ \forall j\}.$$

[9]Beresteanu and Molinari (2006, 2008, Proposition 4.1) use this result to establish sharpness of the identification region of the parameters of a best linear predictor with interval outcome data. Galichon and Henry (2006) use it to define a correctly specified partially identified structural model, and derive a Kolmogorov-Smirnov test for Choquet capacities.

Definition D.1 *Let Z be a random closed set in $\Re^d$, and denote by $\mathcal{K}$ the family of compact subsets of $\Re^d$. The functionals $\mathbf{T}_Z : \mathcal{K} \to [0, 1]$, $\mathbf{C}_Z : \mathcal{K} \to [0, 1]$, and $\mathbf{I}_Z : \mathcal{K} \to [0, 1]$, given by*

$$\mathbf{T}_Z(K) = \mathbf{P}\{Z \cap K \neq \emptyset\}, \quad \mathbf{C}_Z(K) = \mathbf{P}\{Z \subset K\}, \quad \mathbf{I}_Z(K) = \mathbf{P}\{K \subset Z\}, \quad K \in \mathcal{K},$$

*are said to be, respectively, the **capacity functional** of Z, the **containment functional** of Z, and the **inclusion functional** of Z.*

Denoting by K^c the complement of the set K, the following relationship holds:

$$(D.6) \qquad\qquad\qquad \mathbf{C}_Z(K) = 1 - \mathbf{T}_Z(K^c).$$

Example 2 *Consider again the simple two player entry game in Example 1. Figure 1 plots the set Y_θ against the realizations of $\varepsilon_1, \varepsilon_2$. In this case, $\mathbf{T}_{Y_\theta}(\{(0,0)\}) = \mathbf{P}(\varepsilon_1 \leq 0, \varepsilon_2 \leq 0)$, $\mathbf{T}_{Y_\theta}(\{(1,0)\}) = \mathbf{P}(\varepsilon_1 \geq 0, \varepsilon_2 \leq -\theta_2)$, $\mathbf{T}_{Y_\theta}(\{(0,1)\}) = \mathbf{P}(\varepsilon_1 \leq -\theta_1, \varepsilon_2 \geq 0)$, $\mathbf{T}_{Y_\theta}(\{(1,1)\}) = \mathbf{P}(\varepsilon_1 \geq -\theta_1, \varepsilon_2 \geq -\theta_2)$, $\mathbf{T}_{Y_\theta}(\{(1,0),(0,1)\}) = \mathbf{T}_{Y_\theta}(\{(1,0)\}) + \mathbf{T}_{Y_\theta}(\{(0,1)\}) - \mathbf{P}(0 \leq \varepsilon_1 \leq -\theta_1, 0 \leq \varepsilon_2 \leq -\theta_2)$. The capacity functional of the remaining subsets of $\mathcal{Y}$ can be calculated similarly.* $\square$

Notice that given equation (D.6), inequalities (D.5) can be equivalently written as

$$(D.7) \qquad\qquad \mathbf{C}_{Y_\theta|\underline{x}}(K) \leq \mathbf{P}\{y \in K | \underline{x}\} \leq \mathbf{T}_{Y_\theta|\underline{x}}(K) \;\; \forall K \subset \mathcal{Y} \; \underline{x} - a.s.,$$

where the subscript $Y_\theta|\underline{x}$ denotes that the functional is for the random set Y_θ conditional on $\underline{x}$. We return to this representation of inequalities (D.5) when discussing the relationship between our analysis and that of CT. Clearly, if one considers all $K \subset \mathcal{Y}$, the left-hand side inequality in (D.7) is superfluous: when the inequalities in (D.7) are used, only subsets $K \subset \mathcal{Y}$ of cardinality up to half of the cardinality of $\mathcal{Y}$ are needed.

We can re-define the identified set of parameters θ as

$$(D.8) \qquad\qquad \Theta_I = \left\{\theta \in \Theta : \mathbf{P}\{y \in K | \underline{x}\} \leq \mathbf{T}_{Y_\theta|\underline{x}}(K) \;\; \forall K \subset \mathcal{Y} \; \underline{x} - a.s.\right\}.$$

For comparison purposes, we reformulate the definition of the outer regions given by ABJ and CT respectively through the capacity functional and the containment functional:

$$(D.9) \qquad\qquad \Theta_O^{ABJ} = \left\{\theta \in \Theta : \mathbf{P}\{y = t | \underline{x}\} \leq \mathbf{T}_{Y_\theta|\underline{x}}(t) \;\; \forall t \in \mathcal{Y} \; \underline{x} - a.s.\right\},$$

$$(D.10) \qquad \Theta_O^{CT} = \left\{\theta \in \Theta : \mathbf{C}_{Y_\theta|\underline{x}}(t) \leq \mathbf{P}\{y = t | \underline{x}\} \leq \mathbf{T}_{Y_\theta|\underline{x}}(t) \;\; \forall t \in \mathcal{Y} \; \underline{x} - a.s.\right\}.$$

Both ABJ and CT acknowledge that the parameter regions they give are not sharp. Comparing the sets in equations (D.9)-(D.10) with the set in equation (D.8), one observes that Θ_O^{ABJ} is obtained

applying inequality (D.5) only for $K = \{t\}$ for all $t \in \mathcal{Y}$. Similarly, Θ_O^{CT} is obtained applying inequality (D.7) only for $K = \{t\}$ (or, equivalently, applying inequality (D.5) for $K = \{t\}$ and $K = \mathcal{Y} \setminus \{t\}$ for all $t \in \mathcal{Y}$). Clearly both ABJ and CT do not use the information contained in the remaining subsets of $\mathcal{Y}$, while this information is used to obtain Θ_I. Two questions arise: (1) whether Θ_I as defined in equation (D.8) yields the sharp identification region of θ; and (2) if and by how much Θ_I differs from Θ_O^{ABJ} and Θ_O^{CT}. We answer here the first question. Appendix D.1 answers the second question by looking at a simple example.

Theorem D.2 *Assume that players use only pure strategies, and that Assumptions 3.1 and 3.2 in BMM and Assumption D.1 are satisfied. Then for $\underline{x} - a.s.$ these two conditions are equivalent:*
1. $u' \mathbf{P} (y | \underline{x}) \leq \mathbf{E} [h (Q_\theta, u) | \underline{x}] \; \forall \; u \in \Re^{\kappa_\mathcal{Y}}$,
2. $\mathbf{P} \{y \in K | \underline{x}\} \leq \mathbf{T}_{Y_\theta | \underline{x}} (K) \; \forall K \subset \mathcal{Y}$.

