TOPOLOGICAL OBSTRUCTIONS FOR SOBOLEV SPACES

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Abstract. We will delve into various aspects of topological obstructions within the framework of Sobolev spaces. To illustrate fundamental principles, we will initially explore a Sobolev adaptation of the Brouwer Fixed Point theorem. This exploration will naturally lead us to considerations regarding the definition of degree for Sobolev maps between manifolds. Subsequently, we will examine Sobolev maps with restricted rank, alongside examples illustrating topological obstructions encountered in the approximation or extension of Sobolev maps such as homeomorphisms. It is assumed that participants are acquainted with the theory of Sobolev spaces in Euclidean contexts. The topological concept needed will be defined throughout the course.

updated versions: https://sites.pitt.edu/~armin/aachen2024/topandsob2.pdf

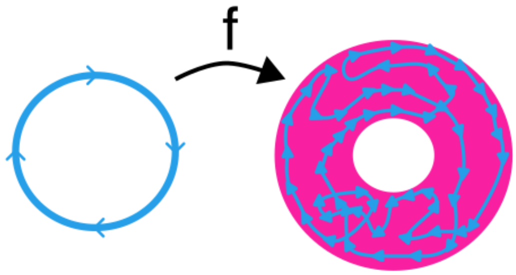
Contents

1. Brouwer Fixed Point theorem – Classical and Sobolev

1.1. **An introductory example: continuous and one-dimensional – the winding number.** Let $\mathbb{S}^1$ be the circle, and consider continuous maps $f : \mathbb{S}^1 \to \mathbb{S}^1$. Having

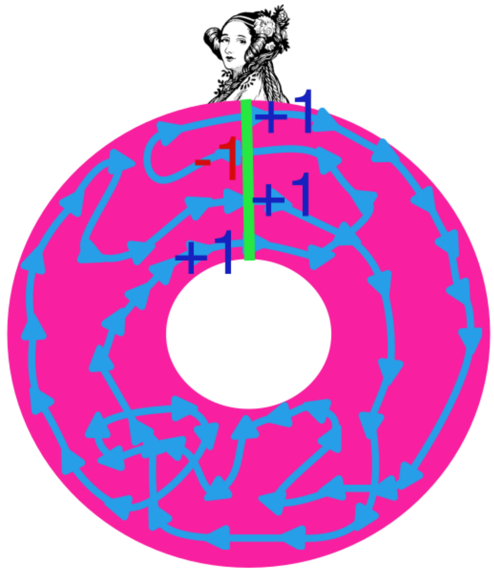
Definition 1.1 (Winding number)**.** We want to define the winding number.

- Any map $f : \mathbb{S}^1 \to \mathbb{S}^1$ can be visualized as a curve in the target space (pink) $\mathbb{S}^1$. For visual reason, to see the strands of f, we fatten $\mathbb{S}^1$. Observe we take the orientation into account, i.e. we move on the domain circle (blue) clockwise and then denote with arrows what the curve is doing in the (pink) target sphere.



- To compute the winding number, stand on top of the circle (we use Ada Lovelace to do this for us). We are going to care about how the curve $f : \mathbb{S}^1 \to \mathbb{S}^1$ behaves at the point where Ada sits, so we draw a line (remember, the pink sphere is drawn as an annulus, but it *is* a onedimensional sphere; so the green line is actually a point.)

- Now we start counting. If the curve goes through the green line clockwise, then we count this as a $+1$. If the curve goes counterclockwise we count this as a -1. Sum all of those numbers up, we get the winding number



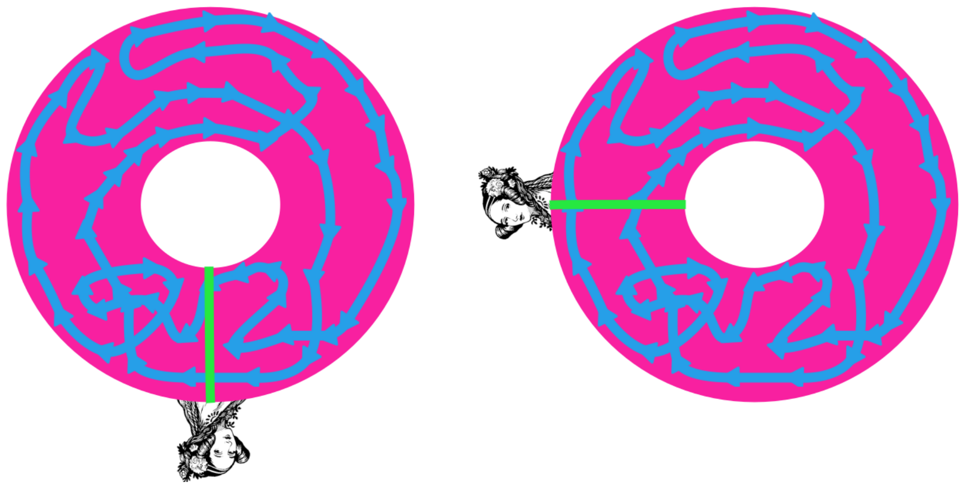
In this picture the winding number is $+2$.

- A few pathological cases:
  - If the curve never crosses the green line, the winding number is 0
  - If the curve just touches the green line, but never crosses this, this is a zero, actually we think of this as a $+1$ followed by a -1 (or vice versa).
  - We only assumed continuity, so there is another pathological case. The curve could pass through the green line infinitely many times. But it *cannot* go around the full pink circle infinitely many times. Because f must be uniformly continuous, so going through the full circle needs at least some δ_0 movement in the domain, this can only happen finitely many times. So if f passes infinitely many times through the green line, all but finitely many times it just comes right back (meaning $+1 - 1 = 0$ for the computation of the winding number).

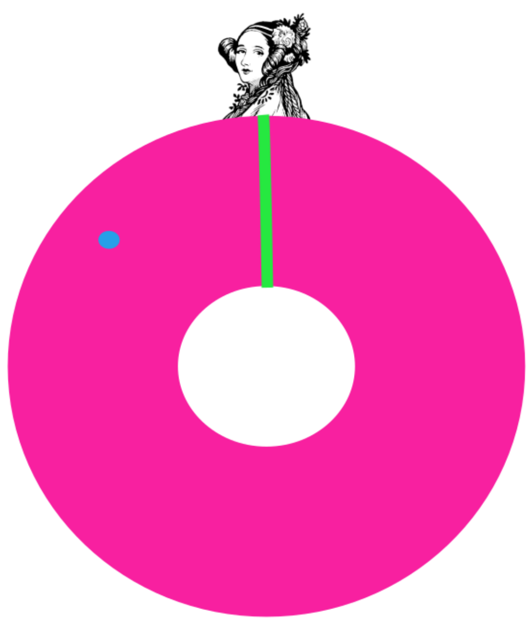
So: The winding number is well-defined.

Let us discuss some properties

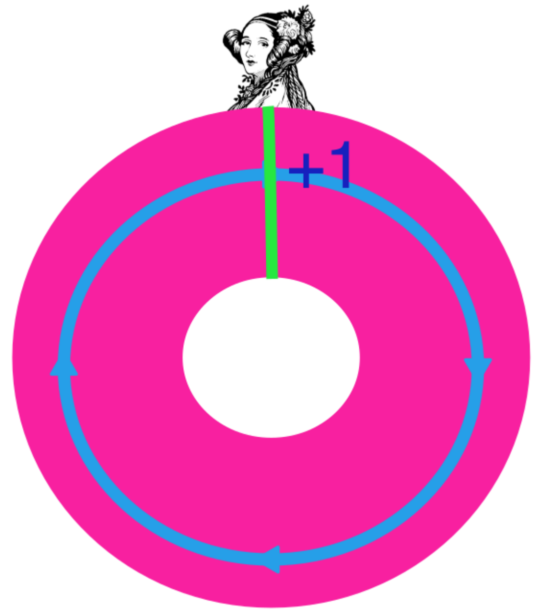
Example 1.2. The winding number does not depend on where Ada sits! To see this (visually), move Ada continuously.



Example 1.3. Winding number of constant map $f : \mathbb{S}^1 \to \mathbb{S}^1$ is 0



Example 1.4. Winding number of identity map is 1

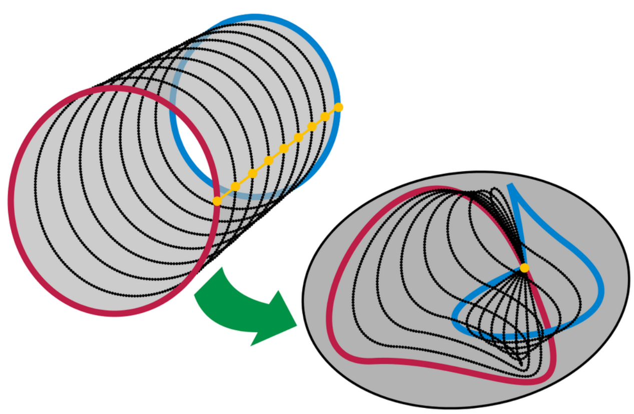


Definition 1.5 (Homotopy)**.** Take two sets $\mathcal{M}$ and $\mathcal{N}$ (think e.g. of two spheres).

A (continuous) curve $f : \mathcal{M} \to \mathcal{N}$ is *homotopic* to a (continuous) curve $g : \mathcal{M} \to \mathcal{N}$ if there exists a homotopy between them:

- $H : [0,1] \times \mathcal{M} \to \mathcal{N}$ continuous such that
- $H(0,x) = f(x)$, $H(1,x) = g(x)$

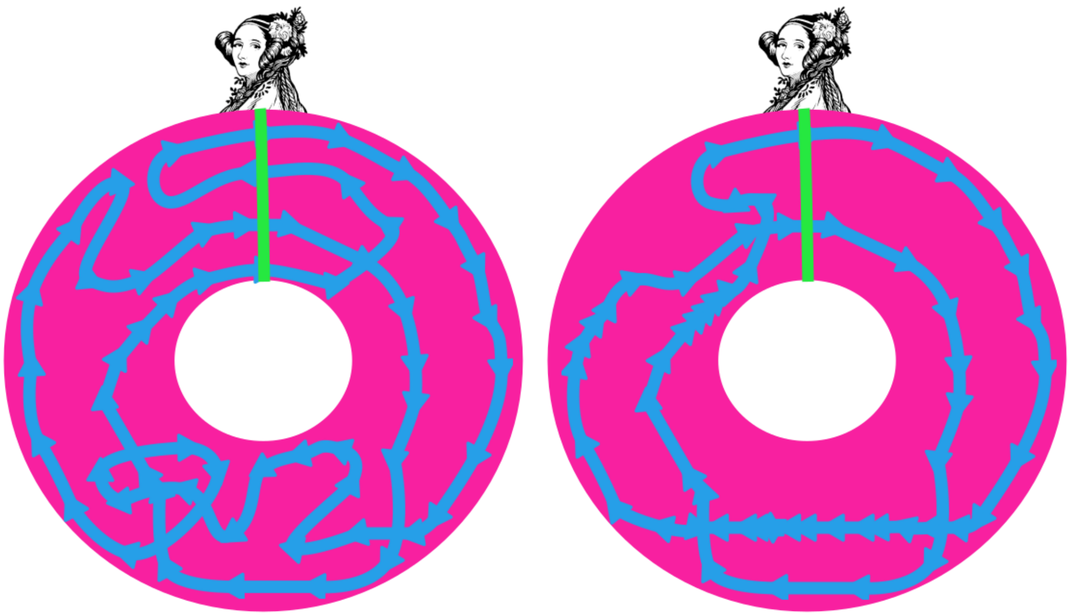
Here is a is a picture[1] of a homotopy: Think of f as the red curve and g as the blue curve and the cylinder on the left is $[0,1] \times \mathcal{M}$ and the right object is $\mathcal{N}$. The homotopy transforms continuously the red curve to the blue curve.



[1]Image: KSmrq, CC BY-SA 3.0 https://creativecommons.org/licenses/by-sa/3.0, via Wikimedia Commons

Proposition 1.6. *Winding number is homotopy invariant. That is if $f : \mathbb{S}^1 \to \mathbb{S}^1$ and $g : \mathbb{S}^1 \to \mathbb{S}^1$ are homotopic, then the winding number of f is the same as the winding number of g.*

Proof. The best way to see this is a visual proof: follow the homotopy!



□

Corollary 1.7. *Assume $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a continuous map which has an extension $F : \mathbb{B}^2 \to \mathbb{S}^1$. Then the winding number of f is zero.*

Proof. Take $H(t, x) := f(tx)$. For $t = 1$ this is just f for $t = 0$ this is constant. This is a homotopy. The winding number for the constant map is zero. So the winding number for f must be zero. □

Another formulation of Corollary 1.7 is:

Corollary 1.8. *Assume a map $f : \mathbb{S}^1 \to \mathbb{S}^1$ has a non-zero winding number.*

Any such map has an extension $F : \mathbb{B}^2 \to \mathbb{R}^2$. However there is no way to find an extension $F : \mathbb{B}^2 \to \mathbb{S}^1$.

