ADVANCED CALCULUS I & II

VERSION: September 19, 2024

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Contents

In Analysis
there are no theorems
only proofs

These lecture notes are based to a substantial extent on the book [Hajłasz, 2020] and the lecture notes [Hajłasz, 2008, Hajłasz, 2009] by Piotr Hajłasz. Several pictures are taken from these notes. They can be accessed online

- http://www.pitt.edu/∼hajlasz/Teaching/Math1530Fall2018/selection.pdf
- http://www.pitt.edu/∼hajlasz/Teaching/Math1530Fall2018/AdvancedCalculusI-Fall2008.pdf
- http://www.pitt.edu/ hajlasz/Teaching/Math1530Fall2018/AdvancedCalculusII-Spring2009.pdf

Other notes include Michael Struwe's lecture notes Analysis I-II (in German), which can be found

- https://people.math.ethz.ch/∼struwe/Skripten/Analysis-I-II-final-6-9-2012.pdf

In particular the section about submanifolds takes content, inspiration, and images from P. Hajłasz' lecture notes on Differential Geometry.

- http://www.pitt.edu/∼hajlasz/Notatki/Differential_Geometry_1.pdf

Pictures that were not taken from above mentioned sources or wikipedia are usually made with geogebra.

Requisite of this course is Math 420 and Math 413 (although we revisit some aspects but with a stronger focus on doing our own proofs).

Thanks to Piotr Hajłasz and Josh Xie for corrections.

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Part 1. Advanced Calculus I

1. Review: Elements of set theory and logic

Basic Logic. In order do rigorous proofs, we recall use the following notation: $\wedge$ (and), $\vee$ (or), $\neg$ (not). Here is the *truth table*

(1.1)

A	B	$A \wedge B$ and	$A \vee B$ or	$\neg A$ not
True	*True*	*True*	*True*	*False*
True	*False*	*False*	*True*	*False*
False	*True*	*False*	*True*	*True*
False	*False*	*False*	*False*	*True*

Recall the *De Morgan's laws*

Theorem 1.1. *Let A, B be mathematical statements (i.e. A, B are each either true or false). Then*

$$\neg(A \vee B) = (\neg A) \wedge \neg(B)$$

$$\neg(A \wedge B) = (\neg A) \vee \neg(B)$$

Exercise 1.2. *Prove Theorem 1.1 by using the truth-table (1.1).*

Let's practice a bit

Exercise 1.3. *Use the equivalence*

$$p \wedge (q \vee r) \quad \equiv \quad (p \wedge q) \vee (p \wedge r) \tag{1.2}$$

to prove

$$p \vee (q \wedge r) \quad \equiv \quad (p \vee q) \wedge (p \vee r).$$

To this end apply (1.2) to $\neg p$, $\neg q$, $\neg r$ in place of p, q, r, and negate the statement using De Morgan's Laws, Theorem 1.1.

A logical statement can imply another logical statement:

- $A \Rightarrow B$ means: If A is true, then B is also true. (if A is false then B can be true or false). We also say A *implies* B.
- We write $A \Leftarrow B$ if $B \Rightarrow A$
- We say A and B are *equivalent* in formulas $A \Leftrightarrow B$ if $A \Rightarrow B$ and $B \Rightarrow A$. We also say "A *if and only if* B".

Exercise 1.4. *Show that*

$$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$$

using the truth tables.

In other words: If we want to show that A implies B, we can just proof that the negation of B implies the negation of A.

Exercise 1.5. *Is the following a true statement (every number in $\mathbb{N}$)?*

$$(1 = 0) \Rightarrow (\textit{There are unicorns})$$

We will use the following notation frequently

- $\forall$: for every/all
- $\exists$: there exists
- $\equiv$: is exactly the same as (usually used for functions, but also similar to $\Leftrightarrow$)

For example the following is a logical statement (and it happens to be TRUE)

$$\forall x \in \mathbb{R} : \Big(x^2 = 1 \Leftrightarrow x \in \{-1, 1\} \Big)$$

In Analysis, we do not want to write every statement in the above form (its annoying and unreadable). However, it is *absolutely important* to be able to translate statements from symbols to words to precise mathematical formulation. For example for two sets A, B

$$\begin{aligned}
& A \subset B \\
\Leftrightarrow \quad & A \text{ is a subset of } B \\
\Leftrightarrow \quad & \text{every element of } A \text{ is also an element of } B \\
\Leftrightarrow \quad & \forall x \in A : x \in B \\
\Leftrightarrow \quad & \forall x : (x \in A \quad \Rightarrow x \in B).
\end{aligned}$$

So while we may often write or say $A \subset B$ or A is a subset of B, in our mind we have to be able to write the precise logical definition!

We write $A \supset B$ if $B \subset A$.

Exercise 1.6. *Negate the statement*[1]

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \in \mathbb{R} \;\; \forall y \in \mathbb{R} \;\; (|x - y| < \delta \;\; \Rightarrow \;\; |\sin x - \sin y| < \varepsilon).$$

Exercise 1.7. *Negate the statement:* For all real numbers x, y satisfying $x < y$, there is a rational number q such that $x < q < y$. *Formulate the negation as a sentence and not as a formula involving quantifiers.*

Exercise 1.8. *Use an argument by contradiction to prove that $\sqrt{3}$ is irrational.*

[1] This is a true statement known as uniform continuity of the function $\sin x$. However, you are not asked to prove the statement only to negate it.

Sets. By $\emptyset$ we denote the empty set consisting of no elements.

We write $x \in A$ if x is an element of the set A.

How to define sets. Sets can be defined by listing all of their elements, e.g.

$$A = \{1, 3, 5\} = \{5, 1, 3\}$$

or they can be defined by some property[2]

$$B = \{x \in A : \quad P(x)\} \quad \text{all elements in } x \in A \text{ that satisfy property } P.$$

Sets we use a lot.

$\mathbb{N} = \{1, 2, 3, \dots\}$ set of *natural numbers* or positive integers

$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\} = \{0\} \cup \mathbb{N}$ set of natural numbers including zero or nonnegative integers

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ set of all *integers*

$\mathbb{Q} = \left\{\frac{p}{q}, \quad p \in \mathbb{Z}, q \in \mathbb{N}\right\}$ set of all *rational numbers*

$\mathbb{R} = \left\{\lim_{k \to \infty} q_k, \quad (q_k)_{k \in \mathbb{N}} \subset \mathbb{Q} \text{ is Cauchy sequence}^{3}\right\}$ set of all *real numbers*

$\mathbb{R}^n = \{(v_1, v_2, \dots, v_n), \quad v_i \in \mathbb{R}\}$ set of n-vectors of real numbers

So for example we work with

$$\{x \in \mathbb{N} : \quad x \text{ is an even number}\} = \{2, 4, 6, \dots\}$$

or

$$\{x \in \mathbb{R} : x^2 \leq 1\} = [-1, 1]$$

or

$$\{x \in \mathbb{R}^n : |x|^2 = 1\} =: \mathbb{S}^{n-1} \text{ the } n-1\text{-dimensional unit sphere.}$$

Definition 1.9. Two sets A, B are equal if they have the same elements, namely

$$x \in A \Leftrightarrow x \in B.$$

Exercise 1.10. *$A = B$ iff $A \subset B$ and $B \subset A$.*

Let us recall further set operations

$$A \cup B = \{x : \quad x \in A \text{ or } x \in B\} \quad \textit{union} \text{ of } A \text{ and } B$$
$$A \cap B = \{x : \quad x \in A \text{ and } x \in B\} \quad \textit{intersection} \text{ of } A \text{ and } B$$

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[2] One has to be a bit careful here. The set

$$B = \{x : P(x)\}$$

may not be well-defined (see *Russel's Paradox*, [Hajłasz, 2020]) It is important that x is specified as an object some given set. I.e. it is ok to define

$$B = \{x \in A : P(x)\}.$$

Lemma 1.11. *If $a \in A$ then $\{a\} \subset A$, and $\{a\} \cap A = \{a\}$.*

Although this seems like absolutely obvious, lets prove this!

Proof. Let $a \in A$.

- We first show $\{a\} \subset A$.
  For this we need to show for any $x \in \{a\}$ we have $x \in A$.
  So let $x \in \{a\}$. Then $x = a$. So $x = a \in A$.
- We now show $\{a\} \cap A = \{a\}$. In view of Exercise 1.10 there are two things we need to show.
  - $\{a\} \cap A \subset \{a\}$. Let $x \in \{a\} \cap A$. This means that $x \in \{a\}$ and $x \in A$. The former implies that $x = a$ (which is compatible with the latter, $x = a \in A$). So $x = a \in \{a\}$ and the first direction is proven.
  - $\{a\} \cap A \supset \{a\}$. Let $x \in \{a\}$. This means $x = a$. By assumptions $x \in A$, and thus $x \in \{a\} \cap A$.

□

We want to use $\cap$ and $\cup$ on more than one element. Observe that for example

$$A \cap B \cup C$$

does not make any sense. It could mean $(A \cap B) \cup C$ or it could mean $A \cap (B \cup C)$[4]

Lemma 1.12. *Let A, B, C be three sets. Then*

(1) (symmetry) $(A \cup B) = (B \cup A)$, $(A \cap B) = (B \cap A)$
(2) (associative) $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$
(3) (distributive) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$[5]

These claims may seem easy, but again, you should be able to *prove* this!

Exercise 1.13. *Prove Lemma 1.12.*

In particular we are going to write $A \cap B \cap C$ and $A \cup B \cup C$ (without parenthesis!).