    Proof. Beresteanu, Molchanov, and Molinari (2008, Theorem 4.1). ∎

D.3   On the Number of Inequalities to Be Checked in the Pure Strategies Case

As discussed in Appendix D.1, when it is assumed that players play only pure strategies, often there is no need to verify the complete set of $2^{\kappa_\mathcal{Y}}$ inequalities, because many are redundant. Using the insight in Theorem D.2, one can show that the result in equation (D.3) can be restated using the set Y_θ and its capacity functional. In particular, if K_1 and K_2 are two disjoint subsets of $\mathcal{Y}$ such that

$$(D.11) \qquad \{\omega : Y_\theta (\omega) \cap K_1 \neq \emptyset | \underline{x}\} \cap \{\omega : Y_\theta (\omega) \cap K_2 \neq \emptyset | \underline{x}\} = \emptyset,$$

that is, the set of ω for which Y_θ intersects both K_1 and K_2 has probability zero, then the inequality $\mathbf{P} \{y \in K_1 \cup K_2 | \underline{x}\} \leq \mathbf{P} \{Y_\theta \cap (K_1 \cup K_2) \neq \emptyset | \underline{x}\}$ does not add any information beyond that provided by the inequalities $\mathbf{P} \{y \in K_1 | \underline{x}\} \leq \mathbf{P} \{Y_\theta \cap K_1 \neq \emptyset | \underline{x}\}$ and $\mathbf{P} \{y \in K_2 | \underline{x}\} \leq \mathbf{P} \{Y_\theta \cap K_2 \neq \emptyset | \underline{x}\}$. Therefore, prior knowledge of some properties of the game can be very helpful in eliminating unnecessary inequalities. For example, in a Bresnahan and Reiss entry model with 4 players, if the number of entrants is identified, the number of inequalities to be verified reduces from 65,536 to at most 100. Theorem D.3 below gives a general result which may lead to a dramatic reduction in the number of inequalities to be checked. While its proof is simple, this result is conceptually and practically important.

Theorem D.3 *Take $\theta \in \Theta$ and let Assumptions 3.1 and 3.2 in BMM and Assumption D.1 hold. Consider a partition of Ω into sets $\Omega^1, \ldots, \Omega^M$ of positive probability. Let $\mathcal{Y}_i$*

$$\mathcal{Y}_i = \cup \{Y_\theta(\omega) : \omega \in \Omega^i\}.$$

denote the range of $Y_\theta(\omega)$ for $\omega \in \Omega^i$. If $\mathcal{Y}_1, \ldots, \mathcal{Y}_M$ are disjoint, then it suffices to check (D.5) only for all subsets K such that there is $i = 1, \ldots, M$ for which $K \subseteq \mathcal{Y}_i$.

    Proof. Beresteanu, Molchanov, and Molinari (2008, Theorem 5.1). ∎

A simple Corollary of Theorem D.3, the proof of which is omitted, is the following:

Corollary D.4 *Take $\theta \in \Theta$ and let Assumptions 3.1 and 3.2 in BMM and Assumption D.1 hold. Assume that $\Omega = \Omega^1 \cup \Omega^2$ with $\Omega^1 \cap \Omega^2 = \emptyset$, such that $Y_\theta(\omega)$ is a singleton almost surely for $\omega \in \Omega^1$. Let $\mathcal{Y}_i = \cup_{\omega \in \Omega^i} Y_\theta(\omega)$, $i = 1, 2$, and assume that $\mathcal{Y}_1 \cap \mathcal{Y}_2 = \emptyset$ and that $\kappa_{\mathcal{Y}_2} \leq 2$. Then inequalities (D.5) hold if*

$$(D.12) \qquad \mathbf{P}\{Y_\theta = \{t\}|\underline{x}\} \leq \mathbf{P}\{y = t|\underline{x}\} \leq \mathbf{P}\{t \in Y_\theta|\underline{x}\}$$

$\underline{x} - a.s.$ for all $t \in \mathcal{Y}$.

    An implication of this Corollary is that in a static entry game with two players in which only pure strategies are played, the outer region proposed by CT coincides with ours, and is sharp.[10] In this example, $\mathcal{Y}_1 = \{(0,0),(1,1)\}$, $\mathcal{Y}_2 = \{(0,1),(1,0)\}$, and $\Omega^2 = \{\omega : Y_\theta \cap \mathcal{Y}_2 \neq \emptyset\}$. An application of equation (D.3) shows that actually the sharp identification region can be obtained by checking only five inequalities which have to hold for $\underline{x} - a.s.$, given by inequalities (D.5) for $K = \{(0,0)\}, \{(1,0)\}, \{(0,1)\}, \{(1,1)\}, \{(1,0),(0,1)\}$. On the other hand, the example in Section 3.4 of BMM shows that CT's approach does not yield the sharp identification region when mixed strategies are allowed for.

    When no prior knowledge of the game such as, for example, that required in Theorem D.3 is available, it is still possible to use the insight in equation (D.11) to determine which inequalities yield the sharp identification region, by decomposing $\mathcal{Y}$ into subsets such that Y_θ does not jointly hit any two of them with positive probability. One may wonder whether in general the set of inequalities yielding the sharp identification region is different from the set of inequalities used by ABJ or CT. The following result shows that in general the answer to this question is "yes".

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[10]A literal application of ABJ's approach does not take into account the fact that in this game $(0,0)$ and $(1,1)$ only occur as unique equilibria of the game, and therefore does not yield the sharp identification region, as ABJ discuss (see page 32).

Theorem D.5 *Let Assumptions 3.1 and 3.2 in BMM and Assumption D.1 hold. Assume that there exists $\theta \in \Theta$, with $Y_\theta \neq \emptyset$ $\mathbf{P} - a.s.$, such that for all $\underline{x} \in \tilde{\mathcal{X}} \subset \mathcal{X}$, with $\mathbf{P}\left(\tilde{\mathcal{X}}\right) > 0$, there exist $t^1, t^2 \in \mathcal{Y}$ with*

(D.13)
$$\mathbf{I}_{Y_\theta|\underline{x}}(t^1, t^2) > 0.$$

(a) If $\mathbf{P}\left\{\{t^1, t^2\} \cap Y_\theta \neq \emptyset \middle| \underline{x}\right\} < 1$ for all $t^1, t^2 \in \mathcal{Y}$, then there exists a random vector z which satisfies inequalities (D.5) for $K = \{t\}$ for all $t \in \mathcal{Y}$ but is not a selection of Y_θ.
(b) If

(D.14)
$$\mathbf{P}\{\kappa_{Y_\theta} > 1 \middle| \underline{x}\} > \mathbf{I}_{Y_\theta|\underline{x}}(t^1) + \mathbf{I}_{Y_\theta|\underline{x}}(t^2) - \mathbf{C}_{Y_\theta|\underline{x}}(t^1) - \mathbf{C}_{Y_\theta|\underline{x}}(t^2),$$

then there exists a random vector z which satisfies inequalities (D.5) for $K = \{t\}$ and $K = \mathcal{Y}\backslash \{t\}$ for all $t \in \mathcal{Y}$ but is not a selection of Y_θ.

Proof. Beresteanu, Molchanov, and Molinari (2008, Theorems 5.2 and 5.3) ∎

These results show that the extra inequalities matter in general, compared to those used by ABJ, and CT, to fully characterize Y_θ and determine if $y \in \text{Sel}(Y_\theta)$. In fact, the assumptions of Theorem D.5-(a) are satisfied whenever the model has multiple equilibria with positive probability, which implies that the expected cardinality of Y_θ given $\underline{x}$ is strictly greater than one, and it has at least three different equilibria. The assumptions of Theorem D.5-(b) are satisfied whenever (1) there are regions of the unobservables of positive probability where two different outcomes can result from equilibrium strategy profiles; and (2) the probability that the cardinality of Y_θ is greater than one exceeds the probability that each of these two outcomes is not a unique equilibrium. It is easy to see that these assumptions are not satisfied in a two player entry game where players are allowed only to play pure strategies, but they are satisfied in the four player, two type game described in Section D.1.