The winding number is also called the Brouwer degree. The reason is the Brouwer Fixed point theorem.

Theorem 1.9. *Let $F : \mathbb{B}^n \to \mathbb{B}^n$ be a continuous map (where $\mathbb{B}^n$ denotes the closed unit ball), then there exists a fixed point $\bar{x}$ with $F(\bar{x}) = \bar{x}$.*

The statement works in any dimension, but we want to work with the winding number, so the reader might want to think about $n = 2$.

Proof. Assume the claim is false. Since F is continuous and $F(x) \neq x$, we them must have

$$\varepsilon := \inf_{x \in \mathbb{B}^n} |F(x) - x| > 0 \tag{1.1}$$

We consider the map

$$H(t,x) := \frac{x - tF(x)}{|x - tF(x)|}$$

We make a few observations, leading to a contradiction

- We see that $H(1,x)$ is a continuous map on $\mathbb{B}^n \to \mathbb{S}^{n-1}$, by (1.1).
- Thus the winding number of $H(1,x)\Big|_{\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is zero, by Corollary 1.7.
- We have $t \in [0,1]$
$$|x - tF(x)| \geq t|F(x) - x| - |t-1||x|.$$
Thus, for $t_0 := \max\{\frac{1}{2}, 1 - \frac{\varepsilon}{4}\} < 1$
$$\inf_{x \in \mathbb{B}^n} \inf_{t \in [t_0,1]} |x - tF(x)| \geq \frac{1}{2}\varepsilon - \frac{\varepsilon}{4} \geq \frac{\varepsilon}{4}.$$
That is
$$H : [t_0, 1] \times \mathbb{B}^n \to \mathbb{S}^{n-1}$$
is well-defined and continuous.
- That is the winding number of $H(t,x)\Big|_{\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is zero, by Corollary 1.7, for any $t \in [t_0, 1]$.
- For any $t \in [0, t_0]$ surely
$$\inf_{t \in [0,t_0], |x|=1} |x - tF(x)| \geq 1 - t_0 \underbrace{|F(x)|}_{\leq 1} \geq 1 - t_0 > 0$$
That is $H\Big|_{[0,t_0] \times \mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is continuous!
- In particular, the winding number of $H(0,\cdot)\Big|_{\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ must be 0, by Proposition 1.6
- But $H(0,x) = x$ for $x \in \mathbb{S}^{n-1}$. Thus the winding number $H(0,\cdot)\Big|_{\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is 1 by Example 1.4.
- We have reached a contradiction.

$\square$

Remark 1.10. The Brouwer fixed point theorem does not give as any information *where* the fixed point is, and also not about how many fixed points there are[2]:

- there could be exactly one: Take $F(x) = -x$.
- there could be infinitely many: Take $F(x) = x$
- Even more crazy: take any nonempty set $\Sigma \subset \mathbb{B}$ with $p \in \Sigma$ and set
$$F(x) := \left(1 - \frac{\operatorname{dist}(x, \Sigma)}{\operatorname{diam}\mathbb{B}^2}\right) x + \frac{\operatorname{dist}(x, \Sigma)}{\operatorname{diam}\mathbb{B}^2} p$$
The $F : \mathbb{B}^2 \to \mathbb{B}^2$ (as convex combination of two elements in $\mathbb{B}^2$), and $F(x) = x$ is equivalent to
$$\frac{\operatorname{dist}(x, \Sigma)}{\operatorname{diam}\mathbb{B}^2}(p - x) = 0$$
i.e. either $p = x$ or $\operatorname{dist}(x, \Sigma) = 0$.

This is all very classical, our question is what can we say if instead of $F \in C^0(\mathbb{B}^n, \mathbb{B}^n)$ we have $F \in W^{1,p}(\mathbb{B}^n, \mathbb{B}^n)$? The point that I am trying to make in this is somewhat that we can "save" some results over to Sobolev spaces.

1.2. **Recall: Basic properties for Sobolev spaces.** For now we discuss only integer Sobolev spaces $W^{1,p}(\mathbb{B}^n, \mathbb{R}^N)$, but for the trace theorem we will need also fractional Sobolev spaces.

- $W^{1,p}(\mathbb{B}^n, \mathbb{R}^N)$ consists of all functions $f \in L^p(\mathbb{B}^n, \mathbb{R}^N)$ such that the distributional derivative $\partial_\alpha f \in L^p(\mathbb{B}^n, \mathbb{R}^N)$ for all $\alpha \in \{1, \dots, n\}$.
- Trace of a Sobolev function: If $f \in W^{1,p}(\Omega, \mathbb{R}^N)$ then it does not make any sense to talk about $f\Big|_{\partial\Omega}$. Indeed, Sobolev maps can be changed on a set of measure zero (because L^p-functions are defined only modulo sets of measure zero). $\partial\Omega$ tends to be measure zero.

  However, if Ω is suitably nice, then there exists "something like" $f\Big|_{\partial\Omega}$: the *trace operator*. The trace operator satisfies the following properties (below: $p > 1$)
  - $\Big|_{\partial\Omega} : W^{1,p}(\Omega, \mathbb{R}^N) \to W^{1-\frac{1}{p},p}(\partial\Omega, \mathbb{R}^N)$ is a surjective linear map, where for $s \in (0, 1)$

$f \in W^{s,p}(\partial\Omega)$

$$:\Leftrightarrow f \in L^p(\partial\Omega, d\mathcal{H}^{d-1}), \quad [f]_{W^{s,p}(\partial\Omega)} := \left(\int_{\partial\Omega}\int_{\partial\Omega} \left(\frac{|f(x) - f(y)|}{|x - y|^s}\right)^p \frac{dx\,dy}{|x - y|^{n-1}}\right)^{\frac{1}{p}} < \infty$$

- We say $f \in W^{1,p}(\mathbb{B}^n, \mathcal{M})$ for a manifold $\mathcal{M} \subset \mathbb{R}^N$ if $f \in W^{1,p}(\mathbb{B}^n, \mathbb{R}^N)$ and $f(x) \in \mathcal{M}$ for a.e. $x \in \mathbb{B}^n$. Similarly for $W^{s,p}$.

[2]very different to the Banach Fixed Point theorem

We recall the Morrey-Sobolev embedding

Theorem 1.11 (Morrey-Sobolev)**.**
- *Assume $f \in W^{1,p}(\mathbb{B}^n, \mathbb{R}^N)$ and $1 - \frac{n}{p} > 0$ (i.e. $p > n$). Then $f \in C^{1-\frac{n}{p}}(\mathbb{B}^n, \mathbb{R}^N)$.*
- *Similarly for $W^{s,p}$: if $s - \frac{n}{p} > 0$ then it embedds into $C^{s-\frac{n}{p}}$ (proof is the same)*
- *if $1 - \frac{n}{p} = -\frac{n}{q} < 0$ for $q \in (1, \infty)$ then $f \in L^q$, but $f \in W^{1,p}$ may not be continuous*
- *If $1 - \frac{n}{p} = 0$, $f \in VMO$, which means*

  but f may not be continuous. The typical example is
$$f(x) = \log \log 2/|x|.$$

Warning: we have to be a bit careful here. If $f \in W^{1,p}$ for $1 - \frac{n}{p} > 0$, f is still an L^p-function and thus only defined modulo sets of measure zero. The statement is that there exists (exactly one) representative $\bar{f}$ of the equivalence class $[f] \in L^p$ such that $\bar{f}$ is $C^{1-\frac{n}{p}}$.

Here is our first Sobolev-version of Theorem 1.9 (spoiler: it is quite trivial)

Theorem 1.12. *Let $F \in W^{1,p}(\mathbb{B}^n, \mathbb{B}^n)$, for $p > n$, then there exists a fixed point $\bar{x}$ with $F(\bar{x}) = \bar{x}$.*

Proof. Any such F is actually continuous (since $p > n$). Thus a fixed point exists – as long as we *mean* by $F(x) = x$ the continuous representative of F. □

So for $p > n$ there is no real excitement.

How about $p < n$?

Answer: the analogue of Theorem 1.9 is very very wrong for $W^{1,p}$, $p < n$, indeed we have a striking counterexample.

Example 1.13 (The hedgehog)**.** Let
$$F(x) := \frac{x}{|x|}$$
Then $f \in W^{1,p}(\mathbb{B}^n, \mathbb{S}^{n-1})$ for any $p < n$. Indeed,
$$|\nabla F(x)| \approx |x|^{-1}$$
which belongs to L^p for $p < n$.

The hedgehog itself $\frac{x}{|x|} : \mathbb{B}^n \to \mathbb{S}^{n-1}$ has of course many fixed points on $\partial \mathbb{B}^n$.

But

Theorem 1.14. *The map $f(x) := -\frac{x}{|x|}$ has no fixed points.*

Ok, so now we could[3] complain that the statement of a fixed point theorem does not make any sense, since Sobolev functions can be changed on a set of measure zero: there may not be any continuous representative of F.

So, what do we actually mean by $F(\bar{x}) = \bar{x}$?

As discussed in Remark 1.10, even if there is one represenative of F that has $F(\bar{x}) = \bar{x}$, we could easily change this representative in a zero measure (one point!) such that $F(\bar{x}) \neq \bar{x}$. And the other way around, If $F(x) \neq x$ for all x, then we can easily find a representative a.e. the same such that $F(0) = 0$.

So instead we might be tempted to consider

Definition 1.15. We say that a measurable $F : \mathbb{R}^n \to \mathbb{R}^n$ has *essentially* a fixed point, if

$$\mathcal{L}^n \left(\{x : |F(x) - x| < \varepsilon\}\right) > 0 \quad \forall \varepsilon > 0.$$

Exercise 1.16. *Assume $F : \mathbb{B}^n \to \mathbb{R}^n$ (important that $\mathbb{B}^n$ is closed!) is continuous and has essentially a fixed point. Then F has a fixed point.*

Example 1.17. The example from before, $F(x) := -x/|x|$, has *not* essentially a fixed point.

So for $W^{1,p}$, $p < n$ - no Brouwer fixed point theorem

For $W^{1,p}$, $p > n$ - yes Brouwer fixed point theorem.

What happens for $p = n$?

Theorem 1.18. *If $F \in W^{1,n}(\mathbb{B}^n, \mathbb{B}^n)$, then F has essentially a fixed point .*

Whether this statement is really useful is debatable, we make it mostly to illustrate the point of topological methods being available for (barely) non-continuous Sobolev spaces.

The proof is kind of the same as Theorem 1.12:

Again we are interested in discussing

$$H(t, x) := \frac{x - tF(x)}{|x - tF(x)|},$$

we need to take care of some technicalities.

Lemma 1.19. *Fix $\varepsilon > 0$, set*

$$t_0 := \max\{1 - \frac{\varepsilon}{4}, \frac{1}{2}\}.$$

Assume $F \in W^{1,n}(\mathbb{B}^n, \mathbb{R}^n)$ satisfies

$$\mathcal{L}^n(\{x : |F(x) - x| < \varepsilon\}) = 0,$$

[3]should!

Set

$$H(t,x) := \frac{x-tF(x)}{|x-tF(x)|}$$

then $H(t,\cdot) \in W^{1,n}(\mathbb{B}^n,\mathbb{R}^n)$ *for any* $t \in [t_0,1]$ *and indeed we have Lipschitz continuity,*

$$\|H(t_1,\cdot)-H(t_2,\cdot)\|_{W^{1,n}(\mathbb{B}^n,\mathbb{R}^n)} \leq |t_1-t_2|\ C(n,\varepsilon)\left(\|F\|_{W^{1,n}(\mathbb{B}^n)}+1\right) \quad \forall t_1,t_2 \in [t_0,1]. \tag{1.2}$$

Proof. Formally this is kind of clear: We have

$$|x-tF(x)| \geq |tx-tF(x)| - |1-t|\ |x|$$

So if $t \geq \frac{1}{2}$ and $t \geq 1-\frac{\varepsilon}{4}$ then for almost every $|x| \leq 1$,

$$|x-tF(x)| \geq \frac{1}{2}|F(x)-x| - \frac{\varepsilon}{4}|x| \geq \frac{1}{2}\varepsilon - \frac{\varepsilon}{4} = \frac{\varepsilon}{4} > 0.$$

Then

$$\nabla_x H(t,x) = \frac{\nabla\left(x-tF(x)\right)}{|x-tF(x)|} - \frac{x-tF(x)}{|x-tF(x)|^2}\langle\frac{x-tF(x)}{|x-tF(x)|},\nabla(x-tF(x))\rangle$$

implies

$$|\nabla_x H(t,x)| \leq C\frac{1}{\varepsilon}\left|\nabla(x-tF(x))\right| \in L^n.$$

Thus $H(t,\cdot) \in W^{1,n}$.