Let us recall a third set operation

$$A \setminus B := \{x : \quad x \in A, x \notin B\} \quad \textit{complement} \text{ of } B \text{ relative to } A$$

[4]stuff like this is often the basis for these silly "nobody can solve this math puzzle" that you find on your grandpa's facebook feed

[5]all of these rules look similar to the rules of addition $+$ and multiplication $\times$, if

$$\cup \leftrightarrow +$$

and

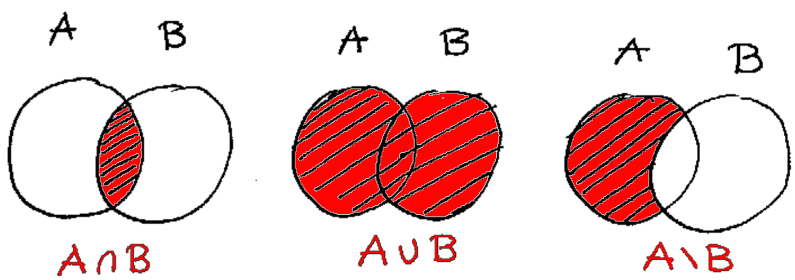
$$\cap \leftrightarrow \times$$

If $\cup$ corresponds to $+$ and $\cap$ corresponds to $\times$ then one might think that $\setminus$ corresponds to $-$. It is important to be a bit careful with this operation, as there are no "negative sets". For example

$$0 - b + b = 0$$

but

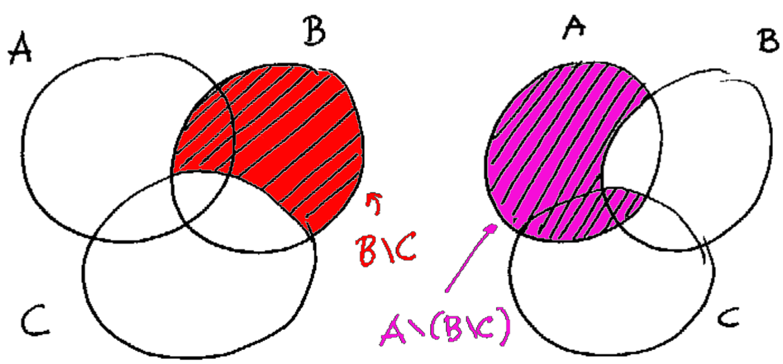
$$(\emptyset \setminus B) \cup B = B.$$



Venn Diagram for set operations.

Exercise 1.14. *Prove that* $A\backslash(B\backslash C) = (A\backslash B) \cup (A \cap C)$.

Picture proof of Exercise 1.14. **Careful, picture proofs can be deceiving. You are allowed to use a picture prove for visualization and intuition. But you are not allowed to actually prove something via a picture.**[6]



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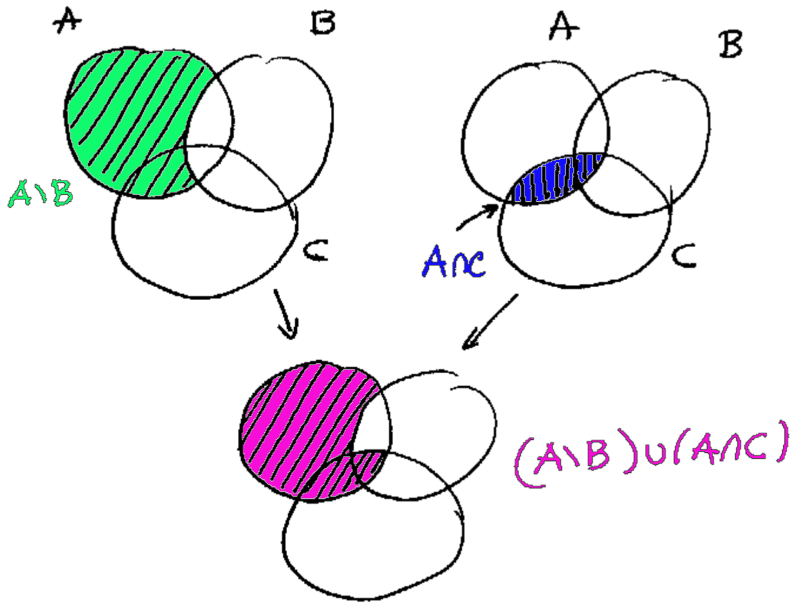
[6] only I can do that



□

Rigorous proof of Exercise 1.14. We show that $x \in A\backslash(B\backslash C)$ is equivalent to $x \in (A\backslash B) \cup (A \cap C)$.

$$\begin{aligned} & x \in A\backslash(B\backslash C) \\ \Leftrightarrow \quad & x \in A \ \wedge \ \neg(x \in B\backslash C) \\ \Leftrightarrow \quad & x \in A \ \wedge \ \neg(x \in B \ \wedge \ x \notin C) \\ \Leftrightarrow \quad & x \in A \ \wedge \ (x \notin B \ \vee \ x \in C) \\ \Leftrightarrow \quad & (x \in A \ \wedge \ x \notin B) \ \vee (x \in A \wedge \ x \in C) \\ \Leftrightarrow \quad & (x \in A\backslash B) \ \vee (x \in A \cap C) \\ \Leftrightarrow \quad & x \in (A\backslash B) \cup (A \cap C) \end{aligned}$$

□

Index sets. Let I be any set. Then we can use I as an *index set* and associate to each $i \in I$ a set A_i. The collection (*family*) of these sets is denoted by $(A_i)_{i \in I}$. We define

$$\bigcup_{i \in I} A_i = \{x : \quad \exists i \in I : \quad x \in A_i\}$$

$$\bigcap_{i \in I} A_i = \{x : \quad \forall i \in I : \quad x \in A_i\}$$

Observe that we have no structure whatsoever assumed on I. It could be the interval $[0, 1]$.

If $I = \{1,2\}$ then

$$\bigcup_{i\in I} A_i = A_1 \cup A_2$$
$$\bigcap_{i\in I} A_i = A_1 \cap A_2.$$

Often we will a *countable family*, namely $(A_i)_{i\in\mathbb{N}}$. In this case we may think of[7]

$$\bigcup_{i\in \mathbb{N}} A_i = A_1 \cup A_2 \cup A_3 \cup \dots$$
$$\bigcap_{i\in \mathbb{N}} A_i = A_1 \cap A_2 \cap A_3 \cup \dots$$

Exercise 1.15. *Prove that for any set A and any family of sets $\{A_i\}_{i\in I}$*

$$A \setminus \bigcup_{i\in I} A_i = \bigcap_{i\in I} (A \setminus A_i)\,,$$
$$A \setminus \bigcap_{i\in I} A_i = \bigcup_{i\in I} (A \setminus A_i)\,.$$

Exercise 1.16. *Prove that if $f : X \to Y$ is a function and $A_1, A_2, A_3, \dots$ are subsets of X, then*

$$f\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f(A_i)\,,$$

and

$$f\left(\bigcap_{i=1}^{\infty} A_i\right) \subset \bigcap_{i=1}^{\infty} f(A_i). \tag{1.3}$$

Provide an example to show that we do not necessarily have equality in (1.3)

1.1. Ordered Pairs.

Sets do not specify any sort of order of elements, e.g.,

$$\{1,2\} = \{2,1\}.$$

If we want to insist on the order of elements we use the notion of an *ordered pair* (a,b). By definition

$$(a,b) = (c,d) \quad :\Leftrightarrow a = c \text{ and } b = d.$$

in particular $(1,2) \neq (2,1)$.

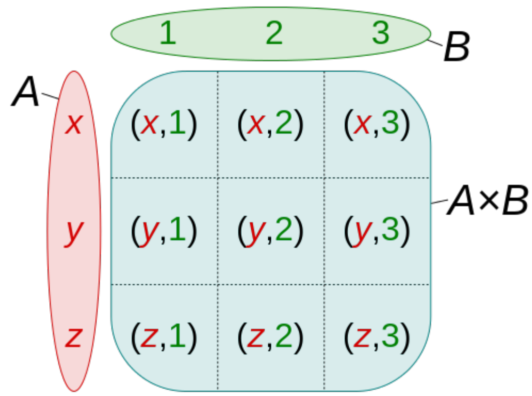
If A, B are two sets, then the *Cartesian product* $A \times B$ is defined as

$$A \times B = \{(a,b), \quad a \in A, b \in B\}\,.$$

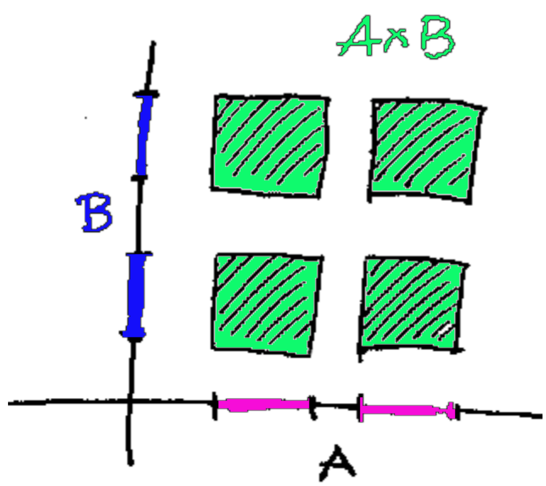
For example

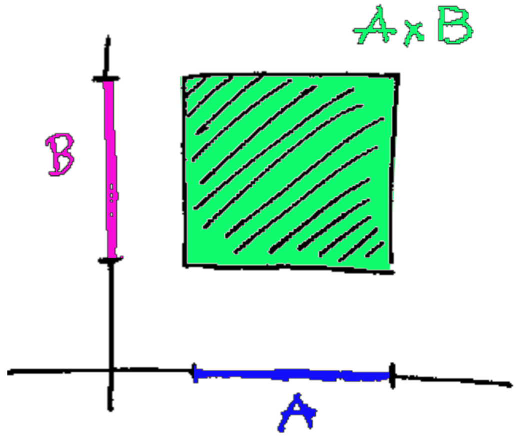
$$\{1,2\} \times \{1,2,3\} = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$$

[7] but what does $\dots$ really mean here? not so clear. Better use the above definition!



Cartesian product of $A = \{x, y, z\}$ and $B = \{1, 2, 3\}$.