E   Extensions to Other Solution Concepts

While in Section 3 of BMM and Appendix D we focus on economic models of games in which Nash Equilibrium is the solution concept employed, our approach easily applies to other solution concepts. Here we consider the case that players are assumed to be only level-1 rational, and the case that they are assumed to play correlated strategies. For simplicity, we exemplify these extensions using a two player simultaneous move static game of entry with complete information.

E.1 Level-1 Rationality

Suppose that players are only assumed to be level-1 rational. The identification problem under this weaker solution concept was first studied by Aradillas-Lopez and Tamer (2008, AT henceforth). Let the econometrician observe players' actions. A level-1 rational profile is given by a mixed strategy for each player that is a best response to one of the possible mixed strategies of her opponent. In this case one can define the θ-dependent set

$$R_\theta\left(\underline{x}, \varepsilon\right) = \left\{ \sigma \in \Sigma\left(\mathcal{Y}\right) : \begin{array}{c} \forall j \; \exists \; \tilde{\sigma}_{-j} \in \Sigma\left(\mathcal{Y}_{-j}\right) \text{ s.t.} \\ \pi_j\left(\sigma_j, \tilde{\sigma}_{-j}, x_j, \varepsilon_j, \theta\right) \geq \pi_j\left(\sigma_j', \tilde{\sigma}_{-j}, x_j, \varepsilon_j, \theta\right) \; \forall \sigma_j' \in \Sigma\left(\mathcal{Y}_j\right) \end{array} \right\}.$$

Omitting the explicit reference to its dependence on $\underline{x}$ and ε, R_θ is the set of level-1 rational strategy profiles of the game. By similar arguments to what we used above, this is a random closed set in $\Sigma\left(\mathcal{Y}\right)$. Figure 4 plots this set against the possible realizations of $\varepsilon_1, \varepsilon_2$, in a simple two player simultaneous move, complete information, static game of entry in which players' payoffs are given by $\pi_j = y_j\left(y_{-j}\theta_j + \varepsilon_j\right)$, $y_j \in \{0, 1\}$, and θ_1, θ_2 are assumed to be negative.

The same approach of Section 3 of BMM allows us to obtain the sharp identification region for θ as

$$\Theta_I = \left\{ \theta \in \Theta : u'\mathbf{P}\left(y|\underline{x}\right) \leq \mathbf{E}\left[h\left(Q\left(R_\theta\right), u\right)\middle|\underline{x}\right] \; \forall \; u \in B \; \underline{x} - a.s. \right\},$$

with

$$Q\left(R_\theta\right) = \left\{\left(\left[q\left(\sigma\right)\right]_k, k = 1, \ldots, \kappa_{\mathcal{Y}}\right) : \sigma \in \mathrm{Sel}\left(R_\theta\right)\right\},$$

where $\left[q\left(\sigma\right)\right]_k, k = 1, \ldots, \kappa_{\mathcal{Y}}$, is defined in Section 3 of BMM.

In our simple example in Figure 4, with omitted covariates, for any $\omega \in \Omega$ such that $\varepsilon\left(\omega\right) \in [0, -\theta_1] \times [0, -\theta_2]$,

$$\left[q\left(\left(\frac{\varepsilon_2\left(\omega\right)}{-\theta_2}, \frac{\varepsilon_1\left(\omega\right)}{-\theta_1}\right)\right)\right] \in \mathrm{co}\left[\left\{\left[q\left(0, 0\right)\right], \left[q\left(1, 0\right)\right], \left[q\left(0, 1\right)\right], \left[q\left(1, 1\right)\right]\right\}\right],$$

and therefore it follows that $\mathbb{E}\left(Q\left(R_\theta\right)\right)$ is equal to $\mathbb{E}\left(Q\left(\tilde{R}_\theta\right)\right)$, with $\tilde{R}_\theta$ restricted to be the set of level-1 rational pure strategies. Hence, by Theorem D.1 below, Θ_I can be obtained by checking a finite number of moment inequalities.

For the case that ε has a discrete distribution, AT (Section 3.1) suggest to obtain the sharp identification region as the set of parameter values that return value zero for the objective function of a linear programming problem. For the general case in which ε may have a continuous distribution, AT apply the same insight of CT and characterize an outer identification region through eight moment inequalities similar to those in equation (D.10). One may also extend ABJ's approach to

63

this problem, and obtain a larger outer region through four moment inequalities similar to those in equation (D.9). Our approach, which yields the sharp identification region, in this simple example requires one to check just 14 inequalities.

As shown in AT (Figure 3), the model with level-1 rationality only places upper bounds on θ_1, θ_2. Figure 5 plots the upper contours of Θ_I, Θ_O^{CT}, and Θ_O^{ABJ} in a simple example with $(\varepsilon_1, \varepsilon_2) \overset{iid}{\sim} N(0,1)$ and $\Theta = [-5, 0]^2$. The data is generated with $\theta_1^\star = -1.15$, $\theta_2^\star = -1.4$, and using a selection mechanism which picks outcome $(0,0)$ for 40% of $\omega : \varepsilon(\omega) \in [0, -\theta_1^\star] \times [0, -\theta_2^\star]$, outcome $(1,1)$ for 10% of $\omega : \varepsilon(\omega) \in [0, -\theta_1^\star] \times [0, -\theta_2^\star]$, and each of outcome $(1,0)$ and $(0,1)$ for 25% of $\omega : \varepsilon(\omega) \in [0, -\theta_1^\star] \times [0, -\theta_2^\star]$. Hence, the observed distribution is $\mathbf{P}(y) = [0.5048 \ 0.2218 \ 0.1996 \ 0.0738]'$. Our methodology allows us to obtain significantly lower upper contours compared to AT (and CT) and ABJ. The upper bounds on θ_1, θ_2 resulting from the projections of Θ_O^{ABJ}, Θ_O^{CT} and Θ_I are, respectively, $(-0.02, -0.02)$, $(-0.15, -0.26)$, and $(-0.54, -0.61)$.

E.2 Objective Correlated Equilibria

Suppose that players play correlated equilibria, a notion introduced by Aumann (1974). A correlated equilibrium can be interpreted as the distribution of play instructions given by some "trusted authority" to the players. Each player is given her instruction privately but does not know the instruction received by others. The distribution of instructions is common knowledge across all players. Then a correlated joint strategy $\gamma \in \Delta^{\kappa_{\mathcal{Y}}-1}$, where $\Delta^{\kappa_{\mathcal{Y}}-1}$ denotes the set of probability distributions on $\mathcal{Y}$, is an equilibrium if, conditional on knowing that her own instruction is to play y_j, each player j has no incentive to deviate to any other strategy y_j', assuming that the other players follow their own instructions. In this case one can define the θ-dependent set

$$C_\theta(\underline{x}, \varepsilon) = \left\{ \gamma \in \Delta^{\kappa_{\mathcal{Y}}-1} : \begin{array}{l} \sum_{y_{-j} \in \mathcal{Y}_{-j}} \gamma(y_j, y_{-j}) \pi_j(y_j, y_{-j}, x_j, \varepsilon_j, \theta) \geq \\ \sum_{y_{-j} \in \mathcal{Y}_{-j}} \gamma(y_j, y_{-j}) \pi_j\left(y_j', y_{-j}, x_j, \varepsilon_j, \theta\right), \ \forall y_j \in \mathcal{Y}_j, \ \forall y_j' \in \mathcal{Y}_j, \ \forall j \end{array} \right\}.$$