You might be a bit concerned about the previous argument. And you should be. A function whose derivative is a.e. in L^n may not be in $W^{1,n}$ (it could have a measure part, think of the Heaviside function).

So the actual argument is as follows:

Consider for $\delta > 0$

$$H_\delta(t,x) := \frac{x-tF(x)}{|x-tF(x)|+\delta}$$

$H_\delta(t,\cdot)$ is clearly in $W^{1,n}$ since $p \mapsto \frac{p}{|p|+\delta}$ is a Lipschitz function, and composition of Lipschitz functions with $W^{1,q}$ is $W^{1,q}$.

Then

$$\nabla_x H_\delta(t,x) = \frac{\nabla_x\left(x-tF(x)\right)}{|x-tF(x)|+\delta} - \frac{x-tF(x)}{\left(|x-tF(x)|+\delta\right)^2}\langle\frac{x-tF(x)}{|x-tF(x)|},\nabla_x(x-tF(x))\rangle$$

Again: we know this is the distributional derivative, since H_δ belongs to $W^{1,1}_{loc}$ and thus a.e. derivative and distributional derivative coincide.

Thus, for a.e. x,

$$|\nabla H_\delta(t,x)| \lesssim \frac{1}{\varepsilon+\delta}|\nabla(F(x)-x)|.$$

Since $\varepsilon > 0$ we find

$$\sup_{\delta\in(0,1)} \|H_\delta(t,\cdot)\|_{W^{1,n}(\mathbb{B}^n)} < \infty,$$

By reflexivity we thus conclude that there must be some $H_0(t,\cdot) \in W^{1,n}(\mathbb{B}^n)$ that is the $W^{1,n}$-weak limit of $H_{\delta_n}(t,\cdot)$ for some $\delta_n \to 0$. By Rellich's theorem, this is a pointwise a.e. convergence, so for every[4] has point x with $tF(x) \neq x$ we have

$$H_0(t,\cdot) = \frac{x-tF(x)}{|x-tF(x)|}$$

That is, $\frac{x-tF(x)}{|x-tF(x)|} \in W^{1,n}(\mathbb{B}^n,\mathbb{R}^n)$.

By the same argument, for the Lipschitz bound (1.2), it suffices to show

$$\|H_\delta(t_1,\cdot)-H_\delta(t_2,\cdot)\|_{W^{1,n}(\mathbb{B}^n,\mathbb{R}^n)} \leq |t_1-t_2|\ C(n,\varepsilon)\left(\|F\|_{W^{1,n}(\mathbb{B}^n)}+1\right)$$

and by the fundamental theorem in calculus for this it is enough to show to consider

$$\partial_t H_\delta(t,x) = \frac{F(x)}{|x-tF(x)|+\delta} - \frac{x-tF(x)}{(|x-tF(x)|+\delta)^2}\langle\frac{x-tF(x)}{|x-tF(x)|},F(x)\rangle$$

Observe that we already know that $\frac{x-tF(x)}{|x-tF(x)|} \in W^{1,n}$ then we find

$$\|\nabla\partial_t H_\delta(t,x)\|_{L^n} \leq C(n,\varepsilon)\left(\|\nabla F\|_{L^n}+1\right)$$

We can conclude. □

In the same way we have

Lemma 1.20. *Assume $t_0 < 1$ and $F \in W^{1,n}(\mathbb{B}^n,\mathbb{R}^n)$ with $|F(x)| \leq 1$ a.e.. Set for $t \in [0,t_0]$*

$$H(t,x) := \frac{x-tF(x)}{|x-tF(x)|},$$

which is well-defined for $|x| \in (t_0,1]$, and indeed we have

$$H(t,\cdot) \in W^{1,n}(B(1)\setminus B(\frac{1-t_0}{2}),\mathbb{R}^n)$$

and

(1.3)
$$\|H(t_1,\cdot)-H(t_2,\cdot)\|_{W^{1,n}(B(1)\setminus B(\frac{1-t_0}{2}),\mathbb{R}^n)} \leq |t_1-t_2|\ C(n,\varepsilon)\left(\|F\|_{W^{1,n}(\mathbb{B}^n)}+1\right) \quad \forall t_1,t_2 \in [t_0,1].$$

Proof of Theorem 1.18. We follow the strategy of the proof of Theorem 1.9:

Assume the claim is false. Then there must be some $\varepsilon > 0$ such that

(1.4)
$$\mathcal{L}^n(\{x : |F(x)-x| < \varepsilon\}) = 0.$$

[4]recall: the set $\{x : tF(x) = x\}$ is a zero-set by assumption for any $t \in [t_0,1]$

Again we consider the map

$$H(t,x) := \frac{x - tF(x)}{|x - tF(x)|}$$

- By Lemma 1.19, for some $t_0 \in (0,1)$ we have that $H(t,\cdot) \in W^{1,n}(\mathbb{B}^n,\mathbb{R}^n)$ for any $t \in [t_0,1]$.
- By Lemma 1.20 and the trace theorem for $t \in [0,t_0]$

$$H(t,\cdot)\Big|_{\mathbb{S}^{n-1}} \in W^{1-\frac{1}{n},n}(\mathbb{S}^{n-1},\mathbb{S}^{n-1}).$$

- H is continuous in t w.r.t. above spaces.

So we are done, if we can extend the notion of winding number to maps $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ which belong to $W^{1-\frac{1}{n},n}(\mathbb{S}^{n-1},\mathbb{R}^n)$: we need

- the winding number (or better: the degree) is defined for $W^{1-\frac{1}{n},n}(\mathbb{S}^{n-1},\mathbb{S}^{n-1})$-maps, and if $f \in C^0 \cap W^{1-\frac{1}{n},n}(\mathbb{S}^{n-1},\mathbb{S}^{n-1})$ then the notion of winding numbers (Sobolev and usual) coincide.
- Homotopies in $C^0_t W^{1-\frac{1}{n},n}$ don't change the winding number
- If $f \in W^{1-\frac{1}{n},n}(\mathbb{S}^{n-1},\mathbb{S}^{n-1})$ is the trace of a map $F \in W^{1,n}(\mathbb{B}^n,\mathbb{S}^{n-1})$ then the winding number of f is zero.

□

2. Degree for (essentially) VMO-maps

Again, if you are unfamiliar with topology it makes sense to think of the case of maps $\mathbb{S}^1 \to \mathbb{S}^1$, but this works for the degree, and more generally any homotopy invariant quantity

Theorem 2.1. *Let $\mathcal{M}^d$ and $\mathcal{N} \subset \mathbb{R}^N$ be two smooth, compact manifolds without boundary (think of two spheres), where $\mathcal{M}^d$ is d-dimensional.*

Consider the space

$$X := \{f : \mathcal{M}^d \to \mathcal{N} \subset \mathbb{R}^N \text{ continuous}\}.$$

Assume for some $Y \subset \mathbb{R}^5$ a map

$$\deg : X \to Y$$

which is homotopy invariant, i.e. which has the property

$$\text{If } f \text{ and } g \text{ are homotopic then } \deg(f) = \deg(g).$$

Fix $s \in (0,1]$, $\frac{d}{s} \neq 1$ and set

$$\tilde{X}_s \equiv W^{s,\frac{d}{s}}(\mathcal{M}^d,\mathcal{N}) := \{f \in W^{s,\frac{d}{s}}(\mathcal{M}^d,\mathbb{R}^N) : \quad f(x) \in \mathcal{N}, \text{ for a.e. } x \in \mathcal{M}^d\}$$

[5] Y could also be something like $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}_2$

Then there exists

$$\widetilde{\deg} : \tilde{X}_s \to Y$$

such that

- $\widetilde{\deg}$ *extends* $\deg$*, i.e.* $\widetilde{\deg}(f) = \deg(f)$ *whenever* $f \in C^0 \cap W^{s,\frac{d}{s}}$
- $\widetilde{\deg}$ *is continuous w.r.t.* $W^{s,\frac{d}{s}}$*-topology. Even more: for any* $f \in W^{s,\frac{d}{s}}(\mathcal{M},\mathcal{N})$ *there exists* $\delta > 0$ *(depending on* f*!) such that if* $g \in W^{s,\frac{d}{s}}(\mathcal{M},\mathcal{N})$ *and*

  $$[f-g]_{W^{s,\frac{d}{s}}(\mathcal{M},\mathbb{R}^N)} < \delta$$

  then $\widetilde{\deg} f = \widetilde{\deg} g$[6]
- *(In particular if we) assume* $H \in C^0([0,1]; W^{s,\frac{d}{s}}(\mathcal{M},\mathcal{N}))$ *i.e.*

  $$H : [0,1] \times \mathcal{M} \to \mathcal{N}$$

  with

  $$H(t,\cdot) \in W^{s,\frac{d}{s}}(\mathcal{M},\mathcal{N})$$

  and for any $t \in [0,1]$ *and any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any* $s \in (t-\delta, t+\delta) \cap [0,1]$ *we have*

  $$[H(s,\cdot) - H(t,\cdot)]_{W^{s,\frac{d}{s}}(\mathcal{M},\mathcal{N})} < \varepsilon.$$

  If moreover $H(0,\cdot) = f$ *and* $H(1,\cdot) = g$ *then*

  $$\widetilde{\deg}(f) = \widetilde{\deg}(g).$$

Remark 2.2. Why $W^{s,\frac{d}{s}}$? Because this is exactly the case when

$$s - \frac{d}{\frac{d}{s}} = 0$$

Like $W^{1,d}(\mathbb{R}^d)$ – its the limiting case:

The above also works for $W^{t,\frac{d}{s}}$ for $t > s$ (because then any Sobolev map is actually continuous), and they tend to fail when $t < s$ (essentially because $\frac{x}{|x|}$ belongs to $W^{t,\frac{d}{s}}$ which is a map that "extends a winding number 1 map homotopically to the constant in $W^{t,\frac{d}{s}}$, $t < s$")

Once we have Theorem 2.1 we can almost conclude the proof of Theorem 1.18, indeed we recall what we needed:

- the winding number (or better: the degree) is defined for $W^{1-\frac{1}{n},n}(\mathbb{S}^{n-1},\mathbb{S}^{n-1})$-maps, and if $f \in C^0 \cap W^{1-\frac{1}{n},n}(\mathbb{S}^{n-1},\mathbb{S}^{n-1})$ then the notion of winding numbers (Sobolev and usual) coincide. (done)

[6]Well, since the image of $\widetilde{\deg}$ is Y which is discrete this is the same as saying

$$|\widetilde{\deg} f - \widetilde{\deg} g| < \varepsilon$$

for a small ε

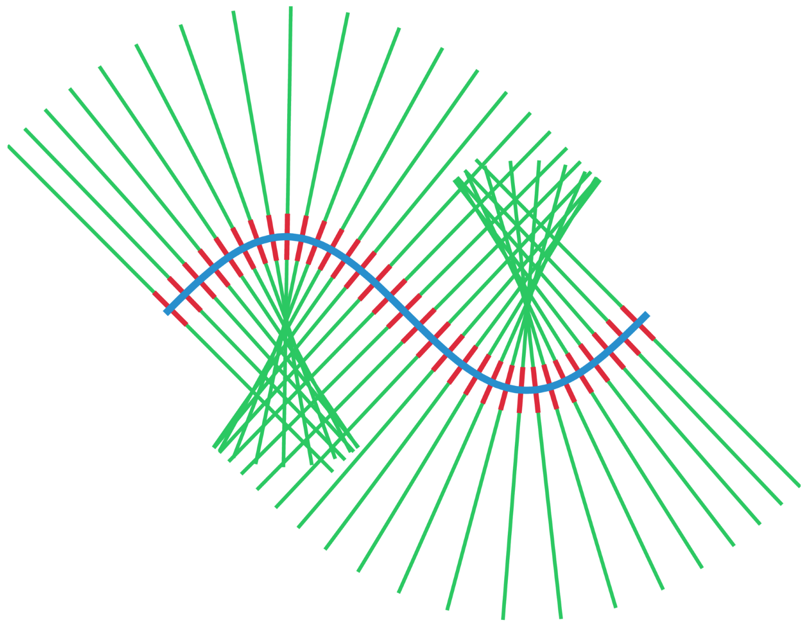


FIGURE 2.1. The nearest point projection from Lemma 2.3: close to the manifold (the blue curve) we can project (even orthogonally, i.e. to the nearest point) onto the manifold (in the red area, along the normal field). Source:Oleg Alexandrov/Wikipedia

- Homotopies in $C^0_t W^{1-\frac{1}{n},n}$ don't change the winding number (done)
- If $f \in W^{1-\frac{1}{n},n}(\mathbb{S}^{n-1},\mathbb{S}^{n-1})$ is the trace of a map $F \in W^{1,n}(\mathbb{B}^n,\mathbb{S}^{n-1})$ then the winding number of f is zero. (to be done!)