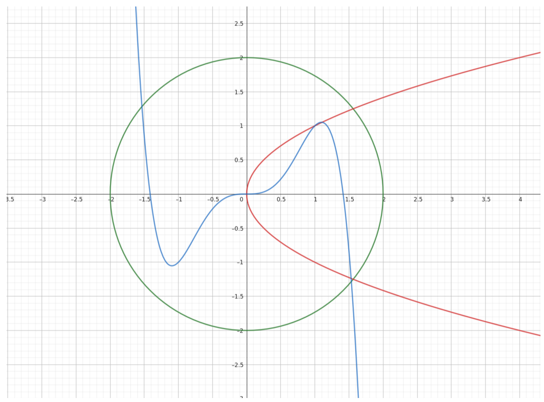




1.2. **Functions.** Let X and Y be two sets. A function $f : X \to Y$ is a rule that assigns to each element $x \in X$ an element $f(x) \in Y$. We call

- X the *domain* of f
- Y the *target* of f
- $f(X) := \{y \in Y : \exists x \in X : f(x) = y\}$ the *range* or the *image* or the *codomain* of f
- The *graph* of f is a subset of $X \times Y$,

$$\text{graph}(f) := \{(x, f(x)) \in X \times Y : \; x \in X\}.$$



the blue curve in $\mathbb{R}^2$ is a graph $(x, f(x))$ of a function. The green circle and the red curve are not graphs of functions over the x-axis. Of course, the red curve is a graph of a function over the y-axis, $(y, g(y))$.

We often identify the function $f : X \to Y$ with its graph, $\text{graph}(f) \subset X \times Y$, in the following way.

Let R be a set $R \subset X \times Y$ with the property such that for every $x \in X$ there exists exactly one $y \in Y$ such that $(x, y) \in R$. If we define $f(x) := y$ then we have recovered our function.

A function f is called *one-to-one* or *injective* if

$$\forall x_1, x_2 \in X : \quad (x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2))$$

(in words: two different points get mapped into two different points), or equivalently (**exercise!**)

$$\forall x_1, x_2 \in X : \quad (f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2)\,.$$

(in words: two points get mapped to the same value if and only if they are the same points or in words: two different point

If a function $f : X \to Y$ is injective, then there exists an *inverse function*,

$$f^{-1} : f(X) \to X$$

such that $f^{-1}(f(x)) = x$ for all $x \in X$ (and $f(f^{-1}(y)) = y$ for all $y \in f(X)$).

A function is called *onto* or *surjective* if $f(X) = Y$, i.e.

$$\forall y \in Y \ \exists x \in X \ f(x) = y.$$

A map which is injective and surjective is called *bijective*

If $f : X \to Y$ and $A \subset Y$ then we define the *preimage* of A under f, $f^{-1}(A) \subset X$ as

$$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

The function $f : X \to X$ defined by $f(x) = x$ is called the *identity*. We sometimes denote it by id_X or I_X.

If the image of f consist of one point, i.e. $f(X) = \{a\}$ then f is called a *constant function*. If this happens, we like to write

$$f \equiv a \quad \text{in } X$$

If $f : X \to Y$ and $g : Y \to Z$ are two functions, then the *composition* of g and f, $g \circ f$, is the function

$$g \circ f : X \to Z$$

defined by the formula

$$(g \circ f)(x) = g(f(x)).$$

If $f : X \to Y$ and $A \subset X$ then the *restriction* of f to A defined as

$$f\Big|_A : A \to Y, \quad f\Big|_A(x) := f(x) \quad x \in A.$$

If $f : X \to Y$ and $g : Z \to Y$ and we have $X \subset Z$ and $f(x) = g(x)$ for all $x \in X$ then g is called an *extension* of f to Z.

Exercise 1.17. *Prove that if $f : X \to Y$ is one-to-one and $A_1, A_2, A_3, \ldots$ are subsets of X, then*

$$f\left(\bigcap_{i=1}^{\infty} A_i\right) = \bigcap_{i=1}^{\infty} f(A_i).$$

(compare this to Exercise 1.16)

2. Cardinality

Cardinality is the question of sizes of sets. It is easy to say that the two sets

$$\{1, 2, 3\} \quad \text{and} \quad \{🐱, 🌎, 🐵\}$$

have the same *cardinality* (namely: 3).

But what about $\mathbb{N}$ and $\mathbb{Z}$ etc.?

Since $\mathbb{N} \subsetneq \mathbb{N}_0$ we could think that the cardinality of $\mathbb{N}$ is strictly less that the cardinality of $\mathbb{N}_0$. But on the other hand we have for each element in $\mathbb{N}_0$ exactly one element in $\mathbb{N}$, namely the function $x \mapsto x+1$ is a bijection from $\mathbb{N}_0$ to $\mathbb{N}$, so they should have the "same amount" of elements.

One could simply say: well cardinality of $\mathbb{N}$ is infinity, so is $\mathbb{N}_0$ so is $\mathbb{Z}$. But $\mathbb{R}$ is *much larger* than $\mathbb{N}$, there is no bijective map $f : \mathbb{N} \to \mathbb{R}$.

So we go with the following definition

Definition 2.1. Two sets X and Y have the same *cardinality* if there exists a bijective map $f : X \to Y$.

Clearly two finite sets have the same cardinality if and only if they have the same amount of elements.

Proposition 2.2. *The sets* $\mathbb{N}$ *and*

$$2\mathbb{N} := \{2n, \quad n \in \mathbb{N}\}$$

have the same cardinality.

Proof. Indeed, let $\varphi(n) := 2n$ then $\varphi : \mathbb{N} \to 2\mathbb{N}$ is clearly injective and surjective. □

Proposition 2.3. *The sets* $\mathbb{N}$ *and* $\mathbb{Z}$ *have the same cardinality.*

Proof. We could construct a bijective map by hand in this case, but lets make it more intuitive.

It suffices to show that we can arrange all elements of $\mathbb{Z}$ as a sequence

$$0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$$

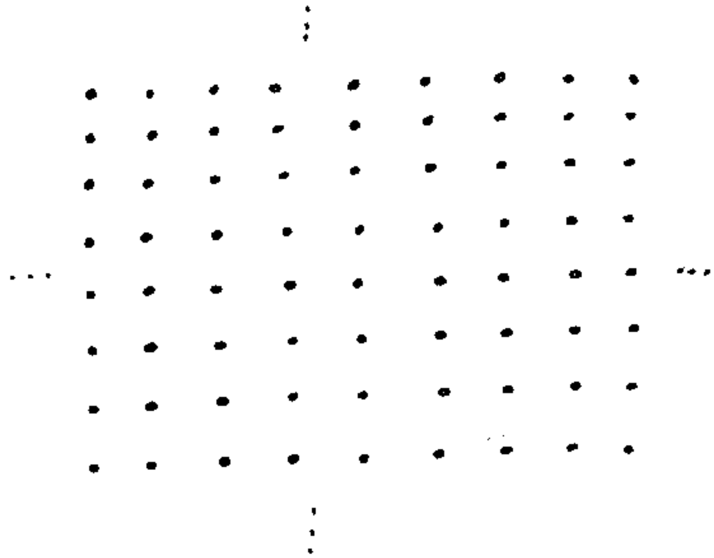
Then the bijection can be defined via

$$\begin{array}{cccccccccccc} \mathbb{N} \ni & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \mathbb{Z} \ni & 0 & 1 & -1 & 2 & -2 & 3 & -3 & 4 & -4 & \ldots \end{array}$$

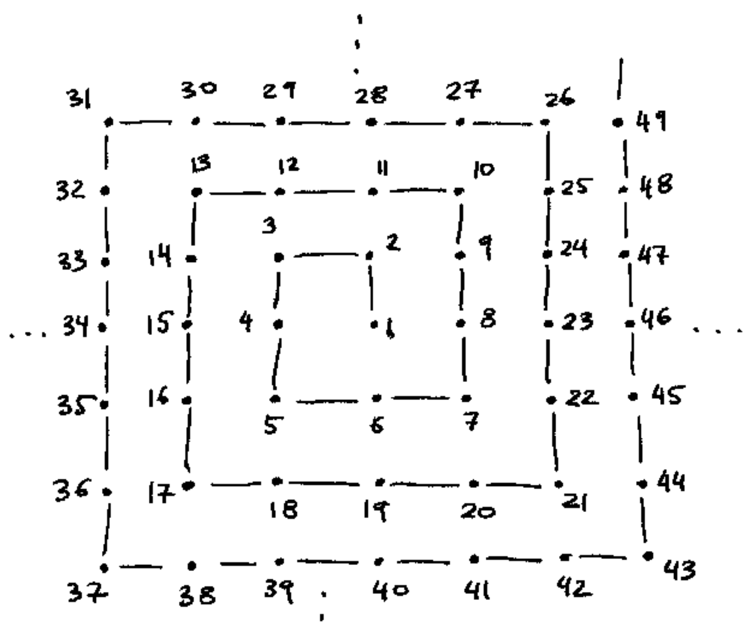
□

Proposition 2.4. *Let* $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$. *Then* $\mathbb{Z}^2$ *has the same cardinality as* $\mathbb{N}$.

Proof. We can interpret $\mathbb{Z}^2$ as the set of all points in $\mathbb{R}^2$ with both coordinates being integers,



Again, we can arrange this as a sequence as follows



□

Theorem 2.5. *The set of rational numbers* $\mathbb{Q}$ *has the same cardinality as* $\mathbb{N}$*.*

Proof. Each rational number $q \in \mathbb{Q}$ can be uniquely represented by a quotient $\frac{n}{m}$ where $n \in \mathbb{Z}$ and $m \in \mathbb{N}$ and the greatest common divisor of $|n|$ and m is 1. The map $\varphi : \mathbb{Q} \to \mathbb{Z}^2$, $\varphi(q) := (n, m)$ is injective.

Now a modification of the spiral argument of Proposition 2.4 constructs the bijection: we simply skip the points of $\mathbb{Z}^2$ that are not contained in $\varphi(\mathbb{Q})$. $\square$

So we see that $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}^2$ have all the same cardinality. Lets give this a name.