Omitting the explicit reference to its dependence on $\underline{x}$ and ε, C_θ is the set of correlated equilibrium strategies of the game. By similar arguments as those used before, it is a random closed set in $\Delta^{\kappa_{\mathcal{Y}}-1}$. Notice that C_θ is defined by a finite number of linear inequalities on the set $\Delta^{\kappa_{\mathcal{Y}}-1}$ of correlated strategies, and therefore it is a non-empty polytope. Yang (2008) is the first to use this fact, along with the fact that $\text{co}[Q(S_\theta)] \subset C_\theta$, to develop a computationally easy-to-implement estimator for an outer identification region of θ, when the solution concept employed is Nash equilibrium. Here we provide a simple characterization of the sharp identification region Θ_I, when the solution concept employed is objective correlated equilibrium. In particular, the same approach

of Section 3 of BMM allows us to obtain the sharp identification region for θ as

$$\Theta_I = \left\{ \theta \in \Theta : u' \mathbf{P} \left(y | \underline{x} \right) \leq \mathbf{E} \left[h \left(C_\theta, u \right) | \underline{x} \right] \ \forall \ u \in B \ \underline{x} - a.s. \right\}.$$

In our simple two player simultaneous move, complete information, static game of entry, $\mathcal{Y}_j = \{0,1\}$, $j = 1,2$, $\mathcal{Y} = \{(0,0),(1,0),(0,1),(1,1)\}$. Omitting again the covariates, we assume that players' payoffs are given by $\pi_j = y_j \left(y_{-j} \theta_j + \varepsilon_j \right)$, where $y_j \in \{0,1\}$ and θ_j is assumed to be negative (monopoly payoffs are higher than duopoly payoffs), $j = 1,2$. Figure 6 plots the set C_θ against the possible realizations of $\varepsilon_1, \varepsilon_2$, for this example. Notice that for $\omega \in \Omega$ such that $\varepsilon(\omega) \notin [0, -\theta_1] \times [0, -\theta_2]$, the game is dominance solvable and therefore $C_\theta(\omega)$ is given by the singleton $Q_\theta(\omega)$ resulting from the unique Nash equilibrium in these regions. For $\omega \in \Omega$ such that $\varepsilon(\omega) \in [0, -\theta_1] \times [0, -\theta_2]$, $C_\theta(\omega)$ is given by a polytope with five vertices, three of which are implied by Nash equilibria, see Calvó-Armengol (2006), and given by

$$\gamma_0(\omega) = [0 \ 0 \ 1 \ 0]',$$

$$\gamma_1(\omega) = \left[1 \ -\frac{\varepsilon_2(\omega)}{\theta_2 + \varepsilon_2(\omega)} \ -\frac{\varepsilon_1(\omega)}{\theta_1 + \varepsilon_1(\omega)} \ 0 \right]' \left(1 - \frac{\varepsilon_1(\omega)}{\theta_1 + \varepsilon_1(\omega)} - \frac{\varepsilon_2(\omega)}{\theta_2 + \varepsilon_2(\omega)} \right)^{-1},$$

$$\gamma_2(\omega) = \left[\left(1 + \frac{\varepsilon_2(\omega)}{\theta_2} \right) \left(1 + \frac{\varepsilon_1(\omega)}{\theta_1} \right) \ -\frac{\varepsilon_2(\omega)}{\theta_2} \left(1 + \frac{\varepsilon_1(\omega)}{\theta_1} \right) \ -\left(1 + \frac{\varepsilon_2(\omega)}{\theta_2} \right) \frac{\varepsilon_1(\omega)}{\theta_1} \ \frac{\varepsilon_2(\omega)}{\theta_2} \frac{\varepsilon_1(\omega)}{\theta_1} \right]',$$

$$\gamma_3(\omega) = \left[0 \ -\frac{\varepsilon_2(\omega)}{\theta_2 + \varepsilon_2(\omega)} \ -\frac{\varepsilon_1(\omega)}{\theta_1 + \varepsilon_1(\omega)} \ \frac{\varepsilon_1(\omega)}{\theta_1 + \varepsilon_1(\omega)} \frac{\varepsilon_2(\omega)}{\theta_2 + \varepsilon_2(\omega)} \right]' \left(\frac{\varepsilon_1(\omega)}{\theta_1 + \varepsilon_1(\omega)} \frac{\varepsilon_2(\omega)}{\theta_2 + \varepsilon_2(\omega)} - \frac{\varepsilon_1(\omega)}{\theta_1 + \varepsilon_1(\omega)} - \frac{\varepsilon_2(\omega)}{\theta_2 + \varepsilon_2(\omega)} \right)^{-1},$$

$$\gamma_4(\omega) = [0 \ 1 \ 0 \ 0]'.$$

Also in this case one can extend the approaches of ABJ and CT to obtain outer regions defined, respectively, by four and eight moment inequalities.

Figure 7 and Table 2 report Θ_I, Θ_O^{CT}, and Θ_O^{ABJ} in a simple example with $(\varepsilon_1, \varepsilon_2) \overset{iid}{\sim} N(0,1)$ and $\Theta = [-5, 0]^2$. In the figure, Θ_O^{ABJ} is given by the union of the yellow, red and black areas, and Θ_O^{CT} by the union of the red and black areas. Θ_I is the black region. The data is generated with $\theta_1^\star = -1.15$, $\theta_2^\star = -1.4$, and using a selection mechanism which picks each of outcome $(0,0)$ and $(1,1)$ for 10% of $\omega : \varepsilon(\omega) \in [0, -\theta_1^\star] \times [0, -\theta_2^\star]$, and each of outcome $(1,0)$ and $(0,1)$ for 40% of $\omega : \varepsilon(\omega) \in [0, -\theta_1^\star] \times [0, -\theta_2^\star]$. Hence, the observed distribution is $\mathbf{P}(y) = [0.26572 \ 0.34315 \ 0.36531 \ 0.02582]'$. Also in this case Θ_I is smaller than Θ_O^{CT} and Θ_O^{ABJ}, although the reduction in the size of the identification region is less pronounced than in the case where mixed strategy Nash equilibrium is the solution concept.

F   Multinomial Choice Models with Interval Regressors Data

This Section of the Supplement applies the methodology introduced in Section 2 of BMM to provide a tractable characterization of the sharp identification region of the parameters θ characterizing random utility models of multinomial choice, when only interval information is available on regressors. In doing so, we extend the seminal contribution of Manski and Tamer (2002), who considered the same inferential problem in the case of binary choice models. For these models, Manski and Tamer (2002) provided a tractable characterization of the sharp identification region, and proposed set estimators which are consistent with respect to the Hausdorff distance. However, their characterization of the sharp identification region does not easily extend to models in which the agents face more than two choices, as we illustrate below.