2.1. **Proof of Theorem 2.1.** In order to see this we need a few observations. First, we need the nearest point projection[7]:

Lemma 2.3 (Nearest point projection)**.** *Let $\mathcal{N} \subset \mathbb{R}^N$ be a compact manifold without boundary.*

There exists a small $\delta > 0$ and smooth projection π from a tubular neighborhood

$$B_\delta(\mathcal{N}) := \{p \in \mathbb{R}^N : \operatorname{dist}(p,\mathcal{N}) < \delta\}$$

such that

$$\pi_{\mathcal{N}} : B_\delta(\mathcal{N}) \to \mathcal{N}; \quad \pi_{\mathcal{N}}(p) = p \quad \forall p \in \mathcal{N}$$

Cf. Figure 2.1.

Lemma 2.4. *Assume that $\mathcal{M}$ and $\mathcal{N} \subset \mathbb{R}^N$ are two smooth compact manifolds without boundary. There exists a number $\varepsilon_0 = \varepsilon_0(\mathcal{N}) > 0$ such that any two continuous maps*

[7] see [Sim96, Section 2.12.3]

$f,g:\mathcal{M}\to\mathcal{N}$ with

$$\|f-g\|_{L^\infty(\mathcal{M})}<\varepsilon_0$$

are homotopic to each other.

Proof. The proof is kind of easy, since we have $\pi_{\mathcal{N}}$ from Lemma 2.3: we use a convex combination as homotopy.

Assume that f,g are continuous and $\|f-g\|_{L^\infty}<\frac{\delta}{2}$, where δ is from Lemma 2.3. Set

$$H(t,x):=\pi_{\mathcal{N}}\left((1-t)f(x)+tg(x)\right)$$

This is well defined and continuous map since

$$\operatorname{dist}\left((1-t)f(x)+tg(x),\mathcal{N}\right)\le|(1-t)f(x)+tg(x)-f(x)|\le|f(x)-g(x)|<\frac{\delta}{2}.$$

That is H is a homotopy. We are done. □

Lemma 2.5. *Let $f\in W^{s,\frac{d}{s}}(\mathcal{M},\mathcal{N})$ then there exist $f_k\in C^\infty(\mathcal{M},\mathcal{N})$ such that*

$$\|f-f_k\|_{W^{s,\frac{d}{s}}(\mathcal{M},\mathbb{R}^N)}\xrightarrow{k\to\infty}0$$

Remark 2.6. Lemma 2.5 is due (for $s=1$, but the proof easily translates to $s<1$) to Schoen and Uhlenbeck, [SU82, Section 3]. It was quickly observed [SU83, Section 4] that this result may fail (depending on topology) if $u\in W^{s,p}$ if $sp<d$. We refer the interested reader to [BZ88, Bet91, HL03, BPVS15, BM15] for this fascinating field.

Proof of Lemma 2.5. We use the usual mollification: Assume that $\mathcal{M}$ is embedded nicely into some $\mathbb{R}^D$ take $\eta\in C_c^\infty(B(0,1))$, $\eta\ge0$, $\eta\Big|_{B(0,1/2))}=1$, and set $\eta_t(z)=t^{-d}\eta(\frac{z}{t})$. We shall additionally assume that η is rotation invariant, i.e. $\eta(z)=\eta(|z|)$. Set

$$g_t(x):=\int_{\mathcal{M}}f(y)\eta_t(x-y)d\mathcal{H}^d(y)$$

Set

$$c_t(x):=\int_{\mathcal{M}}\eta_t(x-y)d\mathcal{H}^d(y)=\int_{\mathcal{M}\cap B(x,t)}\eta_t(x-y)d\mathcal{H}^d(y)$$

We have

$$t^{-d}\mathcal{H}^d(\mathcal{M}\cap B(x,t/2))\le c_t(x)\le\|\eta\|_{L^\infty}t^{-d}\mathcal{H}^d(\mathcal{M}\cap B(x,t))$$

This implies that for any fixed small $t_0>0$ we have

$$0<\inf_{t\in[0,t_0],x\in\mathcal{M}}c_t(x)\le\sup_{t\in[0,t_0],x\in\mathcal{M}}c_t(x)<\infty.$$

Moreover we claim that

$$\lim_{t\to0}c_t(x)=1\quad\forall x\in\mathcal{M}.$$

To see this fix any $x \in \mathcal{M}$, assume $t \ll 1$ so that $B(x,t) \cap \mathcal{M}$ can be parametrized by a diffeomorphism $\Phi : \mathbb{B}^d \to B(x,t) \cap \mathcal{M}$ and up to a rotation of $\mathcal{M} \subset \mathbb{R}^D$ (which does not change η!) we can assume that

$$\Phi(0) = x = \begin{pmatrix} x' \\ 0 \end{pmatrix}, \quad D\Phi(0) = \begin{pmatrix} I_{d\times d} \\ 0_{D-d\times D-d} \end{pmatrix} \in \mathbb{R}^{D\times d}$$

Then we have

$$\int_{\mathcal{M}\cap B(x,t)} \eta_t(x-y) d\mathcal{H}^d(y) = \int_{\mathbb{B}^d \cap \{z: |x-\Phi(z)|<t\}} \eta_t(x-\Phi(z)) \mathrm{Jac}(\Phi(z)) dz$$

By Taylor expansion, for $|x-\Phi(z)| < t$ we have

$$\mathrm{Jac}(\Phi(z)) = 1 + O(t)$$

and since

$$\left|\limsup_{t\to 0^+} \int_{\mathbb{B}^d \cap \{z: |x-\Phi(z)|<t\}} \eta_t(x-\Phi(z)) dz\right| < \infty$$

we find that

$$\int_{\mathcal{M}\cap B(x,t)} \eta_t(x-y) d\mathcal{H}^d(y) = \int_{\mathbb{B}^d \cap \{z: |x-\Phi(z)|<t\}} \eta_t(x-\Phi(z)) dz + o_{t\to 0}(1).$$

Another Taylor expansion tells us that

$$\mathbb{R}^D \ni x - \Phi(z) = \begin{pmatrix} x'-z \\ 0 \end{pmatrix} + O(|x-z|^2)$$

So that we have

$$\eta_t(x-\Phi(z)) = t^{-d} \eta(\frac{|x'-z|}{t}) + t^{-d} t^{-1} O(t^2)$$

We conclude that

$$\int_{\mathcal{M}\cap B(x,t)} \eta_t(x-y) d\mathcal{H}^d(y) = \int_{\mathbb{B}^d \cap \{z: |x-z|<t\}} \eta_t(x'-z) dz + o_{t\to 0}(1).$$

Now we see that

$$\lim_{t\to 0} \int_{\mathcal{M}\cap B(x,t)} \eta_t(x-y) d\mathcal{H}^d(y) = \eta(0) = 1.$$

In particular, since $c_t(x)$ is a smooth function in x, uniformly in t, we see that

$$\|\nabla c_t\|_{L^\infty(\mathcal{M})} \xrightarrow{t\to 0} 0$$

and in particular

$$\|\nabla \left(\frac{1}{c_t}\right)\|_{L^\infty(\mathcal{M})} \xrightarrow{t\to 0} 0$$

Then standard by the usual convolution argument

$$\tilde{f}_t(x) := \frac{1}{c_t(x)} g_t(x) \xrightarrow{t\to 0} f(x) \quad \text{in } W^{s,\frac{d}{s}}(\mathcal{M}^d, \mathbb{R}^N)$$

and $\tilde{f}_t \in C^\infty(\mathcal{M}^d, \mathbb{R}^N)$. *However:* $\tilde{f}_t : \mathcal{M}^d \to \mathbb{R}^N$. We wanted $\tilde{f}_t : \mathcal{M}^d \to \mathcal{N}$. (Also, so far this would have worked for any $W^{s,p}$!)

We observe that for a.e. $z \in \mathcal{M}$, such that $f(z) \in \mathcal{N}$,

$$\operatorname{dist}(\tilde{f}_t(x), \mathcal{N}) \leq |\tilde{f}_t(x) - f(z)|$$

Thus

$$\begin{aligned}\operatorname{dist}(\tilde{f}_t(x), \mathcal{N}) \leq & \frac{1}{c_t(x)} |g_t(x) - c_t(x) f(z)| \\ \leq & \frac{1}{c_t(x)} \left| \int_{\mathcal{M}} (\eta_t(x-y) f(y) - \eta_t(x-y) f(z))\, dy \right| \\ \leq & \frac{1}{c_t(x)} \int_{\mathcal{M}} \eta_t(x-y) |f(y) - f(z)| dy.\end{aligned}$$

Multiply this with $\frac{1}{c_t(x)} \eta_t(x-z)$ and integrate in z (the left hand site integrates to 1 since it doesn't depend on z)

$$\operatorname{dist}(\tilde{f}_t(x), \mathcal{N}) \leq \frac{1}{c_t(x)} \frac{1}{c_t(x)} \int_{\mathcal{M}} \int_{\mathcal{M}} \eta_t(x-y) \eta_t(x-z) |f(y) - f(z)| dy dz$$

This is BMO! For all $t < t_0$

$$\operatorname{dist}(\tilde{f}_t(x), \mathcal{N}) \lesssim \frac{1}{t^{2d}} \int_{\mathcal{M} \cap B(x,t)} \int_{\mathcal{M} \cap B(x,t)} |f(y) - f(z)| dy dz$$

We do, what essentially is Sobolev-Poincaré inequality,

$$\begin{aligned}\operatorname{dist}(\tilde{f}_t(x), \mathcal{N}) \lesssim & \frac{\left(\mathcal{H}^d(\mathcal{M} \cap B(x,t))\right)^{\frac{2}{p'}}}{t^{2d}} \left(\int_{\mathcal{M} \cap B(x,t)} \int_{\mathcal{M} \cap B(x,t)} |f(y) - f(z)|^p dy dz \right)^{\frac{1}{p}} \\ \lesssim & \frac{\left(\mathcal{H}^d(\mathcal{M} \cap B(x,t))\right)^{\frac{2}{p'}}}{t^{2d}} (2t)^{\frac{d+sp}{p}} \left(\int_{\mathcal{M} \cap B(x,t)} \int_{\mathcal{M} \cap B(x,t)} \frac{|f(y) - f(z)|^p}{|y-z|^{d+sp}} dy dz \right)^{\frac{1}{p}}\end{aligned}$$

Now, if t is suitably small (maybe we take t_0 even smaller)

$$\mathcal{H}^d(\mathcal{M} \cap B(x,t)) \approx t^d$$

and we arrive at

$$\begin{aligned}\operatorname{dist}(\tilde{f}_t(x), \mathcal{N}) \lesssim & \frac{\left(t^d\right)^{\frac{2}{p'}}}{t^{2d}} (2t)^{\frac{d+sp}{p}} \left(\int_{\mathcal{M} \cap B(x,t)} \int_{\mathcal{M} \cap B(x,t)} \frac{|f(y) - f(z)|^p}{|y-z|^{d+sp}} dy dz \right)^{\frac{1}{p}} \\ = & t^{\frac{-d+sp}{p}} \left(\int_{\mathcal{M} \cap B(x,t)} \int_{\mathcal{M} \cap B(x,t)} \frac{|f(y) - f(z)|^p}{|y-z|^{d+sp}} dy dz \right)^{\frac{1}{p}}\end{aligned}$$

This works for any $s \in (0,1)$ and $p \in (1, \infty)$. But we care about the case when $p = \frac{d}{s}$, where some magic happens:

$$t^{\frac{-d+sp}{p}} = 1.$$

That is, we have shown

$$\operatorname{dist}(\tilde{f}_t(x), \mathcal{N}) \lesssim \left(\int_{\mathcal{M}\cap B(x,t)} \int_{\mathcal{M}\cap B(x,t)} \frac{|f(y)-f(z)|^{\frac{d}{s}}}{|y-z|^{2d}} dydz \right)^{\frac{s}{d}}$$

Now we recall that by assumption we know

$$\left(\int_{\mathcal{M}} \int_{\mathcal{M}} \frac{|f(y)-f(z)|^{\frac{d}{s}}}{|y-z|^{2d}} dydz \right)^{\frac{s}{d}} < \infty$$

Absolute continuity of the integral now implies for any $\tilde{\delta} > 0$ there exists a t_1 (say $< t_0$) such that

$$\left(\int_{\mathcal{M}\cap B(x,t)} \int_{\mathcal{M}\cap B(x,t)} \frac{|f(y)-f(z)|^{\frac{d}{s}}}{|y-z|^{2d}} dydz \right)^{\frac{s}{d}} < \tilde{\delta} \quad \forall x \in \mathcal{M}.$$

We choose $\tilde{\delta}$ so small so that we can apply the nearest point projection, Lemma 2.3, i.e. so that

$$\tilde{f}_t(x) \in B_\delta(\mathcal{N}) \quad \text{a.e. } x \in \mathcal{M}, \text{ for all } t \in (0, t_1)$$

Let us stress, that t_1 strongly depends on f, not only on the norm $[f]_{W^{s,\frac{d}{s}}}$, but on f itself. This will be very important later.