Definition 2.6. A set which is finite or has the same cardinality as $\mathbb{N}$ is called *countable*. A set which is not countable is called *uncountable*.

Not all sets are countable, indeed our base set in Analysis, $\mathbb{R}$ is uncountable (as we shall see later, Theorem 4.29).

Exercise 2.7. *Let it be given that any real number in $x \in [0, 1)$ can be written as their decimal expansion*[8]

$$x = 0.x_1x_2x_3x_4x_5x_6\dots$$

(1) Show that

$$[0, 1) \text{ has the same cardinality as } [0, 1) \times [0, 1)$$

by considering the map

$$f(x, y) := 0.x_1y_1x_2y_2x_3y_2x_4y_4\dots$$

(2) Show that $\mathbb{R}$ and $\mathbb{R}^2$ have the same cardinality.

3. Induction

A very important method in proving things is the method of *induction*. It is based on the following theorem (which we accept without proof, cf. Math 413).

Theorem 3.1 (Principle of Mathematical Induction)**.** *Let S be a subset of $\mathbb{N}$ that has two properties*

1. $1 \in S$,
2. *For every natural number n, if $n \in S$, then $n + 1 \in S$.*

Then $S = \mathbb{N}$.

A consequence (or version) of Theorem 3.1 is the following.

Theorem 3.2 (Principle of Mathematical Induction)**.** *Let for each $n \in \mathbb{N}$, $P(n)$ be a statement about a natural number n. Suppose also that*

[8] where each $x_i \in \{0, 1 \dots, 9\}$ and there is no N such that $x_i = 9$ for all $i \geq N$.

1. *$P(1)$ is true,*
2. *For every $n \in \mathbb{N}$, if $P(n)$ is true, then $P(n+1)$ is true.*

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Let's put induction into practice.

Proposition 3.3 (Bernoulli's inequality)**.** *For all $n \in \mathbb{N}$ and $a \geq -1$ a real number we have*

$$(1+a)^n \geq 1+na.$$

Proof. So assume $a \geq -1$. Denote by $P(n)$ the statement

$$P(n) := ((1+a)^n \geq 1+na \text{ is true})$$

The first thing to check is

$P(1)$ is true: Indeed, $P(1)$ corresponds to $1+a \geq 1+a$ which is clearly true.

For any $n \in \mathbb{N}$: if $P(n)$ is true, then $P(n+1)$ is true. So fix $n \in \mathbb{N}$ and assume $P(n)$ is true (this is called the *induction hypothesis*). We need to show that then $P(n+1)$ is also true.

$P(n)$ being true means that for our fixed n,

$$(1+a)^n \geq 1+na \tag{P(n)}$$

Now we need to show $P(n+1)$, i.e. we need to show

$$(1+a)^{n+1} \geq 1+(n+1)a \tag{P(n+1)}$$

so we start.

$$\begin{aligned}(1+a)^{n+1} =& \underbrace{(1+a)}_{\geq 0}(1+a)^n \\ \overset{P(n)}{\geq}& (1+a)(1+na) \\ =& 1+na+a+\underbrace{na^2}_{\geq 0} \\ \geq& 1+(n+1)a.\end{aligned}$$

That is (under the assumption that P(n) holds) we have shown P(n+1).

By the method of induction $P(n)$ is thus true for any $n \in \mathbb{N}$.

□

Exercise 3.4. *Prove that $5^{2n}-1$ is divisible by 8 for all $n \in \mathbb{N}$*

We got to be careful with induction though, a common mistake is the following.

Exercise 3.5. *Where is the mistake in the following proof?*

Let us agree that there are finitely many (or countably, if you want) people in the world. Let us also agree that at least on person lives in Pittsburgh. We now prove the following statement.

Claim: *All people in the world live in Pittsburgh.*

We will show this by proving the following statement:

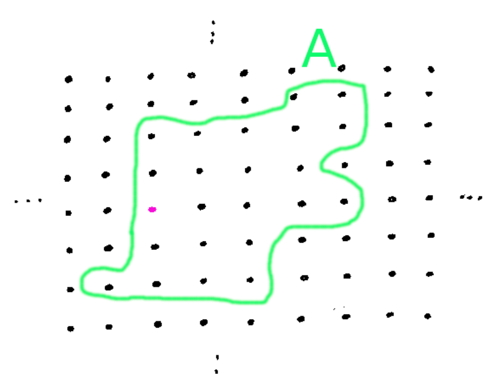
Let $\mathcal{P}$ be the set of people in the world. We prove the following statement for all $n \in \mathbb{N}$ which readily implies the claim.

$P(n)$: *for any set $A \subset \mathcal{P}$ with at most n people: if there exists one person $p \in A$ such that p lives in Pittsburgh, then all people $q \in A$ live in Pittsburgh*

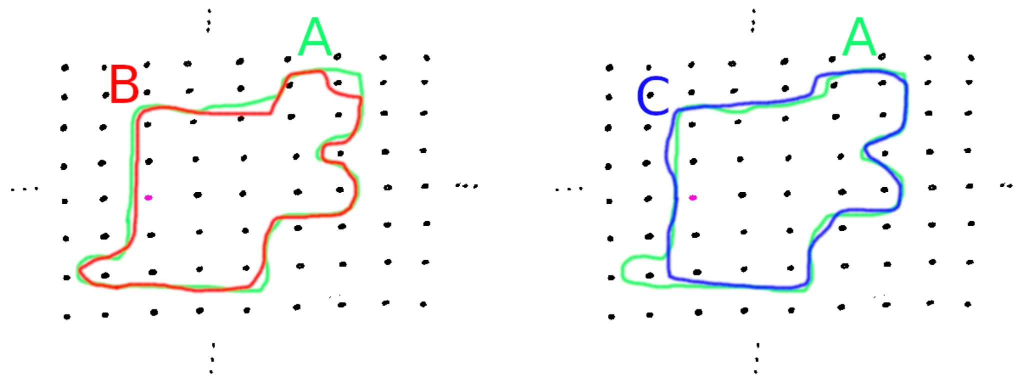
P(1) is true Indeed, if we have a set A consisting of one person, if that person lives in Pittsburgh, all the persons in the A (exactly that one person) does live in Pittsburgh.

If P(n) is true, then P(n+1) is true

So assume that $P(n)$ is true, and let us try to show $P(n+1)$. So let $A \subset \mathcal{P}$ be a set of at most $n+1$ people, and assume we know that one person in A lives actually in Pittsburgh. For illustration purposes assume the set A looks as follows (with the pink dot being the person living in Pittsburgh).



We now construct two sets B and C as follows



Since A consists of at most $n+1$ people, B and C (which have one person removed) consist of at most n people. Moreover the pink dot (the person in Pittsburgh) still belongs to B and C. That is, B and C satisfy the assumptions of $P(n)$, and thus all people in B and all people in C live in Pittsburgh. Since $A = B \cup C$, all people in A live in Pittsburgh.

By induction we include that all people in the world live in Pittsburgh. □

There are many modifications of the method of induction. There are obvious ones (we can start from $n = 3$ or $n = 0$ rather than $n = 1$). One useful modification is the following (proof: exercise. Hint: induction)

Theorem 3.6 (Principle of Mathematical Induction)**.** *Let for each $n \in \mathbb{N}$, $P(n)$ be a statement about a natural number n. Suppose also that*

1. *$P(1)$ is true,*
2. *For every $n \in \mathbb{N}$: if $P(k)$ is true for all $k = 1, \ldots, n$ then also $P(n+1)$ is true.*

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Induction proofs can become quite involved, one classical example is

Theorem 3.7 (Arithmetic-Geometric Mean Inequality)**.** *If $a_1, a_2, \ldots, a_n \geq 0$ then*

$$\underbrace{\sqrt[n]{a_1 \cdot \ldots \cdot a_n}}_{\text{geometric mean}} \leq \underbrace{\frac{a_1 + \ldots + a_n}{n}}_{\text{arithmetic mean}}$$

and the equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

Proof. We will prove the inequality only, but the reader may conclude from the proof that the equality holds if and only if $a_1 = a_2 = \ldots = a_n$. We leave this last conclusion as an exercise.

First we will prove the inequality for $n = 2^k$, $k = 1, 2, 3, \ldots$, i.e., we will prove it for $n = 2, 4, 8, 16, \ldots$

1. For $n = 2^1$ we have

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \quad \Leftrightarrow \quad a_1 - 2\sqrt{a_1 a_2} + a_2 \quad \Leftrightarrow \quad (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0\,.$$

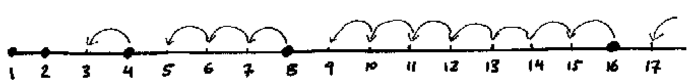
Since the last condition is obviously true, the inequality, as equivalent to the last statement, is also true.

2. Suppose that the inequality is true for $n = 2^k$. We need to prove it for $n = 2^{k+1}$. We have

$$\begin{aligned}
&\sqrt[2^{k+1}]{a_1 \cdot \ldots \cdot a_{2^k} \cdot a_{2^k+1} \cdot \ldots \cdot a_{2^{k+1}}} \\
&= \quad \sqrt{\sqrt[2^k]{a_1 \cdot \ldots \cdot a_{2^k}}\,\sqrt[2^k]{a_{2^k+1} \cdot \ldots \cdot a_{2^{k+1}}}} \\
&\leq \quad \frac{\sqrt[2^k]{a_1 \cdot \ldots \cdot a_{2^k}} + \sqrt[2^k]{a_{2^k+1} \cdot \ldots \cdot a_{2^{k+1}}}}{2} \\
&\leq \quad \frac{\frac{a_1 + \ldots + a_{2^k}}{2^k} + \frac{a_{2^k+1} + \ldots a_{2^{k+1}}}{2^k}}{2} \\
&= \quad \frac{a_1 + a_2 + \ldots + a_{2^{k+1}}}{2^{k+1}}\,.
\end{aligned}$$

The above estimates require some explanations. The first equality is obvious. The second inequality is just a consequence of the arithmetic-geometric inequality for $n = 1$ which was proved in **1.** The third inequality follows from the inductive assumption that the inequality is true for $n = 2^k$ and the last equality is obvious again.