We assume that an agent chooses an alternative y from a finite choice set $\mathcal{C} = \{0, \ldots, \kappa_{\mathcal{C}} - 1\}$ to maximize her utility. The agent possesses a vector of socioeconomic characteristics w. Each alternative $k \in \mathcal{C}$ is characterized by an observable vector of attributes z_k and an attribute ε_k which is observable by the agent but not by the econometrician. The vector $\left(y, w, \{z_k, \varepsilon_k\}_{k=0}^{\kappa_{\mathcal{C}}-1}\right)$ is defined on a non-atomic probability space $(\Omega, \mathfrak{F}, \mathbf{P})$. The agent is assumed to possess a random utility function of known parametric form.

To simplify the exposition, we assume that the random utility is linear, and that w, z_k and ε_k, $k = 0, \ldots, \kappa_{\mathcal{C}} - 1$, are all scalars. However, *all* these assumptions can be relaxed and are in no way essential for our methodology. We let the random utility be $\pi(k; x_k, \varepsilon_k, \theta_k) = \alpha_k + z_k \delta + w \beta_k + \varepsilon_k \equiv x_k \theta_k + \varepsilon_k$, $k \in \mathcal{C}$, with $x_k = [1 \quad z_k \quad w]$ and $\theta_k = [\alpha_k \quad \delta \quad \beta_k]'$. We normalize $\pi(0; x_0, \varepsilon_0, \theta_0) = \varepsilon_0$. For simplicity, we assume that ε_k is independently and identically distributed across choices with a continuous distribution function $F(\varepsilon)$ that is known. We let $\theta = \left[\{\alpha_k\}_{k=1}^{\kappa_{\mathcal{C}}-1} \quad \delta \quad \{\beta_k\}_{k=1}^{\kappa_{\mathcal{C}}-1}\right]' \in \Theta$ be the vector of parameters of interest, with Θ the parameter space. We denote $\varepsilon^k = \varepsilon_k - \varepsilon_0$, $k \in \mathcal{C}$, and $\varepsilon = \left[\{\varepsilon^k\}_{k=1}^{\kappa_{\mathcal{C}}-1}\right]$. Under these assumptions, if the econometrician observes a random sample of choices, socioeconomic characteristics, and alternatives' attributes, the parameter vector θ is point identified.

Here we consider the identification problem arising when the econometrician observes only realizations of $\{y, z_{kL}, z_{kU}, w\}$, but not realizations of z_k, $k = 1, \ldots, \kappa_{\mathcal{C}} - 1$. Following Manski and Tamer (2002), we assume that for each $k = 1, \ldots, \kappa_{\mathcal{C}} - 1$, $\mathbf{P}(z_{kL} \leq z_k \leq z_{kU}) = 1$, and that $\delta > 0$. We let $x_{kL} = [1 \quad z_{kL} \quad w]$, $x_{kU} = [1 \quad z_{kU} \quad w]$, $\underline{x}_k = [1 \quad z_{kL} \quad z_{kU} \quad w]$, and $\underline{x} = [1 \quad \{z_{kL}\}_{k=1}^{\kappa_{\mathcal{C}}-1} \quad \{z_{kU}\}_{k=1}^{\kappa_{\mathcal{C}}-1} \quad w]$. Incompleteness of the data on z_k, $k = 1, \ldots, \kappa_{\mathcal{C}} - 1$, implies that there are regions of values of the exogenous variables where the econometric model predicts that more

than one choice may maximize utility. Therefore, the relationship between the outcome variable of interest and the exogenous variables is a correspondence rather than a function. Hence, the parameters of the utility functions may not be point identified.

In the case of binary choice, Manski and Tamer (2002) establish that the sharp identification region for θ is given by

$$\Theta_I = \left\{\theta \in \Theta : \mathbf{P}\left(x_{1L}\theta + \varepsilon^1 > 0 \middle| \underline{x}\right) \leq \mathbf{P}\left(y = 1 \middle| \underline{x}\right) \leq \mathbf{P}\left(x_{1U}\theta + \varepsilon^1 > 0 \middle| \underline{x}\right), \ \underline{x} - a.s.\right\}.$$

This construction is based on the observation that if the agent chooses alternative 1, this implies that $\varepsilon^1 > -x_1\theta \geq -x_{1U}\theta$. On the other hand, $\varepsilon^1 > -x_{1L}\theta \geq -x_1\theta$ implies that the agent chooses alternative 1.[11] In the case of more than two choices, one may wish to apply a similar insight as in the work of CT, and construct the region

(F.1)

$$\Theta_O = \left\{ \begin{array}{c} \theta \in \Theta : \forall m \in \mathcal{C}, \ \underline{x} - a.s., \\ \mathbf{P}\left(x_m\theta_m + \varepsilon^m \geq x_k\theta_k + \varepsilon^k \ \forall (x_m, x_k) \in [x_{mL}, x_{mU}] \times [x_{kL}, x_{kU}], \ \forall k \in \mathcal{C}, \ k \neq m \middle| \underline{x}\right) \\ \leq \mathbf{P}\left(y = m \middle| \underline{x}\right) \leq \\ \mathbf{P}\left(\exists \ x_m \in [x_{mL}, x_{mU}] \ \text{s.t.} \ \forall k \in \mathcal{C}, \ k \neq m, \exists \ x_k \in [x_{kL}, x_{kU}] \ \text{with} \ x_m\theta_m + \varepsilon^m \geq x_k\theta_k + \varepsilon^k \middle| \underline{x}\right) \end{array} \right\}.$$

The lower bound on $\mathbf{P}\left(y = m \middle| \underline{x}\right)$ in equation (F.1) is given by the probability that ε falls in the regions where choice $m \in \mathcal{C}$ is the only optimal alternative. The upper bound is given by the probability that ε falls in the regions where choice $m \in \mathcal{C}$ is one of the possible optimal alternatives. Similarly to the case of Θ_O^{CT} in the finite games analyzed in Section 3 of BMM, Θ_O is just an outer region for θ, and is not sharp in general. Appendix D.2 provides further insights to explain the lack of sharpness of Θ_O.[12]

We begin our treatment of the identification problem by noticing that, if x_k were observed for each $k \in \mathcal{C}$, one would conclude that a choice $m \in \mathcal{C}$ maximizes utility if

$$\pi(m; x_m, \varepsilon_m, \theta_m) = x_m\theta_m + \varepsilon_m \geq x_k\theta_k + \varepsilon_k = \pi(k; x_k, \varepsilon_k, \theta_k) \ \forall \ k \in \mathcal{C}, \ k \neq m.$$

Hence, for a given $\theta \in \Theta$ and realization of $\underline{x}$ and ε, we can define the following θ-dependent set:

(F.2)
$$\begin{aligned} M_\theta(\underline{x}, \varepsilon) &= \{m \in \mathcal{C} : \exists \ x_m \in [x_{mL}, x_{mU}] \\ \text{s.t.} \ \forall k &\in \mathcal{C}, k \neq m, \exists \ x_k \in [x_{kL}, x_{kU}] \ \text{with} \ x_m\theta_m + \varepsilon^m \geq x_k\theta_k + \varepsilon^k\}. \end{aligned}$$

This is the set of choices associated with a specific value of θ and realization of $\underline{x}$ and ε, which are optimal for some combination of $x_k \in [x_{kL}, x_{kU}]$, $k \in \mathcal{C}$, and therefore form the set of model's

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[11]For $-x_{1U}\theta \leq \varepsilon^1 \leq -x_{1L}\theta$, the model predicts that either alternative 0 or 1 may maximize the agent's utility.