In any case, we now have that

$$f_t := \pi_{\mathcal{N}}(\tilde{f}_t)$$

is well-defined for any $t \in (0, t_1)$, and since $\pi_{\mathcal{N}}$ is smooth, we have that

$$f_t \xrightarrow{t\to 0^+} \pi_{\mathcal{N}}(f) \quad \text{in } W^{s,\frac{d}{s}}.$$

Since $f(x) \in \mathcal{N}$ a.e. we conclude that actually

$$f_t \xrightarrow{t\to 0^+} f \quad \text{in } W^{s,\frac{d}{s}}.$$

We can conclude. □

So if $f \in W^{s,\frac{d}{s}}(\mathcal{M},\mathcal{N})$ (in the sense discussed above) we can approximate it by $f_k \in W^{s,\frac{d}{s}} \cap C^0(\mathcal{M},\mathcal{N})$.

So we want to define

$$\widetilde{\deg}(f) := \lim_{k\to\infty} \deg(f_k)$$

The issue is: who says that is well-defined? It could be that $\widetilde{\deg} f_k$ depends on the choice of approximation etc.

But no, it doesn't.

Lemma 2.7. *Assume $f \in W^{s,\frac{d}{s}}(\mathcal{M},\mathcal{N})$. There exists an $\varepsilon > 0$ (depending on f!) such that whenever $g, h \in C^0 \cap W^{s,\frac{d}{s}}(\mathcal{M},\mathcal{N})$ satisfy*

$$\|f-g\|_{W^{s,\frac{d}{s}}(\mathcal{M},\mathcal{N})} + \|f-h\|_{W^{s,\frac{d}{s}}(\mathcal{M},\mathcal{N})} < \varepsilon \tag{2.1}$$

then h and g are homotopic, and in particular $\deg(h) = \deg(g)$.

Proof. The idea is to mollify g and h (crazy, they are already continuous! but still we shall do it):

So consider first

$$g_t(x) := \pi_{\mathcal{N}}\left(\frac{1}{c_t(x)} \int_{\mathcal{M}} \eta_t(x-y) g(y) dy\right)$$

We discussed above in the proof of Lemma 2.5 that this mollification makes sense for any $t \in (0, t_1]$ where t_1 was such that

$$\sup_{x\in\mathcal{M}, t\in(0,t_1)} \left(\int_{\mathcal{M}\cap B(x,t)} \int_{\mathcal{M}\cap B(x,t)} \frac{|g(y)-g(z)|^{\frac{d}{s}}}{|y-z|^{2d}} dydz\right)^{\frac{s}{d}} < \tilde{\delta}_{\mathcal{N}} \tag{2.2}$$

where $\tilde{\delta}_{\mathcal{N}}$ was a constant we derived from $\mathcal{N}$ so that Lemma 2.3 was applicable.

Since g is continuous, we see that

$$t \mapsto g_t \quad t \in [0, t_1]$$

is a homotopy! So

$$\deg g_t = \deg g \quad t \in [0, t_1].$$

The joke is that t_1 from (2.2) is actually dependent on f, not on g: Namely in view of (2.1).

$$\begin{aligned}&\left(\int_{\mathcal{M}\cap B(x,t)} \int_{\mathcal{M}\cap B(x,t)} \frac{|g(y)-g(z)|^{\frac{d}{s}}}{|y-z|^{2d}} dydz\right)^{\frac{s}{d}} \\ \leq &\left(\int_{\mathcal{M}\cap B(x,t)} \int_{\mathcal{M}\cap B(x,t)} \frac{|f(y)-f(z)|^{\frac{d}{s}}}{|y-z|^{2d}} dydz\right)^{\frac{s}{d}} \\ &+ \varepsilon\end{aligned}$$

So if we choose $\varepsilon < \frac{\tilde{\delta}_{\mathcal{N}}}{2}$ (this does not depend yet on f, but just you wait!) and t_1 (depending on the absolute continuity of the integrals in the norm of f!) such that

$$\left(\int_{\mathcal{M}\cap B(x,t)} \int_{\mathcal{M}\cap B(x,t)} \frac{|f(y)-f(z)|^{\frac{d}{s}}}{|y-z|^{2d}} dydz\right)^{\frac{s}{d}} < \frac{\tilde{\delta}_{\mathcal{N}}}{2}$$

then we have

$$t \mapsto g_t := \pi_{\mathcal{N}}\left(\frac{1}{c_t(x)} \int_{\mathcal{M}} \eta_t(x-y) g(y) dy\right) \quad t \in [0, t_1]$$

is a homotopy.

We can do the same for h, so

$$t \mapsto := h_t := \pi_{\mathcal{N}} \left(\frac{1}{c_t(x)} \int_{\mathcal{M}} \eta_t(x-y) h(y) dy \right) \quad t \in [0, t_1]$$

The point is t_1 is the same in both cases.

Now we want to show that h_t and g_t are homotopic, and we can use Lemma 2.4. Observe that since $\pi_{\mathcal{N}}$ is smooth

$$\begin{aligned} |g_t(x) - h_t(x)| \lesssim & \left| \frac{1}{c_t(x)} \int_{\mathcal{M}} \eta_t(x-y)(h(y) - g(y))\, dy \right| \\ \lesssim & \frac{1}{t^d} \|h - g\|_{L^1(\mathcal{M})} \end{aligned}$$

By (2.1),

$$\sup_{x \in \mathcal{M}} |g_t(x) - h_t(x)| \leq C \frac{1}{t^d} \varepsilon$$

And now here is the point: We can choose $t = t_1$. And assume ε from (2.1) is even smaller so that

$$C \frac{1}{(t_1)^d} \varepsilon < \varepsilon_0$$

where ε_0 is from Lemma 2.4 – that condition on ε heavily depends on t_1, which in turn heavily depends on f (not just the norm of f!). Then,

$$\sup_{x \in \mathcal{M}} |g_{t_1}(x) - h_{t_1}(x)| < \varepsilon_0$$

and thus by Lemma 2.4 f_{t_1} and g_{t_1} are homotopic; thus by assumptions on deg we have

$$\deg g_{t_1} = \deg h_{t_1}.$$

Since

$$[0,1] \ni t \mapsto g_t$$

and

$$[0,1] \ni t \mapsto h_t$$

are homotopies (g and h are continuous!), and by assumption deg is homotopy invariant in the continuous category, we conclude

$$\deg g = \deg g_{t_1} = \deg h_{t_1} = \deg h.$$

□

So, we have established that if $f \in W^{s,\frac{d}{s}}(\mathcal{M}, \mathcal{N})$ then

$$\deg \tilde{f} \quad \text{is the same } \forall\ W^{s,\frac{d}{s}}\text{-close, continuous maps } \tilde{f}.$$

So, for *any* $W^{s,\frac{d}{s}} \cap C^0$-approximation f_k of f the notion

$$\widetilde{\deg}(f) := \lim_{k\to\infty} \deg(f_k)$$

is well-defined for all $f \in W^{s,\frac{d}{s}}(\mathcal{M},\mathcal{N})$-maps. And if $f \in C^0 \cap W^{s,\frac{d}{2}}(\mathcal{M},\mathcal{N})$ then $\deg f = \widetilde{\deg}(f)$.

The last thing to conclude the claim of Theorem 2.1 is: we still need to show the homotopy invariance, but this is easy by now:

Lemma 2.8. *Assume $f \in W^{s,\frac{d}{s}}(\mathcal{M},\mathcal{N})$. Then there exists $\varepsilon > 0$ (depending heavily on f!) such that for all $g \in W^{s,\frac{d}{s}}(\mathcal{M},\mathcal{N})$ with*

$$\|g - f\|_{W^{s,\frac{d}{s}}(\mathcal{M})} < \varepsilon.$$

we have

$$\deg f = \deg g$$

Proof. Approximate f by a continuous $\tilde{f}$, and g by a continuous $\tilde{g}$ so that

$$\deg f = \deg \tilde{f}$$

and

$$\deg g = \deg \tilde{g}.$$

By Lemma 2.5 this is possible and we can even assume

$$\|\tilde{g} - g\|_{W^{s,\frac{d}{s}}(\mathcal{M})} + \|\tilde{f} - f\|_{W^{s,\frac{d}{s}}(\mathcal{M})} \ll 1.$$

In particular we can ensure that

$$\|\tilde{g} - f\|_{W^{s,\frac{d}{s}}(\mathcal{M})} \leq \|\tilde{g} - g\|_{W^{s,\frac{d}{s}}(\mathcal{M})} + \|g - f\|_{W^{s,\frac{d}{s}}(\mathcal{M})} < 2\varepsilon.$$

If ε is suitably small, by Lemma 2.7 we find that

$$\deg(\tilde{g}) = \deg(\tilde{f}).$$

By choice of $\tilde{g}$ and $\tilde{f}$ this implies $\deg(g) = \deg(f)$. □

Theorem 2.1 is established: any homotopy invariant notion for continuous maps can be extended to critical Sobolev maps.

From the above approximation argument we also readily see that the notion of $W^{s,\frac{d}{s}}$-$\tilde{\deg}$ is independent of the specific s:

Corollary 2.9. *Let $\mathcal{M}$,$\mathcal{N}$, $\deg$ be as in Theorem 2.1. Assume $s,t \in (0,1]$ and $f \in W^{s,\frac{d}{s}} \cap W^{t,\frac{d}{s}}(\mathcal{M},\mathcal{N})$. If we consider $\deg_1$ the extension of $\deg$ as above for $W^{s,\frac{d}{s}}$-maps, and $\deg_2$ the degree as above for $W^{t,\frac{d}{t}}$-maps then*

$$\deg_1 f = \deg_2 f.$$

2.2. **Conclusion of the Theorem 1.18.** All that is needed to conclude Theorem 1.18 is the following

Lemma 2.10. *If $f \in W^{1-\frac{1}{n},n}(\mathbb{S}^{n-1},\mathbb{S}^{n-1})$ is the trace of a map $F \in W^{1,n}(\mathbb{B}^n,\mathbb{S}^{n-1})$ then the winding number of f is zero.*

Proof. The point is simply that we can approximate

$$F \in W^{1,n}(\mathbb{B}^n,\mathbb{S}^{n-1})$$

w.r.t. the $W^{1,n}(\mathbb{B}^n,\mathbb{R}^n)$-norm

$$F_k \in C^\infty(\mathbb{B}^n,\mathbb{S}^{n-1})$$

essentially by Lemma 2.5. The only thing different is that our "manifold" $\mathcal{M} = \mathbb{B}^n$ has a boundary, but that is easily dealt with, by a reflection-type argument. E.g. set

$$\tilde{F}(x) := \begin{cases} F(x) & |x| \leq 1 \\ F(x/|x|^2) & |x| \in (1, \frac{3}{2}) \end{cases}$$

This $\tilde{F}$ is in $W^{1,n}_{loc}(B(3/2))$[8], so we can run the argument of Lemma 2.5 to find $F_k \in C^\infty(\mathbb{R}^n,\mathbb{S}^{n-1})$ such that

$$F_k\Big|_{\mathbb{B}^n} \xrightarrow{k\to\infty} F \quad \text{in } W^{1,n}(\mathbb{B}^n,\mathbb{R}^n).$$

Since the trace operator is continuous, we also have

$$F_k\Big|_{\partial\mathbb{B}^n} \xrightarrow{k\to\infty} f \quad \text{in } W^{1-\frac{1}{n},n}(\mathbb{S}^{n-1},\mathbb{R}^n)$$

Since F_k is in particular continuous we find that

$$\deg f = \lim_{k\to\infty} \deg F_k\Big|_{\partial\mathbb{B}^n}.$$

On the other hand $F_k\Big|_{\partial\mathbb{B}^n}$ can be extended to a map $F_k : \mathbb{B}^n \to \mathbb{S}^{n-1}$ so $\deg F_k\Big|_{\partial\mathbb{B}^n} \equiv 0$, Corollary 1.7. Thus

$$\deg f = \lim_{k\to\infty} \deg F_k\Big|_{\partial\mathbb{B}^n} = 0,$$

so we are done. □

[8]Observe $x \mapsto \frac{x}{|x|^2}$ is a C^∞-diffeomorphism from $\mathbb{B}(3/2) \setminus \mathbb{B}(1)$ to $B(1) \setminus B(2/3)$, and if f is Sobolev and τ is a diffeomorphism then $f \circ \tau$ is Sobolev. Observe that $\tilde{F}$ maps still into the same set as F – if, e.g., we were considering $W^{2,p}$ we would have to be more careful with the reflection

3. The degree formula - obstacles to extensions

We slightly shift focus, and consider the previously discussed "topology for Sobolev maps" as non-existence results:

Above we have seen that if we have a (say smooth, for simplicity) map

$$f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$$

which has *nonzero* winding number / degree, then it cannot be extended to a map

$$F \in W^{1,n}(\mathbb{B}^n, \mathbb{S}^{n-1}),$$

because that would violate Lemma 2.10.

