Aufgabe 1 (i)

$$(\hat{f})^{\vee}(y) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{iy\cdot\xi} \hat{f}(\xi)\, d\xi$$

$$= \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^n} e^{iy\cdot\xi} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx\, d\xi$$

$$= \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{i\xi(y-x)}\, d\xi \right) f(x)\, dx$$

Translation + Hinweis
$$= \int_{\mathbb{R}^n} \delta(y-x) f(x)\, dx$$

$$= f(y)$$

(ii)

$$\widehat{\partial_i g}(\xi) = \int (2\pi)^{-\frac{n}{2}} e^{-i\xi x}\, \partial_i g(x)\, dx$$

$$= (2\pi)^{-\frac{n}{2}} \int -\partial_{x_i}\left(e^{-i\xi x}\right) g(x)\, dx$$

$$= (2\pi)^{-\frac{n}{2}} \int i\xi_i\, e^{-i\xi x} g(x)\, dx \qquad = (i\xi_i)\hat{g}(\xi)$$

$$(\widehat{-ix_i f})(\xi) = (2\pi)^{-\frac{n}{2}} \int e^{-i\xi x} (-ix_i) f(x)\, dx$$

$$= (2\pi)^{-\frac{n}{2}} \int \left(\partial_{\xi_i} e^{-i\xi\cdot x}\right) f(x)\, dx$$

$$= \partial_{\xi_i} \left((2\pi)^{-\frac{n}{2}} \int e^{-i\xi\cdot x} f(x)\, dx \right)$$

$$= \partial_{\xi_i} \hat{f}(x)$$

(iii)

$$\widehat{\Delta g} = \widehat{\partial_i \partial_i g} = -i\xi_i \cdot \widehat{\partial_i g} = (-i\xi_i)^2 \cdot \hat{g} = -|\xi|^2 \hat{g}$$

Summe

(iv)

$$\hat{f}_\lambda(\xi) = (2\pi)^{-\frac{n}{2}} \int e^{-i\xi\cdot x} f(\lambda x)\, dx$$

$$\overset{\text{Trafo}}{=} (2\pi)^{-\frac{n}{2}} \lambda^{-n} \int e^{-i\xi\cdot\frac{x}{\lambda}} f(x)\, dx \quad , \qquad e^{-i\xi\cdot\frac{x}{\lambda}} = e^{-i\frac{\xi}{\lambda}\cdot x}$$

$$= \lambda^{-n} \hat{f}\left(\frac{\xi}{\lambda}\right)$$

(v)

$$\hat{f}(\xi) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-i\xi x} e^{-\frac{x^2}{2}}\, dx$$

$$\partial_\xi \hat{f}(\xi) = \widehat{(-ixf)}(\xi)$$

$$= (2\pi)^{-1} \int_{-\infty}^{\infty} \underbrace{-ixe^{-\frac{x^2}{2}}}_{u'} \underbrace{e^{-i\xi x}}_{v} dx$$

$$\overset{\text{part. Int.}}{=} (2\pi)^{-1} \int_{-\infty}^{\infty} -i\, e^{-\frac{x^2}{2}} \left(-i\xi e^{-i\xi x}\right) dx$$

$$= (2\pi)^{-1} \int_{-\infty}^{\infty} (-\xi)\, e^{-\frac{x^2}{2}} e^{-i\xi x}\, dx$$

$$= -\xi \hat{f}(\xi)$$

$$\Rightarrow -\partial_\xi \hat{f}(\xi) = \xi \hat{f}(\xi) \qquad \Leftrightarrow \qquad \frac{\hat{f}'(\xi)}{\hat{f}(\xi)} = -\xi$$

$$\Rightarrow \int_{\xi_0}^{\xi} \frac{\hat{f}'(\eta)}{\hat{f}(\eta)}\, d\eta = \int_{\xi_0}^{\xi} -\eta\, d\eta$$

$$\Longrightarrow \quad \log \hat{f}(\xi) - \log \hat{f}(\xi_0) = \frac{\xi_0^2}{2} - \frac{\xi^2}{2}$$

