SERIE 12

45) $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ IS RADIAL, IF THERE EXISTS $\tilde{f} : [0, \infty] \to \overline{\mathbb{R}}$ SUCH THAT $f(x) = \tilde{f}(|x|)$, $\forall x \in \mathbb{R}^n$.

(i) $\chi_{B_\rho(0)} = \begin{cases} 1, & \text{IF } |x| < \rho \\ 0, & \text{OTHERWISE} \end{cases}$

NOW $\chi_{[0,\rho)} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ IS SUCH THAT

$$\chi_{B_\rho(0)}(x) = \chi_{[0,\rho)}(|x|).$$

$\Rightarrow$ $\chi_{B_\rho(0)}$ IS A RADIAL FUNCTION.

(ii) f, g RADIAL $\Rightarrow \exists\, \tilde{f}, \tilde{g}$ SUCH THAT $\left.\begin{array}{l} f(x) = \tilde{f}(|x|) \\ g(x) = \tilde{g}(|x|) \end{array}\right\}$

a) $|f|(x) = |f(x)| = |\tilde{f}(|x|)| = |\tilde{f}|(|x|)$

$\Rightarrow$ $|f|$ IS RADIAL.

b) $(f \cdot g)(x) = f(x)\, g(x) = \tilde{f}(|x|) \cdot \tilde{g}(|x|) = (\tilde{f} \cdot \tilde{g})(|x|)$

$\Rightarrow$ $f \cdot g$ IS RADIAL.

c) $(\lambda f + \mu g)(x) = \lambda f(x) + \mu\, g(x) = \lambda \tilde{f}(|x|) + \mu\, \tilde{g}(|x|)$

$= (\lambda \tilde{f} + \mu \tilde{g})(|x|)$

$\Rightarrow$ $\lambda f + \mu g$ IS RADIAL.

d) $\max\{f, g\} = (f - g)^+ + g$

WE KNOW THAT g IS RADIAL.

$(f - g)^+(x) = (\tilde{f} - \tilde{g})^+(|x|)$ $\Rightarrow$ $(f - g)^+$ IS RADIAL

USING (c), IT FOLLOWS THAT $\max\{f, g\}$ IS RADIAL.

iii) $f : \mathbb{R}^2 \to [0, \infty]$ IS MEASURABLE, THEN (BEISPIEL 7.3),

$$\int_{\mathbb{R}^2} f \, d\lambda^2 = \int_{r=0}^{\infty} \int_0^{2\pi} r f(r\cos\theta, r\sin\theta) \, d\theta \, dr.$$

SUPPOSE f IS ALSO RADIAL, THAT IS, $f(x) = \tilde{f}(|x|)$.

THEN, $$\int_{\mathbb{R}^2} f \, d\lambda^2 = \int_{r=0}^{\infty} \int_0^{2\pi} r \tilde{f}\Big(\underbrace{\|(r\cos\theta, r\sin\theta)\|}_{r}\Big) \, d\theta \, dr$$

$$= \int_{r=0}^{\infty} \int_0^{2\pi} r \tilde{f}(r) \, d\theta \, dr = 2\pi \int_{r=0}^{\infty} r \tilde{f}(r) \, dr.$$

$$\therefore \int_{\mathbb{R}^2} f \, d\lambda^2 = 2\pi \int_{r=0}^{\infty} r \tilde{f}(r) \, dr.$$

iv) IN $\mathbb{R}^n$, $n \geq 2$, WE HAVE $\displaystyle\int_{\mathbb{R}^n} f \, d\lambda^n = \omega_{n-1} \int_{r=0}^{\infty} r^{n-1} \tilde{f}(r) \, dr$

TAKE $f = |x|^\alpha \chi_{B_\rho(0)} \overset{\text{by (i)}}{=} |x|^\alpha \chi_{[0,\rho)}(|x|)$ WHICH IS RADIAL

THEN, $$\int_{\mathbb{R}^n} |x|^\alpha \chi_{B_\rho(0)} \, dx = \int_{\mathbb{R}^n} |x|^\alpha \chi_{[0,\rho)}(|x|) \, dx$$

$$\int_{B_\rho(0)} |x|^\alpha \, dx \qquad = \omega_{n-1} \int_{r=0}^{\infty} r^{n-1} r^\alpha \chi_{[0,\rho]}(r) \, dr = \omega_{n-1} \int_0^{\rho} r^{n-1} r^\alpha \, dr$$

$$\Rightarrow \int_{B_\rho(0)} |x|^\alpha \, dx = \omega_{n-1} \int_0^\rho r^{n-1} r^\alpha \, dr = \omega_{n-1} \int_0^\rho r^{n-1+\alpha} \, dr$$

IF $n - 1 + \alpha = 0$ (THAT IS, $\alpha = 1 - n$), $\displaystyle\int_{B_\rho(0)} |x|^\alpha \, dx = \omega_{n-1} \int_0^\rho r^0 \, dr = \rho\,\omega_{n-1}$

IF $n - 1 + \alpha > -1$ (THAT IS, $n + \alpha > 0$), $\displaystyle\int_{B_\rho(0)} |x|^\alpha \, dx = \omega_{n-1} \int_0^\rho r^{n-1+\alpha} \, dr$

$$= \omega_{n-1} \left. \frac{r^{n-1+\alpha+1}}{n-1+\alpha+1} \right|_0^\rho$$

$$= \omega_{n-1} \cdot \frac{r^{n+\alpha}}{n+\alpha}\Bigg|_0^{\rho} = \omega_{n-1} \cdot \frac{\rho^{n+\alpha}}{n+\alpha}.$$

IF $n-1+\alpha = -1$, $(n+\alpha = 0)$, $\displaystyle\int_{B_\rho(0)} |x|^\alpha \, dx = \omega_{n-1} \int_0^\rho \frac{1}{r}\, dr$ ~~$= \omega_{n-1}$~~

INTEGRAL DOESN'T EXI
(PROBLEM AT 0)

IF $n-1+\alpha < -1$ $(n+\alpha < 0)$, $\displaystyle\int_{B_\rho(0)} |x|^\alpha \, dx = \omega_{n-1} \cdot \frac{r^{n-1+\alpha+1}}{n-1+\alpha+1}\Bigg|_0^\rho = \frac{\omega_{n-1}}{n+\alpha} \cdot \frac{1}{r^{-(n+\alpha)}}\Bigg|_0^\rho$

(NOTE $-(n+\alpha)>0$)

PROBLEM AT 0, HENC
INTEGRAL DOESN'T EXIST

$\therefore \displaystyle\int_{B_\rho(0)} |x|^\alpha \, dx$ IS CONVERGENT ONLY IF $n+\alpha > 0$.