Quite differently, it is easy to find extensions

$$F \in W^{1,n}(\mathbb{B}^n, \mathbb{R}^n),$$

e.g. we could set

$$F(x) := \eta(x) f(x/|x|)$$

where $\eta \in C^\infty$, $\eta \equiv 0$ around 0 and $\eta \equiv 1$ around $\partial \mathbb{B}^n$.

Our next goal is to prove the following theorem, which is essentially due [HST14]

Theorem 3.1. *Assume $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be a smooth map of nonzero degree. Then there exist no map $F \in W^{1,n}(\mathbb{B}^n, \mathbb{R}^n)$ with the properties*

- $F\Big|_{\partial \mathbb{B}^n} = f$ *in the sense of traces*
- $\operatorname{rank} DF \leq n-1$ *a.e. in* $\mathbb{B}^n$.

Any $W^{1,n}$-map $F : \mathbb{B}^n \to \mathbb{S}^{n-1}$ has $\operatorname{rank} Df \leq n-1$ a.e. ("by" the implicit function theorem). The above theorem says, that being more permissive with conditions on F, namely allowing F to go to all of $\mathbb{R}^n$, we cannot restrict the rank to be $\operatorname{rank} DF \leq n-1$.

Also, by embedding $\mathbb{S}^{n-1} \subset \mathbb{R}^N$ we get the following theorem

Theorem 3.2. *Let $N \geq n$. There is a smooth map $f : \mathbb{S}^{n-1} \to \mathbb{R}^N$ such that there exist no map $F \in W^{1,n}(\mathbb{B}^n, \mathbb{R}^N)$ with the properties*

- $F\Big|_{\partial \mathbb{B}^n} = f$ *in the sense of traces*
- $\operatorname{rank} Df \leq n-1$ *a.e. in* $\mathbb{B}^n$.

Why do we care about maps with constrained rank? Well because there are things more flexible than manifolds: in [HST14] we considered the Heisenberg group $\mathbb{H}_\ell$, and one can show that any map into the Heisenberg group must have $\operatorname{rank} DF \leq \ell$, so the previous result becomes useful.

To prove Theorem 3.2 it is time to introduce the actual degree (not just the winding number). The essential point is that there is a precise formula for the degree that consists of Jacobians (determinants of ∇F) – i.e. algebraic objects which measures quite well rank. So let's do some linear algebra.

4. Analytical version of degree

Essentially the main point of our future arguments are that if we can measure the topology in terms of differential forms (i.e. DeRham-cohomology) then we are in a good shape. Things get quite unclear if this is not the case. This is related to rational homotopy groups (or "torsion free" parts), which by an abstract theorem by Novikov can be represented as cohomology [HR08]. I completely don't understand what the previous topological notions mean. So before I say too many wrong things, let's focus on the degree.

Fun fact 1: The *Winding number* we so beautifully explained with nice pictures, can be written in a way more ugly, unintuitive, but way more analytically useful form:

$$w(f) = \int_{\mathbb{S}^1} \left(f^1 \partial_\tau f^2 - f^2 \partial_\tau f^1 \right) d\sigma,$$

and more general, for maps $f : \mathbb{S}^n \to \mathbb{S}^n$ we set

$$\deg(f) := \int_{\mathbb{S}^n} \sum_{i=1}^{n+1} (-1)^{i-1} f^i \det(Df^1, \ldots Df^{i-1}, Df^{i+1}, \ldots, Df^n) dp$$

This is a bit messy, so we are going to use the language of differential forms and use that

$$\deg(f) = \int_{\mathbb{S}^n} f^*(\mathrm{vol}) : \quad \forall\, f \in C^\infty(\mathbb{S}^n, \mathbb{S}^n).$$

But we need to discuss what this means.

4.1. Crash course on differential forms.

A k-form ω on $\mathbb{R}^N$, we write $\omega \in \bigwedge^k \mathbb{R}^N$, is simply something that can be written as

$$\omega = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq N} \omega_{i_1, \ldots, i_k} dp^{i_1} \wedge \ldots \wedge dp^{i_k}$$

What is the meaning of $dp^{i_1} \wedge \ldots \wedge dp^{i_k}$? We don't really care. ω in the above form is simply a vector in $\bigwedge^k \mathbb{R}^N$.

For convenience we don't want to always have to assume $i_1 < i_2 < \ldots$. So we make the convention

$$dp^i \wedge dp^j = -dp^j \wedge dp^i.$$

ω as above is a vector in a (strange) vector space $\bigwedge^k \mathbb{R}^N$. We can think of a function $\omega : \mathbb{R}^N \to \bigwedge^k \mathbb{R}^N$, which simply means that the coefficients $\omega_{i_1, \ldots, i_k}$ depends on the point

$p \in \mathbb{R}^N$. Then we can discuss notions such as

$$\omega \in W^{1,p}(\mathbb{R}^N)$$

which simply means that

$$\omega = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq N} \omega_{i_1,\ldots,i_k}(p) dp^{i_1} \wedge \ldots \wedge dp^{i_k}$$

and $\omega_{i_1,\ldots,i_k}(x) \in W^{1,p}(\mathbb{R}^N, \bigwedge^k \mathbb{R}^N)$. If someone needs a norm, we will say

$$\|\omega\|_{W^{1,p}} := \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq N} \|\omega_{i_1,\ldots,i_k}\|_{W^{1,p}}.$$

One important operation is the differential. If $\omega : \mathbb{R}^N \to \bigwedge^k \mathbb{R}^N$ can afford a derivative, e.g. if $\omega \in C^1(\mathbb{R}^N, \bigwedge^k \mathbb{R}^N)$, then we set

$$d\omega(p) = \sum_{j=1}^{N} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq N} \frac{\partial}{\partial p^j} \omega_{i_1,\ldots,i_k}(p)\, dp^j \wedge dp^{i_1} \wedge \ldots \wedge dp^{i_k}$$

Then we see

$$d : C^1(\mathbb{R}^N, \bigwedge^k \mathbb{R}^N) \to C^0(\mathbb{R}^N, \bigwedge^{k+1} \mathbb{R}^N)$$

(a curious observation is that $d \circ d = 0$, i.e. $dd\omega = 0$ no matter what ω is)

It is somewhat instructive to see that d encodes the gradient for 0-forms (which are simply functions)

$$f : \mathbb{R}^N \to \mathbb{R} \text{ then } \quad df = \sum_{i=1}^{N} \partial_i f dp^i \quad \in \bigwedge^1 \mathbb{R}^N.$$

If we have a vectorial map

$$\vec{g} : \mathbb{R}^N \to \mathbb{R}^N$$

we could be tempted to "translate" $\vec{g}$ to a map $f : \mathbb{R}^N \to \bigwedge^1 \mathbb{R}^N$,

$$f = \sum_{i=1}^{N} g^i dp^i$$

What is dg? Observe that $d \circ d = 0$, so if $g = df$ for a function f, we must have $dg = 0$. Indeed, it turns out dg is the[9] *curl*.

[9]The divergence of g can also be represented with differential forms, we just the the Hodge star operator, or the co-differential. This is not of interest to us, so we skip this here

Indeed,

$$\begin{aligned}
dg =& \sum_{i=1}^N \sum_{\ell=1}^N \partial_\ell g^i(p)\, dp^\ell \wedge dp^i\\
\overset{dp^\ell\wedge dp^\ell=0}{=}& \sum_{i=1}^N \sum_{\ell<i} \partial_\ell g^i(p)\, dp^\ell \wedge dp^i + \sum_{i=1}^N \sum_{\ell>i} \partial_\ell g^i(p)\, dp^\ell \wedge dp^i\\
=& \sum_{i=1}^N \sum_{\ell<i} \partial_\ell g^i(p)\, dp^\ell \wedge dp^i - \sum_{i=1}^N \sum_{\ell>i} \partial_\ell g^i(p)\, dp^i \wedge dp^\ell\\
=& \sum_{1\le i_1<i_2\le N} \partial_{i_1} g^{i_2}(p)\, dp^{i_1} \wedge dp^{i_2} - \sum_{1\le i_1<i_2\le N} \partial_{i_2} g^{i_1}(p)\, dp^{i_1} \wedge dp^{i_2}\\
=& \sum_{1\le i_1<i_2\le N} \Big(\partial_{i_1} g^{i_2}(p) - \partial_{i_2} g^{i_1}(p)\Big)\, dp^{i_1} \wedge dp^{i_2}
\end{aligned}$$

The second important operation is the *pullback* $\Phi^*(\omega)$.

Let $\Phi : \mathbb{R}^n \to \mathbb{R}^N$ be any smooth (enough) map, and take $\omega \in C^\infty(\mathbb{R}^N, \bigwedge^k \mathbb{R}^N)$,

$$\omega = \sum_{1\le i_1<i_2<\ldots<i_k\le N} \omega_{i_1,\ldots,i_k}(p)\, dp^{i_1} \wedge \ldots \wedge dp^{i_k}.$$

Then we define the pullback $\Phi^*(\omega) \in C^\infty(\mathbb{R}^n, \bigwedge^k \mathbb{R}^n)$

$$\Phi^*(\omega) := \sum_{1\le i_1<i_2<\ldots<i_k\le N} \omega_{i_1,\ldots,i_k}(\Phi(x))\, d\Phi^{i_1}(x) \wedge \ldots \wedge d\Phi^{i_k}(x)$$

Here $\Phi^{i_k}(x)$ is the i_k-th component of $\Phi(x) = (\Phi^1(x), \Phi^2(x), \ldots, \Phi^N(x))$. Each $\Phi^i(x)$ is a zero-form in $\mathbb{R}^n$, and then[10]

$$d\Phi^i(x) = \frac{\partial}{\partial x^j}\Phi^i(x)\, dx^j$$

So what is

$$d\Phi^{i_1}(x) \wedge \ldots \wedge d\Phi^{i_k}(x)?$$

Well it is a k-form so it can be written as

$$d\Phi^{i_1}(x) \wedge \ldots \wedge d\Phi^{i_k}(x) = \sum_{1\le \alpha_1<\alpha_2<\ldots<\alpha_k\le n} J^{i_1,\ldots,i_k}_{\alpha_1,\ldots,\alpha_k}(D\Phi(x))\, dx^{\alpha_1} \wedge \ldots dx^{\alpha_k}$$

The question is what is J?

It turns out (and that is really the beauty of these differential forms!) it is a Jacobian of a submatrix of $D\Phi$. If we agree on the notation

$$((D\Phi)_{j\alpha})_{j\in\{1,\ldots,N\},\alpha\in\{1,\ldots,n\}} := \Big(\partial_\alpha \Phi^j(x)\Big)_{j\in\{1,\ldots,N\},\alpha\in\{1,\ldots,n\}}.$$

[10] we sum over repeated indices!

That is $D\Phi$ is a $\mathbb{R}^{N\times n}$-matrix. We can restrict it to a $k \times k$ matrix by only considering the rows $i_1, \ldots, i_k$ and the colums $\alpha_1, \ldots, \alpha_k$.

$$\left(D\Phi(x)^{i_1,\ldots,i_k}_{\alpha_1,\ldots,\alpha_k}\right)_{j\in\{i_1,\ldots,i_k\},\alpha\in\{\alpha_1,\ldots,\alpha_k\}} := \left(\partial_\alpha \Phi^j(x)\right)_{j\in\{i_1,\ldots,i_k\},\alpha\in\{\alpha_1,\ldots,\alpha_k\}}.$$

And it turns out (I am not going to prove it, but its pure algebra)

$$J^{i_1,\ldots,i_k}_{\alpha_1,\ldots,\alpha_k}\left(D\Phi(x)\right) = \pm \det_{k\times k}\left(D\Phi(x)^{i_1,\ldots,i_k}_{\alpha_1,\ldots,\alpha_k}\right)_{j\in\{i_1,\ldots,i_k\},\alpha\in\{\alpha_1,\ldots,\alpha_k\}}$$

The determinant of $D\Phi$ is called Jacobian, so we call these object *sub-Jacobians* (you can compute the sign on your own. We don't really care). Recall that $\det_{k\times k}(A)$ can be interpreted as the area of the unit cube under the transform $A \in \mathbb{R}^{k\times k}$ (with some orientation taken into account, because the determinant can be negative). That is the *Jacobian is the area of the linearization of* Φ – this is the basis for the area formula, which for diffeomorphisms $\Phi : \mathcal{M} \to \mathbb{R}^N$ now reads as

$$\int_{\Phi(\mathcal{M})} \omega = \int_{\mathcal{M}} \Phi^*(\omega)$$

Observe that this gives us a way of integrating differential forms on manifolds: in $\mathbb{R}^n$ we call $dx^1 \wedge \ldots dx^n$ *the volume form*, and set

$$\int_{\mathbb{R}^n} f(x) dx^1 \wedge \ldots dx^n = \int_{\mathbb{R}^n} f(x) d\mathcal{L}^n(x).$$

So if Φ is a local parametrization from $U \subset \mathbb{R}^n$ of a manifold $\mathcal{M} \subset \mathbb{R}^N$ then we define

$$\int_{\mathcal{M}} \omega := \int_U \Phi^*(\omega).$$

Good news is this coincides with our usual notion of integration on manifolds. Observe that we can only integrate k-forms on k-manifolds.