$$\Longleftrightarrow \quad \log \hat{f}(\xi) = -\frac{\xi^2}{2} + C$$

$$\Longrightarrow \quad \hat{f}(\xi) = C' e^{-\frac{\xi^2}{2}}$$

Bestimmung von C':

$$\hat{f}(0) = (2\pi)^{-\frac{1}{2}} \int e^{-i \cdot 0 \cdot x} f(x)\,dx$$

$$= (2\pi)^{-\frac{1}{2}} \underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{=1,\ \text{da} \int e^{-\xi^2} d\xi = \sqrt{\pi}}$$

(vi)

Sei $\lambda := (2t)^{-\frac{1}{2}}$. Dann ist $(2\pi)^{\frac{n}{2}} \hat{f}(\frac{\xi}{\lambda}) = e^{-t\frac{|\xi|^2}{\lambda^2}} = e^{-\frac{|\xi|^2}{2}}$

Nach (iv) ist $\hat{f}(\frac{\xi}{\lambda}) = \lambda^n \widehat{f(\lambda x)}$ und nach (v) ist $e^{-\frac{\xi^2}{2}} = \widehat{e^{-\frac{x^2}{2}}}$. Damit:

$$\lambda^n \widehat{f(\lambda x)} = \widehat{e^{-\frac{x^2}{2}}} \cdot (2\pi)^{-\frac{n}{2}}$$

$$\Longrightarrow \quad \lambda^n f(\lambda x) = e^{-\frac{x^2}{2}} (2\pi)^{-\frac{n}{2}}$$

$$\Longrightarrow \quad f(x) = (2\pi\lambda^2)^{-\frac{n}{2}} e^{-\frac{x^2}{2\lambda^2}}$$

$$= (4\pi t)^{-\frac{n}{2}} e^{-\frac{x^2}{4t}}$$

(vii)

$$(2\pi)^n \int \hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta = \iiint e^{-i(\xi-\eta)x} f(x) e^{-i\eta y} g(y)\,dx\,dy\,d\eta$$

$$= \iiint e^{-i\xi x} e^{-i\eta(x-y)} f(x) g(y)\,d\eta\,dx\,dy$$

$$= \iint (2\pi)^n \delta(x-y) e^{-i\xi x} f(x) g(y)\,dx\,dy$$

$$= (2\pi)^n \int e^{-i\xi x} f(x) g(x)\,dx$$

$$= (2\pi)^{\frac{3}{2}n} \widehat{(fg)}(\xi) \qquad \Longrightarrow \qquad \hat{f} * \hat{g} = (2\pi)^{\frac{n}{2}} \widehat{(fg)}$$

Aufgabe 2:

(i)

$$\partial_t \Phi(x,t) = \frac{|x|^2 e^{-\frac{|x|^2}{4t}}}{4t^2 (4\pi t)^{\frac{n}{2}}} - \frac{n e^{-\frac{|x|^2}{4t}}}{2t(4\pi t)^{\frac{n}{2}}}$$

$$\partial_{x_i} \Phi(x,t) = \frac{-x_i e^{-\frac{|x|^2}{4t}}}{2t(4\pi t)^{\frac{n}{2}}}$$

$$\partial_{x_i}^2 \Phi(x,t) = \frac{x_i^2 e^{-\frac{|x|^2}{4t}}}{4t^2(4\pi t)^{\frac{n}{2}}} - \frac{e^{-\frac{|x|^2}{4t}}}{2t(4\pi t)^{\frac{n}{2}}}$$

Identität folgt durch Summation über i.