We proved the inequality for $n = 2, 4, 8, 16, \ldots$. In order to prove that the inequality is true for all integers it suffices to prove that if it is true for n, then it is also true for $n - 1$ (reverse induction).



Thus suppose that the inequality is true for n. We will prove it is true for $n - 1$. We have

$$\begin{aligned}
&\sqrt[n]{a_1 \cdot \ldots \cdot a_{n-1} \cdot \left(\frac{a_1 + \ldots + a_{n-1}}{n-1}\right)} \\
&\leq \quad \frac{a_1 + \ldots a_{n-1} + \left(\frac{a_1 + \ldots + a_{n-1}}{n-1}\right)}{n} \\
&= \quad \frac{a_1 + \ldots + a_{n-1}}{n-1}\,.
\end{aligned}$$

The first inequality above follows from the assumption that the arithmetic-geometric inequality is true for n. Hence

$$\sqrt[n]{a_1 \cdot \ldots \cdot a_{n-1}} \; \sqrt[n]{\frac{a_1 + \ldots + a_{n-1}}{n-1}} \leq \frac{a_1 + \ldots + a_{n-1}}{n-1},$$

so

$$\Big(a_1 \cdot \ldots \cdot a_{n-1}\Big)^{1/n} \leq \Big(\frac{a_1 + \ldots a_{n-1}}{n-1}\Big)^{1-1/n}$$

and finally

$$\Big(a_1 \cdot \ldots \cdot a_{n-1}\Big)^{\frac{1}{n-1}} \leq \frac{a_1 + \ldots + a_{n-1}}{n-1}$$

which is what we wanted to prove. □

3.1. **Exercises on Induction.**

Exercise 3.8. *Prove that for $n \geq 1$*

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{3n} + \frac{1}{3n+1} > 1$$

Exercise 3.9. *Show that $4^n + 15n - 1$ is divisible by 9 for all $n \geq 1$*

Exercise 3.10. *Show that $4^n + 15n - 1$ is divisible by 9 for all $n \geq 1$*

Exercise 3.11. *Prove for all $n \geq 0$ that*

$$1 \cdot (1!) + 2 \cdot 2! + \ldots + n \cdot (n!) = (n+1)!$$

Exercise 3.12. *Prove that $n^2 - 1$ is divisible by 8 for all odd positive integers n.*

Exercise 3.13. *Show that $n! > 2^n$ for all $n \geq 4$*

Exercise 3.14. *Let a_i be a recursive sequence given as*

$$a_1 = 2$$

$$a_{n+1} = \frac{1}{2}(2 + \frac{1}{a_n})$$

Then $a_n \in [\frac{1}{2}, 2]$ for all $n \in \mathbb{N}$.

Exercise 3.15. *Let $f(x) := x^2 e^x$ show that $f^{(n)}(x) = (x^2 + 2nx + n(n-1))e^x$.*

Exercise 3.16. *If A is a set, then the power set 2^A is the collection of all subsets, i.e.*

$$2^A = \{B \subset A\}.$$

Show that if A contains n elements, then 2^A contains 2^n elements.

Exercise 3.17. *Let*

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Show that the n-th power of the matrix, i.e.

$$A^n = \underline{A \cdot A \cdot A \cdot \ldots A}_{ntimes}$$

is

$$A^n = \begin{pmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise 3.18. *Let $x_1, x_2 \ldots, x_n \geq 0$ such that*

$$x_1 \cdot x_2 \cdot \ldots \cdot x_{n-1} \cdot x_n = 1.$$

Without using the Arithmetic-Geometric Mean Inequality, Theorem 3.7, show that then

$$x_1 + x_2 + \ldots + x_{n-1} + x_n \geq n$$

Exercise 3.19. *Prove that $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$.*

Exercise 3.20. *Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be positive numbers. Prove that*

$$\prod_{i=1}^{n} (a_i + b_i)^{1/n} \geq \prod_{i=1}^{n} a_i^{1/n} + \prod_{i=1}^{n} b_i^{1/n}.$$

Hint: *Use the arithmetic-geometric mean inequality, Theorem 3.7.*

4. Real numbers, metric spaces, converging sequences

4.1. **Fields, order, murder.** The most natural set of numbers are the *natural numbers*

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \quad \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}.$$

The numbers in $\mathbb{N}$ can be added and multiplied. If we extend the natural numbers $\mathbb{N}$ to *integers* $\mathbb{Z}$,

$$\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$$

we can also subtract.

Within the set of *rational numbers* $\mathbb{Q}$,

$$\mathbb{Q} = \left\{ \frac{p}{q} : \quad p, q \in \mathbb{Z} : \quad q > 0 \right\}$$

we can also divide (by any number but 0).

$\mathbb{Q}$ is what we call a *field*.

Definition 4.1 (Field)**.** A *field* is a set $\mathbb{F}$ with operations of addition '+' and multiplication '·' that satisfies the following 10 axioms:

(A1) $x + y = y + x$.
(A2) $x + (y + z) = (x + y) + z$.
(A3) There is an element denoted by 0 such that for every x, $x + 0 = x$.
(A4) For every x there is an element denoted by $-x$ such that $x + (-x) = 0$.
(A5) $x \cdot y = y \cdot x$
(A6) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
(A7) There is an element denoted by 1 such that $x \cdot 1 = x$.
(A8) For every $x \neq 0$ there is an element denoted by x^{-1} such that $x \cdot (x^{-1}) = 1$.
(A9) $x \cdot (y + z) = x \cdot y + x \cdot z$.
(A10) $1 \neq 0$.

Here we understand that conditions are satisfied for all $x, y, z \in \mathbb{F}$. We did not include quantifiers in the conditions to make them more transparent.

Remark 4.2. In more linear algebraic terms, a field is a *abelian group* with respect to the $+$ operation (A1)-(A4), an abelian group with respect to the $\cdot$-operation, (A5)-(A8). $+$ and $\cdot$-operation are compatible as in (A9). And (A10) is a nontriviality assumption.

Example 4.3.
- $\mathbb{N}$, $\mathbb{Z}$ are not fields, since they do not have inverse with respect to multiplication (e.g. $2^{-1} \notin \mathbb{N}$).
- The smallest field is $\mathbb{F} = \{0, 1\}$ with the multiplication/addition table

$$\begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \quad \text{and} \quad \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$

- $\mathbb{Z}_p = \{0, 1, 2, \dots, p - 1\}$ is a field with the operations

$$x \cdot y := x \cdot y \mod p$$

$$x + y := (x + y) \mod p$$

if and only if p is a prime number (otherwise we have no multiplicative inverse)

More is true of $\mathbb{Q}$, its *totally ordered* (i.e. if $x, y \in \mathbb{Q}$ we know that either $x \leq y$ or $y \leq x$).

Definition 4.4. A field $\mathbb{F}$ (as in Definition 4.1) equipped with a relation $\leq$ is called an *ordered field* if

(A11) $x \leq x$.
(A12) $x \leq y$ and $y \leq x \Rightarrow x = y$.
(A13) $x \leq y$ and $y \leq z \Rightarrow x \leq z$.
(A14) $x \leq y \Rightarrow x + z \leq y + z$
(A15) $0 \leq x$ and $0 \leq y \Rightarrow 0 \leq xy$.

We also write $x \geq y$ if $y \leq x$.

The field is called *totally ordered* if

(A16) For every x, y either $x \leq y$ or $y \leq x$.

Example 4.5. The notion of totally ordered is a nontrivial assumption.

- For example the subset-relation is a partial order, but not a total order, since there are sets A, B with $A \not\subseteq B$ and $B \not\subseteq A$.
- Let p be a prime number, then $\mathbb{Z}_p$ is a field. But it cannot be ordered.

At first (and thats what the ancient Greek's thought until Hippasus[9]) it may seem that $\mathbb{Q}$ contains all numbers, and any number may be represented by an element in $\mathbb{Q}$. However Hippasus proved[10] that there is no number $q \in \mathbb{Q}$ such that $q \cdot q = 2$. Observe this does not contradict any of our field axioms. However this means that there is no solution (in $\mathbb{Q}$) to the question of the length of the diagonal of a square of sidelength 1. Now it feels like $\mathbb{Q}$ is *incomplete*, and this is why we need the real numbers!

What does complete mean? It means that what should converge (*Cauchy sequences*) does converge.

In $\mathbb{Q}$ it means: We can approximate the lenght of the diagonal of a square of sidelengths 1 by rational numbers. We want that approximate number "converge" to a *real number* (pun intended).

4.2. **The metric: absolute value; also: convergence.** For $x \in \mathbb{R}$ we define the *absolute value* $|x|$ as[11]

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x \leq 0. \end{cases}$$

The absolute value is incredibly important for the Analysis in $\mathbb{R}$ (later $\mathbb{R}^n$), because it gives $\mathbb{R}$ a *metric*: we can use it to measure the (a reasonable) *distance* between to points $x, y \in \mathbb{R}$. Indeed, $d(x, y) := |x - y|$

Definition 4.6 (metric)**.** Let X be any set. A map $d : X \times X \to \mathbb{R}$[12] is called a metric for X if

- $d(x, y) = d(y, x)$ for all $x, y \in X$ (*symmetry*)
- $d(x, y) \geq 0$ for all $x, y \in X$ (*positivity*)

[9]historians seem to believe that Egyptians and Babylonians had figured this out before

[10]and according to legend was murdered by Pythagoras for such heresy

[11]observe, this definition makes sense for any totally ordered field, in particular $\mathbb{Q}$

[12]ok, here is some technicality that we already should know what $\mathbb{R}$ is to define what a metric is, I will brush over this because actually we have learned what $\mathbb{R}$ is in Math413. To be precise one should first define the *metric* on $\mathbb{Q}$ to map into $\mathbb{Q}$, then by metric completion this extends to a $\mathbb{R}$-metric etc.