Let us also mention the integration by parts formula, *Stokes' theorem*, which for differential forms now reads as

$$\int_{\partial B} \omega = \int_B d\omega.$$

Observe that if B has no boundary, say its the sphere $\mathbb{S}^{n-1}$ then Stokes theorem simply is

$$\int_{\mathbb{S}^{n-1}} d\omega = 0.$$

Lastly, we mention (a direct calculation) that differential and pullback commute,

$$d\Phi^*(\omega) = \Phi^*(d\omega)$$

We also discuss two important notions: *closed forms* and *exact forms*.

A k-form ω is *closed* if $d\omega = 0$. It is *exact* if $\omega = d\eta$ for a $(k-1)$-form η. In particular any exact form is necessarily closed, but in principle a closed form could be not exact. The k-th *DeRham cohomology group* consists of the closed k-forms, where two forms are considered the same if their difference is exact.

4.2. **Back to the degree.** Let

$$\mathrm{vol} \in \bigwedge^{n-1} \mathbb{R}^n$$

be the volume form of the sphere. Namely assume vol is a $n-1$-form such that

$$\int_{\mathbb{S}^{n-1}} \mathrm{vol} = \mathcal{H}^{n-1}(\mathbb{S}^{n-1}).$$

The volume form is not unique (indeed, by stokes theorem $\mathrm{vol} + d\alpha$ is still a volume form)

Usually we like to use

$$\mathrm{vol} = \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \ldots dx^{i-1} \wedge dx^{i+1} \wedge \ldots dx^n.$$

The point of all this is that the winding number and the degree formulas we introduced earlier are simply:

Theorem 4.1. *Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be smooth then*

$$\deg(f) = \int_{\mathbb{S}^n} f^*(\mathrm{vol}) \tag{4.1}$$

In the case of the winding number, quite handwavingly one could try to prove this as follows. Fix $\bar{p} \in \mathbb{S}^1$, the point where we want to put our "observation point" for the winding number.

Assume for some $x \in \mathbb{S}^1$ we have $f(x) = \bar{p}$. Then

$$\begin{aligned} f^*\mathrm{vol}_{\mathbb{S}^1} \text{ ``=''} &\Big(f^1(x)\partial_\tau f^2(x) - f^2\partial_\tau f^1(x)\Big) \\ &= \begin{pmatrix} -f^2(x) \\ f^1(x) \end{pmatrix} \cdot \partial_\tau f(x) \\ &= \bar{p}^\perp \cdot \partial_\tau f(x) \end{aligned}$$

and observe that since $\bar{p} \in \mathbb{R}^2$, $\bar{p}^\perp = \begin{pmatrix} -\bar{p}^2 \\ \bar{p}^1 \end{pmatrix}$ is a tangent vector to $\mathbb{S}^1$ at the point p, and $\partial_\tau f$ is also tangent. So $\bar{p}^\perp \cdot \partial_\tau f(x)$ is positive if f passes in direction $\bar{p}^\perp$ and negative if it passes in direction $-\bar{p}^\perp$.

The following is the crucial observation: $f \mapsto \int f^*\omega$ is "homotopically invariant" under "homotopies" which have a rank condition instead of mapping into a manifold!

Lemma 4.2. *Fix any ℓ-dimensional manifold $\mathcal{M}$ without boundary (think about $\mathbb{S}^{n-1}$).*

Fix $\omega \in C^\infty(\bigwedge^\ell \mathbb{R}^N)$ any smooth ℓ-form (like the volume form on $\mathbb{S}^{n-1}$ for $\ell = n-1$, $N = n$), and define for smooth maps

$$f : \mathcal{M} \to \mathbb{R}^N$$

the operation

$$\widetilde{\deg}(f) := \int_{\mathcal{M}} f^*(\omega).$$

Then $\widetilde{\deg}$ *is invariant under homotopies with rank restricted by* ℓ*. More precisely:*

If $f, g : \mathcal{M} \to \mathbb{R}^N$ *are smooth maps, and there exists* $H : [0,1] \times \mathcal{M} \to \mathbb{R}^N$ *also smooth with the property*

$$\operatorname{rank} \nabla_{x,t} H \leq \ell \quad a.e.$$

then

$$\widetilde{\deg}(f) = \widetilde{\deg}(g)$$

Proof. We have $[0,1] \times \mathcal{M}$ is a manifold with boundary $\{0\} \times \mathcal{M}$ and $\{1\} \times \mathcal{M}$ (which have opposite orientation). Thus

$$\begin{aligned}
&\int_{\mathcal{M}} f^*(\omega) - \int_{\mathcal{M}} g^*(\omega) \\
=& \int_{\partial([0,1]\times\mathcal{M})} H^*(\omega) \\
\overset{\text{Stokes}}{=}& \int_{[0,1]\times\mathcal{M}} dH^*(\omega) \\
=& \int_{[0,1]\times\mathcal{M}} H^*(d\omega)
\end{aligned}$$

And here comes the joke:

$$H^*(d\omega)$$

is the pull-back of the $(\ell{+}1)$-form $d\omega$, that is any entry in $H^*(d\omega)$ can be written as the determinant of an $(\ell{+}1)\times(\ell{+}1)$-submatrix of $\nabla_{x,t} H$. But we assumed that $\operatorname{rank} \nabla_{x,t} H \leq \ell$, so

$$H^*(d\omega) \equiv 0.$$

Thus we have shown

$$\int_{\mathcal{M}} f^*(\omega) = \int_{\mathcal{M}} g^*(\omega)$$

□

4.3. **Proof of Theorem 3.1.** This approach gives us also a proof of Theorem 3.1, first in the smooth category

- Let $F \in C^\infty(\mathbb{B}^n, \mathbb{R}^n)$ with $f := F\Big|_{\partial\mathbb{B}^n} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$.
- If $\operatorname{rank} DF \leq n-1$ then F is a homotopy to a constant map $g := F(0)$, $H(t,\theta) := F(t\theta)$, so by the previous Lemma
$$\int_{\mathbb{S}^{n-1}} f^*(\omega) = 0$$
for any $n-1$-form ω.
- By Theorem 4.1 this means that $\deg f = 0$.

Did we use anywhere that F is smooth? No. It suffices that $F^*(d\omega)$ is integrable, since $d\omega$ is an n-form, $F \in W^{1,n}(\mathbb{B}^n, \mathbb{R}^N)$ suffices.

5. Obstacles to Sobolev-homeomorphisms with restricted rank

For our next trick we want to push Theorem 3.1 a bit further:

Theorem 5.1. *Assume $f : \mathbb{S}^{n-1} \to \mathbb{R}^N$ be a smooth diffeomorphism onto its own target. Then there exist no map $F \in W^{1,n}(\mathbb{B}^n, \mathbb{R}^N)$ with the properties*

- $F\Big|_{\partial \mathbb{B}^n} = f$ *in the sense of traces*
- $\operatorname{rank} DF \leq n-1$ *a.e. in* $\mathbb{B}^n$.

This is a pretty sharp statement

Example 5.2. For any $\alpha \in (0,1)$, there exists a map $F : \mathbb{B}^n \to \mathbb{R}^n$, *bi-$C^\alpha$-homeomorphic*, such that $\operatorname{rank} DF \leq n-1$ a.e. and $DF \in L^{(n,\infty)}$[11]

This result was proven via methods of convex integration in [FMCO18]. For very related results using more explicit constructions see also, e.g., [LM16] and [Hen11].

This F cannot belong to L^n – this follows from the area formula

$$|\mathcal{H}^n(F(B^n))| \leq \int_\Omega J_F.$$

Here J_F is the Jacobian of f, $\sqrt{\det(\nabla F)}$. This formula makes sense if $DF \in L^n$ and $J_f \in L^1$. – And indeed $J_F = 0$ a.e. if $\operatorname{rank} DF \leq n-1$, that is we have

$$|\mathcal{H}^n(F(B^n))| \equiv 0.$$

Since $F : \mathbb{B}^n \to \mathbb{R}^n$ is a homeomorphism it maps open sets to open sets, and thus $F(B^n)$ must have positive measure – contradiction.

The proof of Theorem 5.1 was given (in the context of Hölder mappings into the Heisenberg group) in [Sch20, HS23] and fractional Sobolev mappings [PS24]. It is pretty easy once we have the following

Lemma 5.3. *Let $f \in C^0 \cap W^{1,n}(\mathbb{S}^{n-1}, \mathbb{R}^N)$ be a homeomorphism. Then there exists $\omega \in C_c^\infty(\bigwedge^{n-1} \mathbb{R}^N)$ such that*

$$\int_{\mathbb{S}^{n-1}} f^*(\omega) = 1$$

[11] this is the weak L^n-space (also called Marcinkiewicz space): $f \in L^{p,\infty}$ if and only if Chebychef-type inequality holds for superlevel sets

$$|\{x : |f(x)| > t\}| \leq \frac{\|f\|_{L^{p,\infty}}}{t^p} \quad \forall t > 0.$$

A map in $L^{(p,\infty)}$ is in particular in L^q_{loc} for any $q < p$.

Once we have Lemma 5.3 we just argue as in the proof of Theorem 3.1:

Proof of Theorem 5.1. For any $F \in W^{1,n}(\mathbb{B}^n, \mathbb{R}^N)$ we have by Stokes' theorem

$$1 = \int_{\mathbb{S}^{n-1}} f^*(\omega) = \int_{\mathbb{B}^n} F^*(d\omega)$$

Since $d\omega$ is an n-form, if $\operatorname{rank} DF \leq n-1$, we have $F^*(d\omega) = 0$ since it is a bunch of $n \times n$-determinants of DF. So we have $1 = 0$ and a contradiction. $\square$

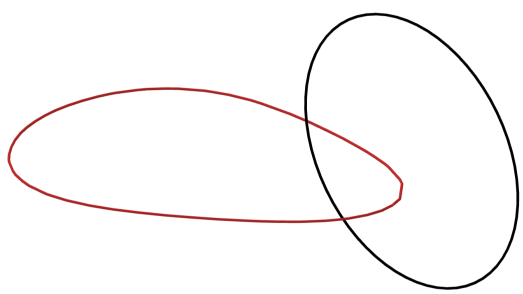
5.1. **Proof of Lemma 5.3 - Linking number.** So how do we prove Lemma 5.3? If you know algebraic topology, the proof is essentially a copy of the fact that the homology class $H_{N-(n-1)-1}(\mathbb{R}^N \setminus f(\mathbb{S}^{n-1}))$ that can be found in the early chapters of any algebraic topology book, see e.g. [Vic94, Corollary 1.29]. The proof below is indeed a translation of exactly that argument to cohomology (via Poincaré duality this is always possible) with explicit computations of the operators in the Mayer-Vietoris sequence (see remark below). The idea is that we can think of

$$[f, d\omega] := \int_{\mathbb{S}^n} f^*(\omega)$$

as the "Linking number" between the $f(\mathbb{S}^n)$ and the $d\omega$ (which is an element of the cohomology group $H^{N-1}(\mathbb{R}^N \setminus f(\mathbb{S}^{n-1}))$) and thus equivalent by Poincaré duality to an element of the homology group $H_{N-(n-1)-1}(\mathbb{R}^N \setminus f(\mathbb{S}^{n-1}))$), i.e. we are finding an $N-(n-1)-1$-dimensional surface that links with $f(\mathbb{S}^{n-1})$.