[12]Appendix D.2 focuses on the lack of sharpness of Θ_O^{CT} in finite games with multiple pure strategy Nash equilibria. The same reasoning applies to the set Θ_O in equation (F.1).

predictions. As we did in Section 3 of BMM, we write the set $M_\theta(\underline{x}, \varepsilon)$ and its realizations, respectively, as M_θ and $M_\theta(\omega) \equiv M_\theta(\underline{x}(\omega), \varepsilon(\omega))$, omitting the explicit reference to $\underline{x}$ and ε. Because M_θ is a subset of a discrete space, and any event of the type $\{m \in M_\theta\}$ can be represented as a combination of measurable events determined by ε_k, $k \in \mathcal{C}$, M_θ is a random closed set in $\mathcal{C}$, see Definition A.1 of BMM.

We now apply to the random closed set M_θ the same logic that we applied to the random closed set S_θ in Section 3 of BMM. The treatment which follows is akin to the treatment of static, simultaneous move finite games of complete information, when players use only pure strategies.

For a given parameter value $\theta \in \Theta$ and realization $m(\omega)$, $\omega \in \Omega$, of a selection $m \in \mathrm{Sel}(M_\theta)$, the individual chooses alternative $k = 0, \ldots, \kappa_\mathcal{C} - 1$ if and only if $m(\omega) = k$. Hence, we can use a selection $m \in \mathrm{Sel}(M_\theta)$ to define a random point $q(m)$ whose realizations have coordinates $[q(m(\omega))]_k = 1(m(\omega) = k)$, $k = 0, \ldots, \kappa_\mathcal{C} - 1$, with $1(\cdot)$ the indicator function of the event in parenthesis. Clearly, the random point $q(m)$ is an element of the unit simplex in the space of dimension $\kappa_\mathcal{C}$, denoted $\Delta^{\kappa_\mathcal{C}-1}$. Because M_θ is a random closed set in $\mathcal{C}$, the set resulting from repeating the above construction for each $m \in \mathrm{Sel}(M_\theta)$ and given by

$$Q(M_\theta) = \{([q(m)]_k, \ k = 0, \ldots, \kappa_\mathcal{C} - 1) : m \in \mathrm{Sel}(M_\theta)\},$$

is a closed random set in $\Delta^{\kappa_\mathcal{C}-1}$. Hence we can define the set

$$\mathbb{E}(Q(M_\theta)|\underline{x}) = \{\mathbf{E}(q|\underline{x}) : q \in \mathrm{Sel}(Q(M_\theta))\} = \{(\mathbf{E}([q(m)]_k|\underline{x}), k = 0, \ldots, \kappa_\mathcal{C} - 1) : m \in \mathrm{Sel}(M_\theta)\}.$$

Because the probability space is non-atomic and the random set $Q(M_\theta)$ takes its realizations in a subset of the finite dimensional space $\Re^{\kappa_\mathcal{C}}$, the set $\mathbb{E}(Q(M_\theta)|\underline{x})$ is a closed convex set for $\underline{x} - a.s.$ By construction, it is the set of probability distributions over alternatives conditional on $\underline{x}$ which are consistent with the maintained modeling assumptions, i.e., with *all* the model implications. If the model is correctly specified, there exists at least one value of $\theta \in \Theta$ such that the observed conditional distribution of y given $\underline{x}$, $\mathbf{P}(y|\underline{x})$, is a point in the set $\mathbb{E}(Q(M_\theta)|\underline{x})$ for $\underline{x} - a.s.$, where $\mathbf{P}(y|\underline{x}) \equiv [\mathbf{P}(y = k|\underline{x}), k = 0, \ldots, \kappa_\mathcal{C} - 1]$.

Using the same mathematical tools leading to Theorem 3.2 of BMM, we obtain that the set of observationally equivalent parameter values which form the sharp identification region is given by

(F.3) $\quad \Theta_I = \left\{ \theta \in \Theta : \max_{u \in B} \left(u'\mathbf{P}(y|\underline{x}) - \mathbf{E}[h(Q(M_\theta), u)|\underline{x}] \right) = 0 \ \underline{x} - a.s. \right\},$

with B the unit ball in $\Re^{\kappa_\mathcal{C}}$.

Notice that the set $Q\left(M_\theta\right)$ assumes at most a finite number of values, and its realizations lie in the subsets of the vertices of Δ^{κ_c-1}. The conditional Aumann expectation of $Q\left(M_\theta\right)$ is given by the weighted Minkowski sum of the possible realizations of $\mathrm{co}\left[Q\left(M_\theta\right)\right]$. Each of these realizations is a polytope, and therefore $\mathbb{E}\left(\left.Q\left(M_\theta\right)\right|\underline{x}\right)$ is a closed convex polytope. By Theorem D.1, a candidate θ belongs to Θ_I as defined in equation (F.3) if and only if $u'\mathbf{P}\left(\left.y\right|\underline{x}\right) \leq \mathbf{E}\left[\left.h\left(Q\left(M_\theta\right),u\right)\right|\underline{x}\right]$ for each of the 2^{κ_c} possible u vectors whose entries are either equal to zero or to one. Hence, Θ_I can be obtained through a finite set of moment inequalities which have to hold for $\underline{x}-a.s.$

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G   Tables and Figures

Table 1: Projections of Θ_O^{ABJ}, Θ_O^{CT} and Θ_I, and reduction in bounds width compared to ABJ. Four player, two type entry game with pure strategy Nash equilibrium as solution concept.

	True Values	Projections of:		
		Θ_O^{ABJ}	Θ_O^{CT}	Θ_I
$\theta_{11}^\star$	-0.15	$[-0.154, -0.144]$	$[-0.153, -0.146]$	$[-0.152, -0.147]$
			(27%)	(54%)
$\theta_{21}^\star$	-0.20	$[-0.206, -0.195]$	$[-0.204, -0.197]$	$[-0.203, -0.198]$
			(27%)	(54%)
$\theta_{22}^\star$	-0.10	$[-0.106, -0.096]$	$[-0.104, -0.097]$	$[-0.103, -0.098]$
			(27%)	(54%)

Table 2: Projections of Θ_O^{ABJ}, Θ_O^{CT} and Θ_I, reduction in bounds width (in parentheses), and area of the identification regions compared to ABJ. Two player entry game with correlated equilibrium as solution concept.