- $d(x,y) = 0$ if and only if $x = y$ (*non-degeneracy*)
- $d(x,y) \le d(x,z) + d(y,z)$ for all $x,y,z \in X$ (*triangular inequality*).

A set X with a metric d is called a *metric space*.

Almost everything we do with respect to convergence, continuity has a metric space generalization. The proofs are the same, the theorem changes. Differentiability, however, becomes more tricky, then more structure on d and X is helpful (e.g. a "norm" structure).

Example 4.7.
- $d(x,y) = 2|x-y|$ is still a metric, nothing changes.
- $d(x,y) = \sqrt{|x-y|}$ is still a metric
- $d(x,y) = |x-y|^2$ is no metric (triangular inequality is false)
- $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$ is a metric in the above sense

Exercise 4.8. *Prove that* $(\mathbb{R}^n, \varrho)$*, where*

$$\varrho(x,y) = \frac{|x-y|}{1+|x-y|}$$

is a metric space.

Exercise 4.9. *Let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence*[13] *in* $\mathbb{R}$.

Let $p \in (0,\infty)$*. We say that a sequence belongs to* ℓ^p, $(a_n)_{n\in\mathbb{N}} \in \ell^p$, *if*[14]

$$\|(a_n)_{n\in\mathbb{N}}\|_{\ell^p} := \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}} < \infty$$

For $p = \infty$ *we say that a sequence belongs to* ℓ^∞, $(a_n)_{n\in\mathbb{N}} \in \ell^\infty$, *if*

$$\|(a_n)_{n\in\mathbb{N}}\|_{\ell^\infty} := \sup_{n\in\mathbb{N}} |a_n| < \infty.$$

That is we set quite tautologically

$$\ell^p := \{(a_n)_{n\in\mathbb{N}} \in \ell^p\}.$$

Define for two sequences $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *their* ℓ^p*-distance*

$$d_p\left((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}\right) := \|(a_n - b_n)_{n\in\mathbb{N}}\|_{\ell^p}.$$

Show that

(1) (ℓ^1, d_1) *is a metric space*
(2) (ℓ^∞, d_∞) *is a metric space*

(Actually, (ℓ^p, d_p) *is a metric space for any* $p \in [1,\infty]$ *– but for triangular inequality we need Minkowski inequality that we shall not prove here, see Exercise 15.19).*

[13] If you forgot what a sequence is, see Section 4.3

[14] If you forgot what convergence of a series means, Definition 4.62

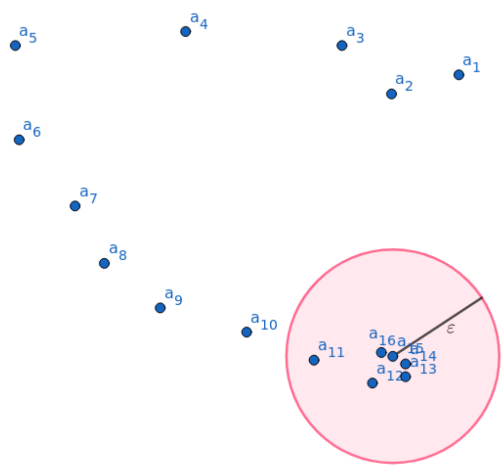


FIGURE 4.1. A convergent sequence

4.3. **Sequences.** A *sequence*, usually denoted by $(a_n)_{n=1}^{\infty} \subset X$, is a map $a : \mathbb{N} \mapsto X$. But instead of writing $a(n)$ we prefer to write a_n. Every sequence induces a set $a(\mathbb{N}) := \{a_n, n \in \mathbb{N}\}$ (but not the other way around, since we do not know which element of the set to take first). Thus we can use set operations on sequences: e.g., if $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ then

$$\sup(a_n)_{n\in\mathbb{N}} = \sup\{a_1, a_2, \ldots\}.$$

Definition 4.10 (Convergence of sequences)**.** Let (X, d) be a metric space. We say that a sequence $(a_n)_{n\in\mathbb{N}}$ of elements in X are convergent if there exists $b \in X$ such that (cf. Figure 4.1)

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} : \quad \forall n \geq n_0 \quad d(a_n, b) < \varepsilon.$$

We write

$$\lim_{n\to\infty} a_n = b.$$

Observe that *two different metrics induce two different notion of convergence* – and those can be really different. So whenever we write $\lim_{n\to\infty} a_n = b$ we assume that we already agree on the underlying metric! Indeed *Different metrics means different spaces*!

Exercise 4.11. *Consider* $\mathbb{R}$ *equipped with two metrics:*

$$X = (\mathbb{R}, d_e), \quad Y = (\mathbb{R}, d_0)$$

where d_e *is the usual* Euclidean metric

$$d_e(x, y) := |x - y|$$

and d_0 *is the following*

$$d_0(x, y) := \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

(1) Show that d_0 *is indeed a metric.*
(2) Give an example of a sequence $(x_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *such that*

$$\lim_{n\to\infty} x_n = 0 \quad \textit{w.r.t. } d_e$$

but

$$\lim_{n\to\infty} x_n \neq 0 \quad \textit{w.r.t. } d_0$$

Exercise 4.12. *Let* (X, d_X) *be a metric space and* $f : Y \to X$ *be an injection, i.e. an injective map. Show that*

$$(Y, d_Y)$$

is a metric space, where

$$d_y(a, b) := d_X(f(a), f(b)).$$

Exercise 4.13. *Consider* $X = (\mathbb{R}, d_e)$ *and* $Y := (\mathbb{R}, d_2)$ *be two metric spaces equipped with*

$$d_e(x, y) := |x - y|$$

and

$$d_2(x, y) := |f(a) - f(b)|,$$

respectively, where we set

$$f(x) := \begin{cases} 1 & x = 0 \\ 0 & x = 1 \\ x & \textit{otherwise.} \end{cases}$$

Set $a_n := \frac{1}{n}$. *Show that*

$$\lim_{n\to\infty} a_n = 0 \quad \textit{in } X$$

but

$$\lim_{n\to\infty} a_n = 1 \quad \textit{in } Y$$

Definition 4.14 (Bounded sequences)**.** Let (X, d) be a metric space. A sequence $(a_n)_{n\in\mathbb{N}} \subset X$ is *bounded*, if there exists $x_0 \in X$ and $M \in \mathbb{R}$ such that

$$d(x_0, x_n) < M \quad \forall n \in \mathbb{N}.$$

Lemma 4.15. *Convergent sequences are bounded.*

Proof. Assume that $(a_n)_{n\in\mathbb{N}}$ is convergent to some $b \in X$. By definition this means (take $\varepsilon := 1$) that there exists $n_1 \in \mathbb{N}$ such that

$$d(a_n, b) < 1 \quad \forall n \geq n_1.$$

So let

$$M := \max\{d(a_1, b), \ldots, d(a_{n_1}, b), 1\} + 1.$$

This max exists because these are finitely many values!

Then

$$d(a_n, b) < M \quad \forall n \in \mathbb{N}.$$

□

Bounded sequences may not be convergent (e.g. $a_n = (-1)^n$ in $\mathbb{R}$ is bounded, but not convergent).

A special type of property for sequences is if they are *Cauchy sequences*. This means that the sequence elements *want to converge*, (but there might be a hole in our space, so they actually dont).

Definition 4.16 (Cauchy sequence)**.** Let (X, d) be a metric space. A sequence $(a_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence if

$$\forall \varepsilon > 0 \exists N = N(\varepsilon) \in \mathbb{N} : d(a_n, a_m) < \varepsilon \quad \forall n, m \geq N$$

Lemma 4.17. *Let (X, d) be any metric space. Every convergent sequence is a Cauchy sequence.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a convergent sequence to some $x \in X$ and let $\varepsilon > 0$. Then there exist $N \in \mathbb{N}$ such that

$$d(x_n, x) < \frac{\varepsilon}{2} \quad \forall n \geq N.$$

Now let $n, m \geq N$. Then by triangular inequality,

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \varepsilon.$$

That is $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. □

In a general space not every Cauchy sequence needs to be convergent.

Exercise 4.18. *Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{Q}$ be the n-th digital expansion of π. I.e.*

$$x_1 = 3, \quad x_2 = 3.1, \quad x_3 := 3.14 \quad \ldots$$

Show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathbb{Q}, d_e)$. Show that $(x_n)_{n\in\mathbb{N}}$ does not converge in $\mathbb{Q}$.

Exercise 4.19. *Let (X, d) be a metric space and $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ two Cauchy sequences in (X, d). Show that*

$$(d(x_n, y_n))_{n\in\mathbb{N}}$$

is a Cauchy sequence in $\mathbb{R}$ (or $\mathbb{Q}$)

Definition 4.20. A metric space (X, d) is called *complete* if all Cauchy sequences are convergent.

Theorem 4.21. *Let (X, d) be any metric space. Then there exists a unique* metric completion *$(\tilde{X}, \tilde{d})$.*

Namely there exists a set $\tilde{X} \supset X$ and a metric $\tilde{d}$ on $\tilde{X}$ such that the following holds:

- *$(\tilde{X}, \tilde{d})$ is complete*
- *(extension) $\tilde{d}(x, y) = d(x, y)$ for all $x, y \in X$*

- *(density) For any $y \in \tilde{X}$ there exists a sequence $(x_k)_{k\in\mathbb{N}} \subset X$ that converges to y (with respect to $\tilde{d}$).*

Uniqueness means that for any two metric completions $(\tilde{X}_1, \tilde{d}_1)$, $(\tilde{X}_2, \tilde{d}_2)$ of (X, d) there exists an isometry $\varphi : \tilde{X}_1 \to \tilde{X}_2$ namely a bijection such that $\tilde{d}_1(\varphi(x_1), \varphi(x_2)) = \tilde{d}_2(x_1, x_2)$ for all $x_1, x_2 \in \tilde{X}_1$.