Here is the picture statement that sounds very credible:

Lemma 5.4. *Let $K \subset \mathbb{R}^3$ be any set homeomorphic to $\mathbb{S}^1$. Then there exists a curve γ that links with K.*

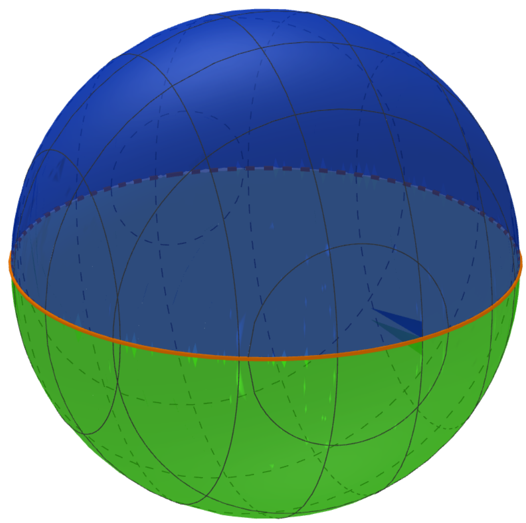


Assume the set K (red) is homeomorphic to a 1-sphere – then we find a linking curve (black).

This result, and the knowledge that Algebraic Topology knows how to prove this result in any dimension, was the motivation for the following proof[12]

[12]As a side remark: this picture of "we find a linked surface with $f(\mathbb{S}^{n-1})$ is not fully correct. After completing the math of [HS23], Hajłasz showed that the "linked surface" we find here may actually not

Proof of Lemma 5.3. We argue by induction on the dimension of the domain sphere $\mathbb{S}^\ell$. Here we consider $\mathbb{S}^{\ell-1}$ the equator (any choice) of the sphere $\mathbb{S}^\ell$. Since the choice of the equator does not really matter we shall assume w.l.o.g. that $f\Big|_{\mathbb{S}^\ell} \in W^{1,n}(\mathbb{S}^\ell, \mathbb{R}^N)$ which is always doable by Fubini's theorem.



We split the sphere $\mathbb{S}^\ell$ into the equator $\mathbb{S}^{\ell-1}$ (red), the closed upper hemisphere $\mathbb{S}^\ell_+$ and the closed lower hemisphere $\mathbb{S}^\ell_-$.

The induction claim is

For $\ell = 0, 1, \ldots, n-1$ there exists an $\omega_\ell \in C^\infty_c(\bigwedge^\ell \mathbb{R}^N)$ such that

- $d\omega_\ell \equiv 0$ in a neighborhood of $f(\mathbb{S}^\ell)$
- $\int_{\mathbb{S}^\ell} f^*(\omega_\ell) \neq 0$

First we consider the <u>case $\ell = 0$</u>

By the decomposition above, $\mathbb{S}^0$ are simply two points, which we may denote with $\{p_+, p_-\}$. Since f is a homeomorphism, $f(p_+) \neq f(p_-)$. So we just pick $\omega_0 \in C^\infty_c(\bigwedge^0 \mathbb{R}^N)$ a 0-form (i.e. function on $\mathbb{R}^N$) to be constantly 1 around $f(p_+)$ and constantly 0 around $f(p_-)$ and also constantly zero outside a big ball in $\mathbb{R}^N$. Then $d\omega_0 = 0$ around $f(\mathbb{S}^0)$, and

$$\int_{\mathbb{S}^0} f^*(\omega_0) = \omega_0(f(p_+)) - \omega_0(f(p_-)) = 1 \neq 0.$$

That was easy.

be a real link, but may consist of several surfaces that combined links [Ha19]. Of course this is not really relevant for our purposes, but a very cool fun fact.

Case $(\ell-1) \to \ell$:

We assume that we have found an $(\ell-1)$-form $\omega_{\ell-1} \in C_c^\infty(\bigwedge^{\ell-1} \mathbb{R}^N)$ such that[13]

$$\eta_{\ell-1} := d\omega_{\ell-1} \in C_c^\infty(\bigwedge^{\ell} \mathbb{R}^N \setminus f(\mathbb{S}^{\ell-1})),$$

and

$$\int_{\mathbb{S}^{\ell-1}} f^*(\omega_{\ell-1}) \neq 0.$$

We first construct a closed $(\ell+1)$-form η_ℓ.

We define open subsets of $\mathbb{R}^N$ as follows:

$$U := \mathbb{R}^N \backslash f(\mathbb{S}^\ell_+), \quad V := \mathbb{R}^N \backslash f(\mathbb{S}^\ell_-).$$

Then

$$U \cup V = \mathbb{R}^N \setminus \{f(\mathbb{S}^\ell_+) \cap f(\mathbb{S}^\ell_-)\} = \mathbb{R}^N \setminus f(\mathbb{S}^{\ell-1}).$$

since f is one-to-one.

Morever,

$$U \cap V = \mathbb{R}^N \setminus \{f(\mathbb{S}^\ell_+) \cup f(\mathbb{S}^\ell_-)\} = \mathbb{R}^N \setminus f(\mathbb{S}^\ell). \tag{5.1}$$

The support of $\eta_{\ell-1}$ is bounded away from $f(\mathbb{S}^{\ell-1})$, thus

$$\operatorname{supp} \eta_{\ell-1} \subset \mathbb{R}^N \backslash f(\mathbb{S}^{\ell-1}) = U \cup V.$$

Any open neighborhood W of $f(\mathbb{S}^\ell_+)$ will cover a bit of $f(\mathbb{S}^\ell_-)$. However, we can choose an open neighborhood W of $f(\mathbb{S}^\ell_+)$ such that the part that W covers of $f(\mathbb{S}^\ell_-)$ does not belong to the support of $\eta_{\ell-1}$ (which is zero around $f(\mathbb{S}^\ell_+) \cap f(\mathbb{S}^\ell_+) = f(\mathbb{S}^{\ell-1})$).

More precisely, take two small open neighborhoods $W_1 \subset\subset W_2$ of $f(\mathbb{S}^\ell_+)$ that

so that

$$\operatorname{supp} \eta_{\ell-1} \cap \overline{W_2} \subset V$$

(i.e. $\operatorname{supp} \eta_{\ell-1} \cap \overline{W_2}$ does not see anything from the southern hemisphere $f(\mathbb{S}^\ell_+)$). Again, this is possible because $f(\mathbb{S}^\ell_+) \cap f(\mathbb{S}^\ell_-) = f(\mathbb{S}^{\ell-1})$ and $\eta_{\ell-1}$ vanishes around these points. Cf. Figure 5.1.

Take your favorite cutoff function $\chi \in C_c^\infty(W_2)$ with $\chi \equiv 1$ in W_1 and set

$$\gamma_V := \chi \eta_{\ell-1}, \quad \gamma_U := (1-\chi)\eta_{\ell-1}.$$

Then

$$\operatorname{supp} \gamma_U \subset \operatorname{supp}(1-\chi) \subset \mathbb{R}^N \setminus W_1 \subset \mathbb{R}^N \setminus f(\mathbb{S}^\ell_+) = U$$

and

$$\operatorname{supp} \gamma_V \subset \operatorname{supp} \chi \cap \operatorname{supp} \eta_{\ell-1} \subset \overline{W_2} \cap \operatorname{supp} \eta_{\ell-1} \subset V.$$

[13]Observe that $f(\mathbb{S}^\ell)$ is a compact set since f is continuous, so that the above means $d\omega \equiv 0$ in particular around $f(\mathbb{S}^\ell)$

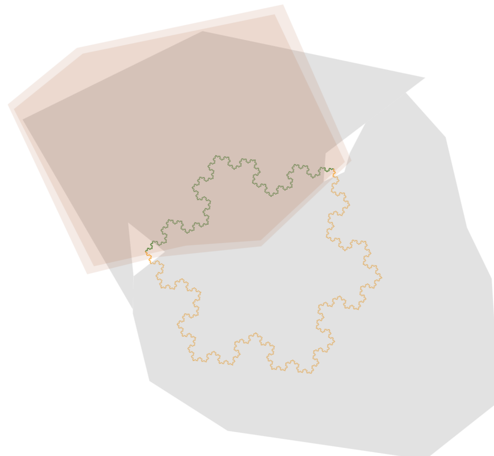


FIGURE 5.1. The Koch snowflake represents $f(\mathbb{S}^\ell)$, the green part is $f(\mathbb{S}^\ell_+)$, the orange part $f(\mathbb{S}^\ell_-)$. The grey area is the support of $\eta_{\ell-1}$, which vanishes around $f(\mathbb{S}^\ell_+) \cap f(\mathbb{S}^\ell_-) = f(\mathbb{S}^{\ell-1})$. The sets W_1 and W_2 (red) contain $f(\mathbb{S}^\ell_+)$, but the only part of $f(\mathbb{S}^\ell_-)$ they touch is where $\eta_{\ell-1}$ vanishes – so that $\operatorname{supp} \eta_\ell \cap \overline{W_i}$ does not contain anything of $f(\mathbb{S}^\ell_-)$, $i = 1, 2$.

and we have

$$\eta_{\ell-1} = \gamma_U + \gamma_V \tag{5.2}$$

We define

$$\omega_\ell := \gamma_U, \quad \eta_\ell := d\gamma_U.$$

Since $d\eta_{\ell-1} = d\,(d\omega_{\ell-1}) = 0$ we actually have

$$\eta_\ell = d\gamma_U = -d\gamma_V.$$

In particular, with (5.1),

$$\operatorname{supp} \eta_\ell \subset \operatorname{supp} \gamma_U \cap \operatorname{supp} \gamma_V \subset U \cap V \subset \mathbb{R}^N \backslash f(\mathbb{S}^\ell).$$

Thus, we have found ω_ℓ, which is a smooth ℓ-form, such that $d\omega_\ell \equiv \eta_\ell = 0$ around $f(\mathbb{S}^\ell)$.

That is, the induction step is completed, once we confirm

$$\int_{\mathbb{S}^\ell} f^*(\omega_\ell) \neq 0. \tag{5.3}$$

So let us establish (5.3). In view of the support of γ_U and γ_V and (5.2)

$$\int_{\mathbb{S}^\ell} f^*(\omega_\ell) = \int_{\mathbb{S}^\ell_-} f^*(\gamma_U) = \int_{\mathbb{S}^\ell_-} f^*(\eta_{\ell-1} - \gamma_V) = \int_{\mathbb{S}^\ell_-} f^*(\eta_{\ell-1}).$$

Now we use Stokes' theorem on $\mathbb{S}^\ell_-$: Observe that by construction of the spheres,

$$\partial \mathbb{S}^\ell_- = \mathbb{S}^{\ell-1}.$$

Then Stokes' theorem tells us

$$\int_{\mathbb{S}^\ell_-} f^*(\eta_{\ell-1}) = \int_{\mathbb{S}^\ell_-} f^*(d\omega_{\ell-1}) = \pm \int_{\mathbb{S}^{\ell-1}} f^*(\omega_\ell) \neq 0$$

The sign $\pm$ comes from the orientation of $\partial \mathbb{S}^\ell_- = \mathbb{S}^{\ell-1}$. The $\neq 0$ is the induction hypothesis. (5.3) is proven. $\square$

Remark 5.5. Let us put the above argument into the perspective of algebraic topology. By the induction hypothesis, $\eta_{\ell-1}$ is an element of the cohomology group $H^\ell(\mathbb{R}^N \backslash f(\mathbb{S}^{\ell-1}))$. We just used the exact *Mayer-Vietoris sequence*,

$$\ldots \to H^\ell(U) \oplus H^\ell(U) \to H^\ell(U \cup V) \xrightarrow{c} H^{\ell+1}(U \cap V) \to H^{\ell+1}(U) \oplus H^{\ell+1}(U) \to \ldots$$

where we observe $U \cap V = \mathbb{R}^N \backslash f(\mathbb{S}^\ell)$, $U \cup V = \mathbb{R}^N \backslash f(\mathbb{S}^{\ell-1})$. Also, since U and V are homeomorphic to $\mathbb{R}^N$ (that is $\mathbb{S}^N$) with a cube taken away,

$$H^{\ell+1}(U) = H^{\ell+1}(V) = H^\ell(U) = H^\ell(V) = 0.$$

Thus, the Mayer-Vietoris sequence is simply

$$0 \to H^\ell(\mathbb{R}^N \backslash f(\mathbb{S}^{\ell-1})) \xrightarrow{c} H^{\ell+1}(\mathbb{R}^N \backslash f(\mathbb{S}^\ell)) \to 0.$$

This just means that the connecting homomorphism $c : H^\ell(\mathbb{R}^N \backslash \varphi(\mathbb{S}^{\ell-1})) \xrightarrow{c} H^{\ell+1}(\mathbb{R}^N \backslash \varphi(\mathbb{S}^\ell)))$ is an isomorphism. On the other hand c is known, and all we did above is set $\eta_\ell := c(\eta_{\ell-1})$.

Actually one can show that $\eta = d\omega \mapsto \int_{\mathbb{S}^k} \varphi^*(\omega)$ is an isomorphism on $H^{k+1}(\mathbb{R}^N \backslash \varphi(\mathbb{S}^k))$.

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