(ii)

$$\frac{e^{-\frac{c}{t}}}{t} \longrightarrow 0, \qquad t \longrightarrow 0$$

(iii) $t^{-\frac{n}{2}} \longrightarrow \infty, \qquad t \longrightarrow 0$

~~Def: $f \in \mathcal{S}(\mathbb{R}^n) \iff \sup |x^\alpha \partial^\beta f(x)| < \infty \ \forall \alpha,\beta \in \mathbb{N}^n \wedge f \in C^\infty(\mathbb{R}^n)$~~

Schwartzraum:

$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) \;\middle|\; \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty \ \forall \alpha,\beta \in \mathbb{N}_0^n \right\}$$

Aufgabe 1 (ii) $\Longrightarrow$ $\mathcal{F}(\mathcal{S}(\mathbb{R}^n)) \subset \mathcal{S}(\mathbb{R}^n)$

Lemma: Seien $\cancel{f,g \in \mathcal{S}}(\mathbb{R}^n)$. Dann gilt:

$$\langle \hat{f}, \hat{g} \rangle_{L^2} = \langle f, g \rangle_{L^2}$$

Beweis:

$$\int \hat{f}\,\overline{\hat{g}}\, d\xi = \int f\, \mathcal{F}(\overline{\mathcal{F}g})\, dx$$

$$= \int f\, \mathcal{F}\left((2\pi)^{-\frac{n}{2}} \int \overline{e^{-i\xi\cdot y} g(y)}\, dy \right) dx$$

$$= \int f\, \mathcal{F}(\mathcal{F}^{-1}\overline{g})\, dx$$

$$= \int f\,\overline{g}\, dx$$

Satz: Sei $f \in S(\mathbb{R}^n)$. Dann gilt:

$$\mathcal{F}(\mathcal{F} f)(x) = f(-x)$$

Beweis:

Bem: Fubini $\Rightarrow$ $\int \hat{f} g \, dx = \int f \hat{g} \, dx$ $\quad (*)$

Sei $\gamma(x) := e^{-\frac{x^2}{2}}$, $\gamma_a(x) = \gamma(ax)$, $a > 0$. Dann gilt:

$$\mathcal{F}(e^{-ix\xi_0}\gamma_a)(\xi) = (2\pi)^{-\frac{n}{2}} \int e^{-ix\xi_0} \gamma(ax) e^{-ix\xi} \, dx$$

$$= \hat{\gamma}_a(\xi + \xi_0) \quad (**)$$

Betrachte nun:

$$(2\pi)^{-\frac{n}{2}} \int \hat{f}(x) e^{-ix\xi_0} \gamma(ax) \, dx \overset{(**)}{\underset{(*)}{=}} (2\pi)^{-\frac{n}{2}} \int f(x) \hat{\gamma}_a(x + \xi_0) \, dx$$

$$= (2\pi)^{-\frac{n}{2}} a^{-n} \int f(x) \hat{\gamma}\left(\frac{x+\xi_0}{a}\right) dx$$

$$= (2\pi)^{-\frac{n}{2}} \int f(ay - \xi_0) \underbrace{\hat{\gamma}(y)}_{= \gamma(y)} \, dy$$

Linke Seite:

$$\left.\begin{array}{l} \hat{f}(x) e^{-ix\xi_0} \gamma(ax) \xrightarrow[a \to 0]{} \hat{f}(x) e^{-ix\xi_0} \quad \forall x \\ \text{und } |\hat{f} e^{-ix\xi_0} \gamma_a| \leq |\hat{f}| \end{array}\right\} \text{Lebesgue} \Rightarrow \text{Integral konvergiert}$$

Rechte Seite:

$$\left.\begin{array}{l} f(ay - \xi_0)\gamma(y) \to f(-\xi_0)\gamma(y) \quad \forall y \\ \text{und } |f(ay-\xi_0)\gamma| \leq \|f\|_\infty \gamma \end{array}\right\} \text{Lebesgue} \Rightarrow \text{Integral konvergiert.}$$

$$\Longrightarrow \qquad (2\pi)^{-\frac{n}{2}} \int \hat{f}(x)\, e^{-ix\xi_0}\, dx = (2\pi)^{-\frac{n}{2}} f(-\xi_0) \underbrace{\int \gamma(y)\, dy}_{= (2\pi)^{\frac{n}{2}}}$$

$$\Longleftrightarrow \qquad \hat{\hat{f}}(\xi_0) = f(-\xi_0)$$