	True Values	Projections of:		
		Θ_O^{ABJ}	Θ_O^{CT}	Θ_I
$\theta_1^\star$	-1.15	$[-4.475, -0.485]$	$[-4.475, -0.585]$	$[-4.125, -0.595]$
			(2.5%)	(11.5%)
$\theta_2^\star$	-1.40	$[-4.585, -0.625]$	$[-4.585, -0.725]$	$[-4.425, -0.735]$
			(2.4%)	(6.8%)
Approximate Reduction in Total Area Compared to Θ_O^{ABJ}			(7.9%)	(23.1%)

Figure 1: The random set of pure strategy NE profiles, S_θ, and the random set of pure strategy NE outcomes, Y_θ, as a function of $\varepsilon_1, \varepsilon_2$ in a two player entry game. In this simple example the two sets coincide.
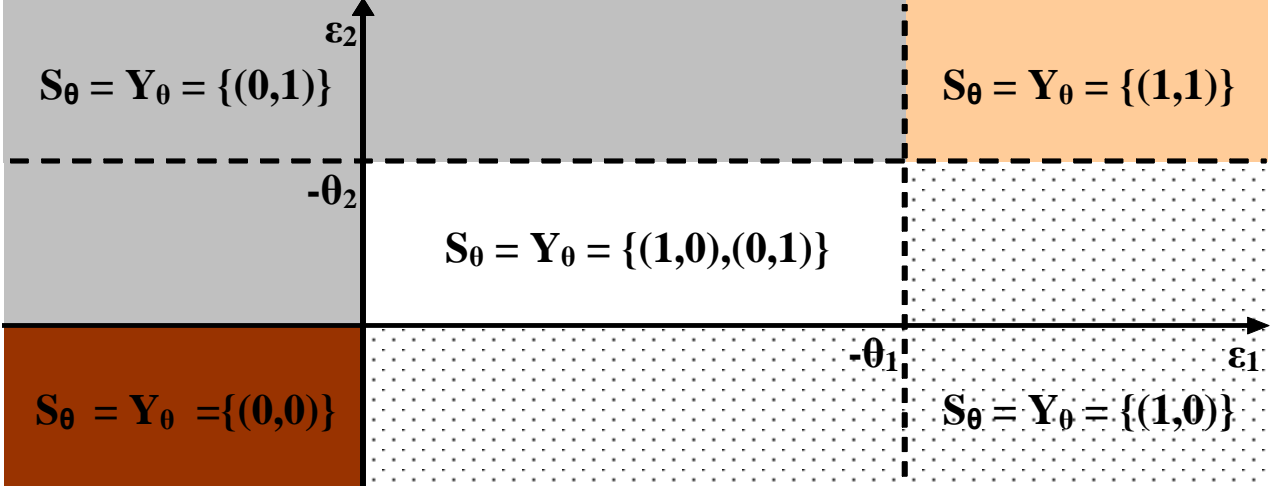

ε_2

$S_\theta = Y_\theta = \{(0,1)\}$

$S_\theta = Y_\theta = \{(1,1)\}$

$-\theta_2$

$S_\theta = Y_\theta = \{(1,0),(0,1)\}$

$-\theta_1$

ε_1

$S_\theta = Y_\theta = \{(0,0)\}$

$S_\theta = Y_\theta = \{(1,0)\}$

Figure 2: The random set of pure strategy NE outcomes as a function of $\varepsilon_1, \varepsilon_2$ in a four player, two type entry game.

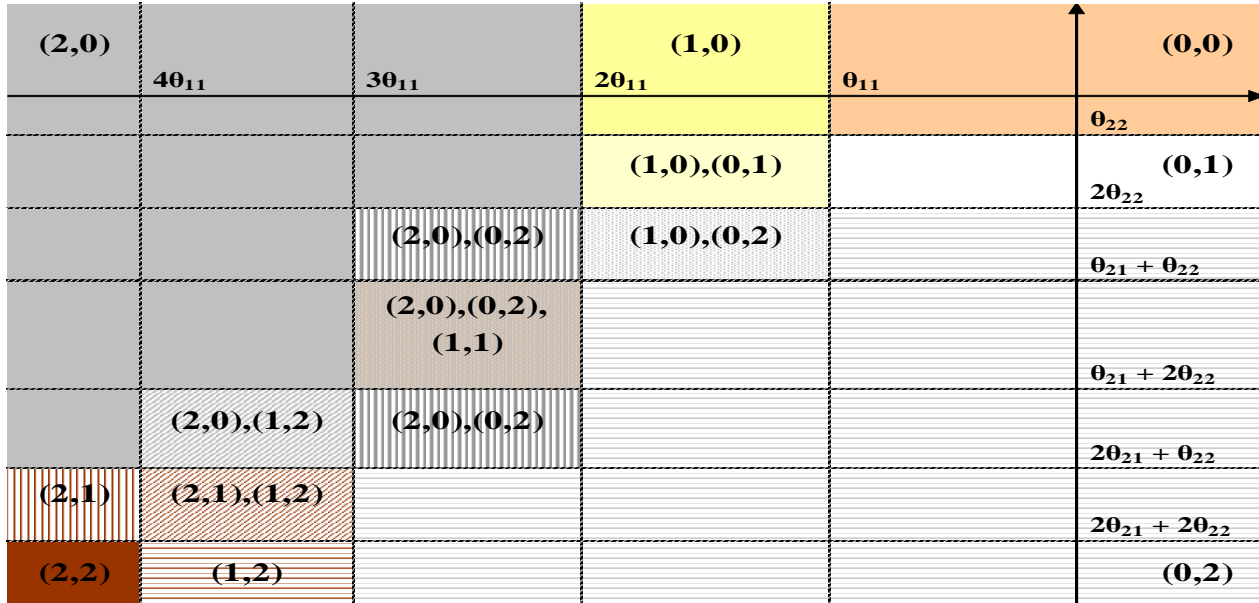| $4\theta_{11}$	$3\theta_{11}$	$2\theta_{11}$	θ_{11}		
(2,0)			**(1,0)**	**(0,0)**	
					θ_{22}
			(1,0),(0,1)	**(0,1)**	
					$2\theta_{22}$
		(2,0),(0,2)	**(1,0),(0,2)**		
					$\theta_{21} + \theta_{22}$
		(2,0),(0,2), (1,1)			
					$\theta_{21} + 2\theta_{22}$
	(2,0),(1,2)	**(2,0),(0,2)**			
					$2\theta_{21} + \theta_{22}$
(2,1)	**(2,1),(1,2)**				
					$2\theta_{21} + 2\theta_{22}$
(2,2)	**(1,2)**			**(0,2)**	

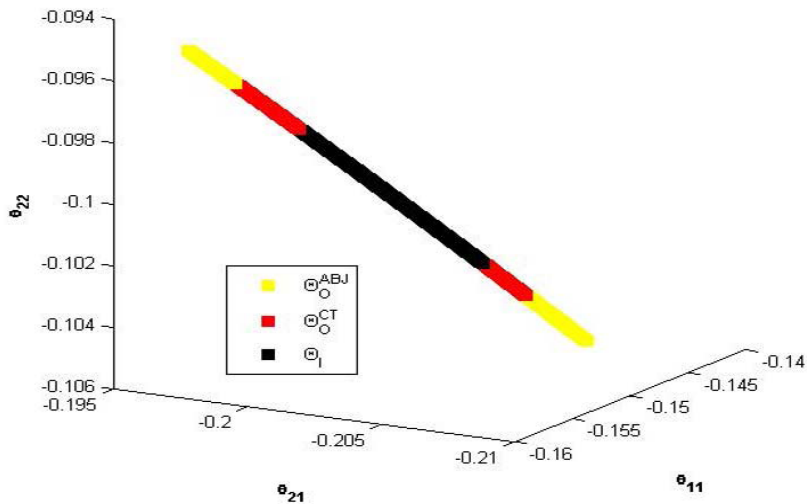Figure 3: Identification regions in a four player, two type entry game with pure strategy Nash equilibrium as solution concept.