If X is a totally ordered field and d corresponds to $|\cdot|_X$, then $\tilde{X}$ inherits the metric field structure of X, i.e. for any $x, y, z \in \tilde{X}$ and any sequence x_n, y_n, z_n in X converging to x, y, z, respectively we have

$$(x + y) \cdot z = \lim_{n\to\infty} (x_n + y_n) \cdot z_n.$$

We will not prove Theorem 4.21 for completely but here is the idea.

Sketch of the proof of Theorem 4.21. Consider Z the set of Cauchy sequences $(x_n)_{n\in\mathbb{N}} \subset X$. This is a way to big set, since many Cauchy sequences "converge" to the same number. So we identify two Cauchy sequences in X if "they converge to the same limit".

$$(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \quad :\Leftrightarrow \lim_{n\to\infty} d_X(x_n, y_n) = 0.$$

The set $\tilde{X}$ is the collection of all such Cauchy sequences.

$$\tilde{X} = Z/\sim,$$

and we equip it with the metric[15]

$$d_{\tilde{X}}((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) := \lim_{n\to\infty} d_X(x_n, y_n)$$

Then $(\tilde{X}, d_{\tilde{X}})$ is a metric space. And any element of X identified with the (convergent) Cauchy sequence $(x, x, x, x, x, x, x, \ldots)$ is an element of $\tilde{X}$.

Now the point is that we can show that $\tilde{X}$ is complete: If we have a Cauchy sequence $(\tilde{x}_k)_{k\in\mathbb{N}} \subset \tilde{X}$, and each

$$\tilde{x}_k = ((x_{n;k})_{n\in\mathbb{N}})_{k\in\mathbb{N}} \subset X$$

Then by the choice of metric $\tilde{d}_{\tilde{X}}$ we can build a Cauchy sequence in X via a diagonal argument.

Fix some $k \in \mathbb{N}$. Since $(x_{n;k})_{n\in\mathbb{N}}$ is a Cauchy sequence, there exists some N_k such that

$$d(x_{n;k}, x_{m;k}) < \frac{1}{k} \quad \forall n, m \geq N_k. \tag{4.1}$$

W.l.o.g. we can assume

$$N_{k+1} \geq N_k$$

[15]careful, a little bit cheating is going on here: for this we have to show that this limit (in $\mathbb{R}$) makes sense, i.e. we already need $\mathbb{R}$ to be complete. If we'd known this then we can show that $(d(x_n, y_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, Exercise 4.19. So technically one first defines $\mathbb{R}$ and then metric completion!

Set

$$\bar{x}_1 := x_{1;1}.$$

and

$$\bar{x}_n := x_{N_n;n}.$$

We need to show that $(\bar{x}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. So let $\varepsilon > 0$. Since $(\tilde{x}_k)_{k\in\mathbb{N}}$ is Cauchy there exists some $K \in \mathbb{N}$ such that

$$\tilde{d}(\tilde{x}_k, \tilde{x}_j) < \frac{\varepsilon}{4} \quad \forall k, j \geq K.$$

That is,

$$\limsup_{n\to\infty} d(x_{n;k}, x_{n;j}) < \frac{\varepsilon}{4} \quad \forall k, j \geq K. \tag{4.2}$$

If we take $K_2 > K$ such that $\frac{1}{K_2} < \frac{\varepsilon}{8}$ then we have for any $k, j \geq K_2$ and any $n \geq \max\{N_k, N_j\}$

$$\begin{aligned} d(\bar{x}_k, \bar{x}_j) = d(x_{N_k;k}, x_{N_j;j}) \leq & d(x_{N_k;k}, x_{n;k}) + d(x_{n;k}, x_{n;j}) + d(x_{N_j;j}, x_{n;j}) \\ \overset{(4.1)}{\leq} & \frac{1}{k} + d(x_{n;k}, x_{n;j}) + \frac{1}{j} \\ \overset{k,j\geq K_2}{\leq} & \frac{\varepsilon}{4} + d(x_{n;k}, x_{n;j}). \end{aligned}$$

Again, this holds for any $n \geq \max\{N_k, N_j\}$, and the left-hand side is not dependent on n. So we can take the the $limsup_{n\to\infty}$ on both sides and in view of (4.2) we have

$$d(\bar{x}_k, \bar{x}_j) \leq \frac{\varepsilon}{4} + \limsup_{k\to\infty} d(x_{n;k}, x_{n;j}) \overset{(4.2)}{\leq} \frac{\varepsilon}{2} \quad \text{for any } k, j \geq K_2$$

That is we have shown: For any $\varepsilon > 0$ there exists a $K_2 = K_2(\varepsilon)$ such that

$$d(\bar{x}_k, \bar{x}_j) < \varepsilon \quad \text{for any } k, j \geq K_2$$

That is, the sequence $(\bar{x}_n)_{n\in\mathbb{N}} \subset X$ is a Cauchy sequence. The last part to prove is that the "Cauchy sequence of Cauchy sequences" $(\tilde{x}_k)_{k\in\mathbb{N}}$ converges (in the sequence space $\tilde{X}$) to $(\bar{x}_n)_{n\in\mathbb{N}}$, i.e. we need to show

$$\forall \varepsilon > 0 \quad \exists K(\varepsilon) : \quad \tilde{d}(\tilde{x}_k, (\bar{x}_n)_{n\in\mathbb{N}}) < \varepsilon \quad \forall k \geq K.$$

Equivalently, by the definition of $\tilde{d}$ and $\bar{x}_n$, we need to show

$$\forall \varepsilon > 0 \quad \exists K(\varepsilon) : \quad \forall k \geq K : \ \exists \Gamma = \Gamma(\varepsilon, k) : d(x_{n;k}, x_{N_n;n}) < \varepsilon \quad \forall n \geq \Gamma. \tag{4.3}$$

So fix $\varepsilon > 0$. Take $K_1 > 0$ such that

$$\tilde{d}(\tilde{x}_k, \tilde{x}_j) < \frac{\varepsilon}{4} \quad \forall k, j \geq K_1.$$

Since $(\bar{x}_m)_{m\in\mathbb{N}}$ is a Cauchy sequence, we also find some M such that

$$d(x_{N_m;m}, x_{N_n;n}) \equiv d(\bar{x}_m, \bar{x}_n) \leq \frac{\varepsilon}{4} \quad \forall m, n \geq M \tag{4.4}$$

We have
$$d(x_{n;k}, x_{N_n;n}) \leq d(x_{n;k}, x_{N_k;k}) + d(x_{N_k;k}, x_{N_n;n}).$$
so
$$d(x_{n;k}, x_{N_n;n}) \leq \frac{\varepsilon}{4} + d(x_{N_k;k}, x_{N_n;n}) \quad \forall n \geq N_k, \quad k \geq K_1$$
By (4.4) we conclude that
$$d(x_{n;k}, x_{N_n;n}) \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon \quad \forall n \geq \underbrace{\max\{N_k, M\}}_{=:\Gamma(\varepsilon,k)}, \quad k \geq \underbrace{\max\{K_1, M\}}_{=:K}$$
This implies (4.3). □

Metric completion should be seen as "plugging all infinitesimal holds", i.e. adding numbers like $\sqrt{2}$ to $\mathbb{Q}$ to make all Cauchy sequences converging.

4.4. **Sequences in Euclidean space.** Since often we work in Euclidean spaces with $d(x, y) = |x - y|$, lets record that convergence (in $\mathbb{Q}^n$, $\mathbb{Z}^n$, $\mathbb{R}^n$) means
$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} : \qquad \forall n \geq n_0 : \quad |a_n - b| < \varepsilon,$$
i.e. the notion of convergence that we learned in Intro to Analysis!

Now let's go back to $\mathbb{Q}$ and lets look at bounded, but *monotonely increasing* sequence $(a_n)_{n\in\mathbb{N}} \subset (0, \infty) \cap \mathbb{Q}$.

Definition 4.22. A sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{Q}$ is

- (monotone) *increasing* if $a_1 \leq a_2 \leq a_3 \leq \ldots$ or more precisely, $a_i \leq a_k$ for $i \leq k$
- (monotone) *decreasing* if $a_1 \geq a_2 \geq a_3 \geq \ldots$ or more precisely, $a_i \geq a_k$ for $i \leq k$
- *strictly (monotone) increasing* if $a_1 < a_2 < a_3 < \ldots$ or more precisely, $a_i < a_k$ for $i < k$
- *strictly (monotone) decreasing* if $a_1 > a_2 > a_3 > \ldots$ or more precisely, $a_i > a_k$ for $i < k$

If a sequence is either monote increasing or decreasing, we simply say "*monotone*".

Exercise 4.23. *Prove that the sequence*
$$\left(1 + \frac{1}{n}\right)^{n+1}$$
is decreasing.

Definition 4.24. An totally ordered field $\mathbb{F}$ is called *complete* if every monotone and bounded sequence is convergent (with respect to the metric induced by $|\cdot|$).