Figure 4: The random set of level-1 rational profiles as a function of $\varepsilon_1, \varepsilon_2$ in a two player entry game.
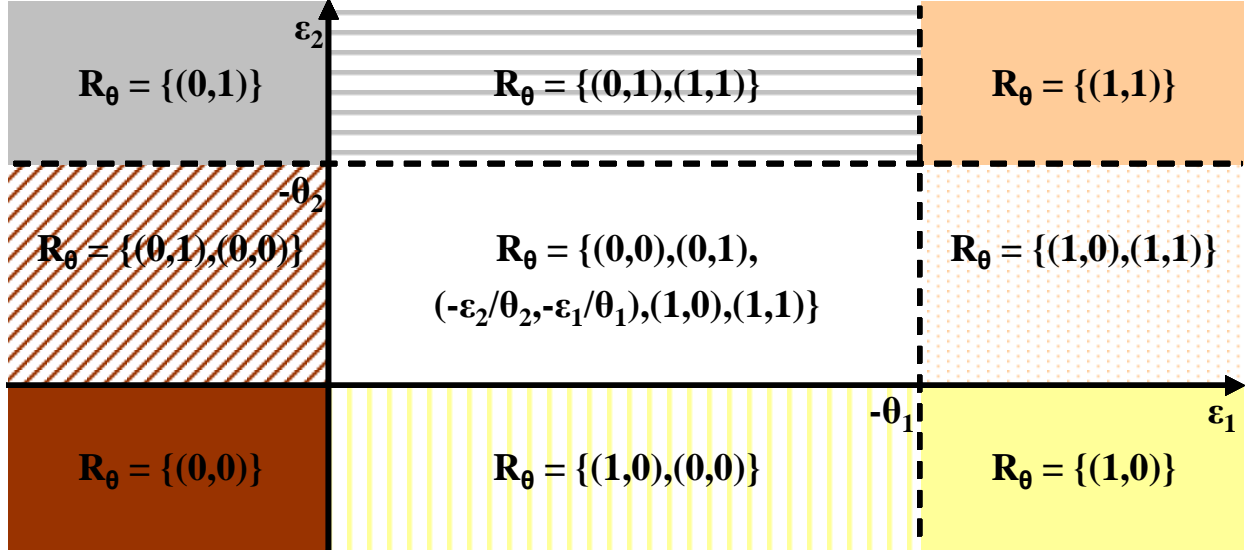

$\mathbf{R_\theta} = \{(0,1)\}$	$\mathbf{R_\theta} = \{(0,1),(1,1)\}$	$\mathbf{R_\theta} = \{(1,1)\}$
$\mathbf{R_\theta} = \{(0,1),(0,0)\}$	$\mathbf{R_\theta} = \{(0,0),(0,1),$ $(-\varepsilon_2/\theta_2,-\varepsilon_1/\theta_1),(1,0),(1,1)\}$	$\mathbf{R_\theta} = \{(1,0),(1,1)\}$
$\mathbf{R_\theta} = \{(0,0)\}$	$\mathbf{R_\theta} = \{(1,0),(0,0)\}$	$\mathbf{R_\theta} = \{(1,0)\}$

Figure 5: Upper contours of the identification regions in a two player entry game with level-1 rationality as solution concept.
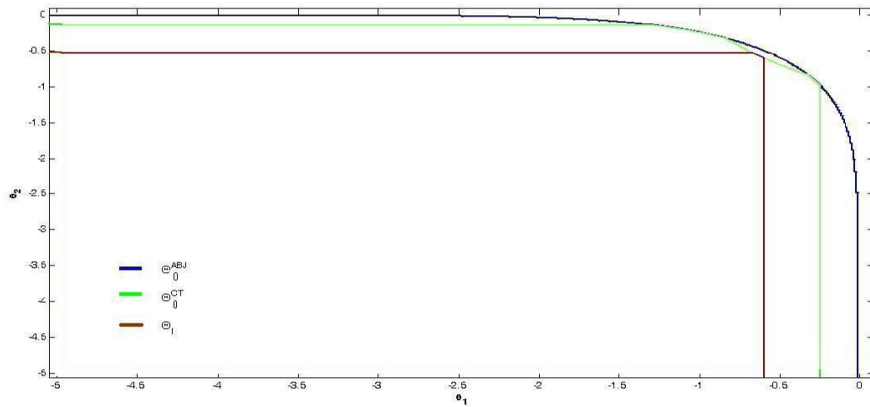
Figure 6: The random set of correlated equilibria as a function of $\varepsilon_1, \varepsilon_2$ in a two player entry game. The correlated equilibria $\gamma_1, \gamma_2, \gamma_3$ are defined in Section E.2.
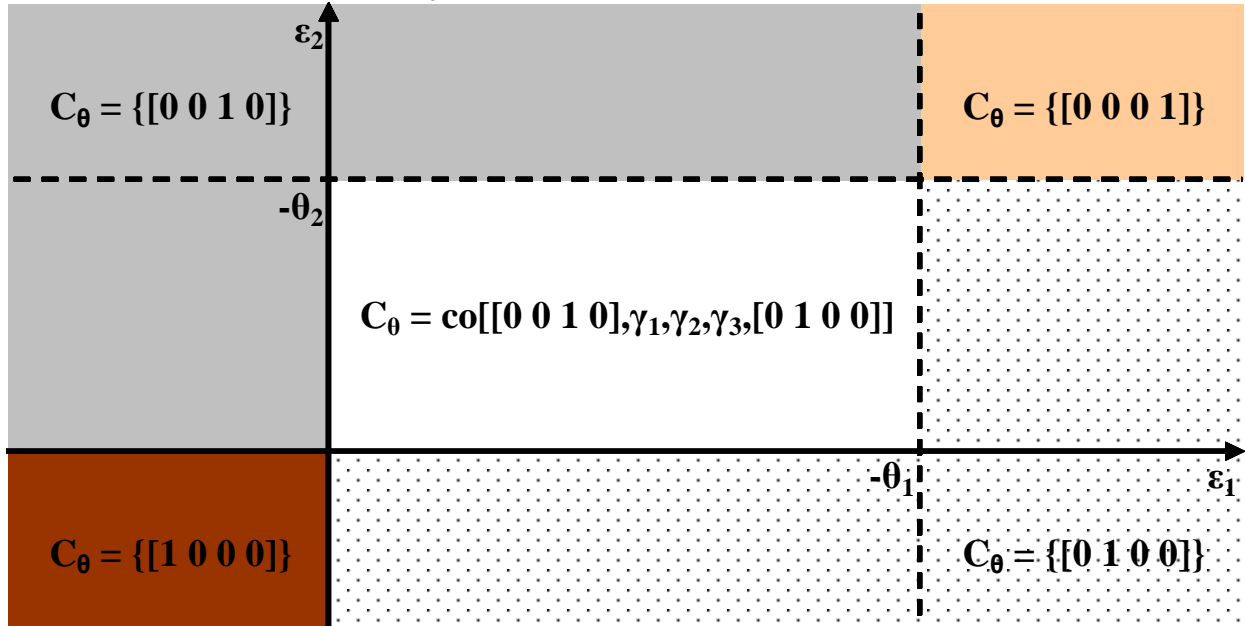

Figure 7: Identification regions in a two player entry game with correlated equilibrium as solution concept.