Monotone, bounded sequences in $\mathbb{Q}$ have a special property, they are *Cauchy sequences*, Definition 4.16

Lemma 4.25. *Any monotone and bounded sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{Q}$ is a Cauchy sequence (with respect to the metric induced by $|\cdot|$)*

Proof. W.l.o.g. let $(a_n)_{n\in\mathbb{N}}$ be increasing. Assume $(a_n)_{n\in\mathbb{N}}$ is *not* a Cauchy sequence. That is,

$$\neg\left(\forall\varepsilon>0\exists N=N(\varepsilon)\in\mathbb{N}: d(a_n,a_m)<\varepsilon \quad \forall n,m\geq N\right).$$

Let us carefully negate that statement, then we have

$$\begin{aligned}
&\exists\varepsilon>0 \quad \neg\left(\exists N\in\mathbb{N}: d(a_n,a_m)<\varepsilon \quad \forall n,m\geq N\right)\\
&\exists\varepsilon>0 \quad \forall N\in\mathbb{N} \quad \neg\left(d(a_n,a_m)<\varepsilon \quad \forall n,m\geq N\right)\\
&\exists\varepsilon>0 \quad \forall N\in\mathbb{N} \quad \neg\left(\forall n,m\geq N: d(a_n,a_m)<\varepsilon\right)\\
&\exists\varepsilon>0 \quad \forall N\in\mathbb{N} \quad \exists n,m\geq N: \quad \neg\left(d(a_n,a_m)<\varepsilon\right)\\
&\exists\varepsilon>0 \quad \forall N\in\mathbb{N} \quad \exists n,m\geq N: \quad d(a_n,a_m)\geq\varepsilon
\end{aligned}$$

Now $n=m$ will not happen (because then $d(a_n,a_n)=0<\varepsilon$), so we can rephrase this as

$$\exists\varepsilon>0 \quad \forall N\in\mathbb{N} \quad \exists n,m\geq N,\, n>m: \quad d(a_n,a_m)\geq\varepsilon$$

Now lets get back to our situation, we are in an ordered field, $a_n\geq a_m$ if $n>m$ and thus $d(a_n,a_m)=|a_m-a_n|=a_n-a_m$. If moreover, $m\geq N$ then we have $a_m\geq a_N$. So the above implies

$$\exists\varepsilon>0 \quad \forall N\in\mathbb{N} \quad \exists n>N: \quad a_n-a_N\geq\varepsilon.$$

Having this there exists a sequence $n_i\xrightarrow{i\to\infty}\infty$ such $a_{n_{i+1}}\geq a_{n_i}+\varepsilon$ and we may choose $a_{n_1}:=a_1$. Then $a_{n_i}\geq a_1+i\varepsilon$ (induction!) which means that $(a_{n_i})_i$ is an unbounded sequence. So (a_n) is an unbounded sequence, contradiction. □

Lemma 4.26. *Let (X,d) be any metric space and $\tilde{X}$ be its metric completion of Theorem 4.21. If X is a totally ordered field and d corresponds to $|\cdot|_X$, then $\tilde{X}$ inherits the metric field structure of X, i.e. for any $x,y,z\in\tilde{X}$ and any sequence x_n,y_n,z_n in X converging to x,y,z, respectively we have*

$$(x+y)\cdot z=\lim_{n\to\infty}(x_n+y_n)\cdot z_n.$$

Since we kind of used $\mathbb{R}$ to define metric completion, the following is a bit circiular (but good enough for us)

Definition 4.27. $\mathbb{R}$ as the metric completion of $(\mathbb{Q},|\cdot|)$, which is the totally ordered field we know and love.

Exercise 4.28 (decimal expansion)**.** *Let $n_i\in\{0,\ldots,9\}$, $i\geq 1$ and $n_0\in\mathbb{Z}$. Then the rational number*

$$x_\ell:=n_0+n_1\frac{1}{10}+\ldots+n_\ell\frac{1}{10^\ell}\in\mathbb{Q}$$

converges to a number in $\mathbb{R}$ as $\ell\to\infty$.

Conversely, any real number can be approximated by a decimal expansion as x_ℓ.

Proof.
- (x_ℓ) is a Cauchy sequence.
- for any real number $r > 0$ let y_ℓ be $y_\ell =$ last digit of $\lfloor r \cdot 10^\ell \rfloor$ where $[\cdot]$ denotes the Gaussian bracket. Then $x_\ell := y_\ell 10^{-\ell}$ is the decimal expansion.

□

Observe that the decimal expansion above is not unique, as $1 = 0.\overline{9}$.

Theorem 4.29 (Cantor, 1874)**.** *$\mathbb{R}$ is uncountable.*

We first observe

Exercise 4.30. *Let X be a set, and let $(Y_i)_{i=1}^\infty$ be each a subset $Y_i \subset X$ such that*

$$X = \bigcup_{i=1}^\infty Y_i.$$

Show the following: X is uncountable if and only if there exists an $i \in \mathbb{N}$ such that Y_i is uncountable.

Exercise 4.31. *Prove the following statement without using that $\mathbb{R}$ is uncountable:*

$\mathbb{R}$ is countable if and only if $(0,1)$ is countable. You can use Exercise 4.30.

Proof of Theorem 4.29. Suppose not, i.e. suppose that $\mathbb{R}$ is countable, then (almost obvious) $(0,1)$ is countable.

Suppose $(0,1)$ is countable, so we can arrange all real numbers from $(0,1)$ into a sequence. Suppose

$$x_1,\, x_2,\, x_3,\, x_4, \ldots$$

is such a sequence, i.e. assume

$$(0,1) = \bigcup_{i \in \mathbb{N}} \{x_i\}. \tag{4.5}$$

We write the decimal expansion of each x_i as follows

$$x_1 = 0.a_{11}a_{12}a_{13}a_{14}\ldots$$

$$x_2 = 0.a_{21}a_{22}a_{23}a_{24}\ldots$$

$$x_3 = 0.a_{31}a_{32}a_{33}a_{34}\ldots$$

$$\ldots$$

Now we build a new number

$$y := 0.b_1b_2b_3\ldots$$

where we choose $b_1 \in \{1,\ldots,8\}\backslash a_{11}$, $b_2 \in \{1,\ldots,8\}\backslash a_{22}$, and in general $b_i \in \{1,\ldots,8\}\backslash\{a_{ii}\}$.

$$\begin{array}{lllllll} y & = & 0. & b_1 & b_2 & b_3 & b_4 & \dots \\ x_1 & = & 0. & a_{11} & a_{12} & a_{13} & a_{14} & \dots \\ x_2 & = & 0. & a_{21} & a_{22} & a_{23} & a_{24} & \dots \\ x_3 & = & 0. & a_{31} & a_{32} & a_{33} & a_{34} & \dots \\ x_4 & = & 0. & a_{41} & a_{42} & a_{43} & a_{44} & \dots \end{array}$$

We get that $y \notin \bigcup_{i\in\mathbb{N}}\{x_i\}$ (here it helps that we ensured $b_i \neq 9$, since $0.\overline{9} = 1$, we can now show that $|y - x_i| > 0$ for all $i \in \mathbb{N}$). Also $y \neq 0$, $y \neq 1$, so $y \in (0,1)\setminus \bigcup_{i\in\mathbb{N}}\{x_i\}$, which is a contradiction to (4.5). □

4.5. suprema and infima.

Definition 4.32. Let $X \subset \mathbb{R}$. A number $M \in \mathbb{R}$ is called an *upper bound* of X if

$$x \leq M \quad \forall x \in X.$$

If X has an upper bound, then X is called *bounded from above*.

A number $s \in \mathbb{R}$ is called the *least upper bound* (or *supremum*) if

- s is an upper bound for X and
- any upper bound M for X satisfies $M \geq s$.

We then write $\sup X := s$.

Similarly we define lower bounds:

A number $M \in \mathbb{R}$ is called an *lower bound* of X if

$$x \geq M \quad \forall x \in X.$$

If X has a lower bound, then X is called *bounded from below*.

A number $i \in \mathbb{R}$ is called the *largest lower bound* (or *infimum*) if

- i is a lower bound for X and
- any lower bound M for X satisfies $M \leq i$.

We then write $\inf X := i$.

If a set is bounded from above and below, we simply say its bounded.

We also define $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

Assume we were working in $\mathbb{Q}$, and consider the set

$$A := \{q \in \mathbb{Q} : q^2 \leq 2\}.$$

Then the set A is bounded, but the supremum does not exists *in* $\mathbb{Q}$. Here is where the completeness of $\mathbb{R}$ shows its advantage: in $\mathbb{R}$ suprema and infima exist (or are $\pm\infty$)

Theorem 4.33. *If $X \neq \emptyset$ is bounded from above then* $\sup X \in \mathbb{R}$ *exists (and is unique). If X is bounded from below then* $\inf X \in \mathbb{R}$ *exists, and is unique.*

Proof. We construct two sequences, $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}} \in \mathbb{R}$ such that

- $x_k \in X$ for all k, and x_k is increasing.
- y_k is an upper bound of X for all k, and y_k is decreasing
- For $k \geq 2$,

$$|x_k - y_k| \leq \frac{1}{2}|x_{k-1} - y_{k-1}|. \tag{4.6}$$

Assume first we have such two sequence. Then in particular we have

$$x_1 \leq x_2 \leq \ldots \leq x_k \leq \ldots \leq y_k \leq y_{k-1} \leq \ldots \leq y_2 \leq y_1$$

Thus for any $k \leq \ell$

$$|x_k - x_\ell| = x_\ell - x_k \leq y_k - x_k = |x_k - y_k|.$$

and similarly for any $k \leq \ell$

$$|y_k - y_\ell| = y_k - y_\ell \leq y_k - x_k = |x_k - y_k|.$$

From (4.6) we obtain (by induction)

$$|x_k - y_k| \leq 2^{1-k}|x_1 - y_1|.$$

so that we have

$$|x_k - x_\ell|, |y_k - y_\ell| \leq 2^{1-\max\{k,\ell\}}|x_1 - y_1|.$$

This readily implies that $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ are both Cauchy sequences, so they are convergent by completeness of $\mathbb{R}$. From (4.6) we obtain that they converge to the same point $\bar{y} \in \mathbb{R}$. We claim that $\bar{y} = \sup X$. For this we have to show

- $\bar{y}$ is an upper bound: Fix any $x \in X$. Then $y_k \geq x$ (since all y_k are upper bounds). Then

  $$x \leq \bar{y} + y_k - x \xrightarrow{k\to\infty} \bar{y}.$$

  This holds for any $x \in X$, so $\bar{y}$ is an upper bound.
- $\bar{y}$ is the lowest upper bound
  Assume there is $\tilde{y}$ is another upper bound of X. Then

  $$x_k \leq \tilde{y} \quad \forall X$$

  That is

  $$\bar{y} \leq \tilde{y} + \bar{x} - x_k \xrightarrow{k\to\infty} \tilde{y}.$$

  That is, $\bar{y} \leq \tilde{y}$ and thus $\bar{y}$ is the lowest upper bound.

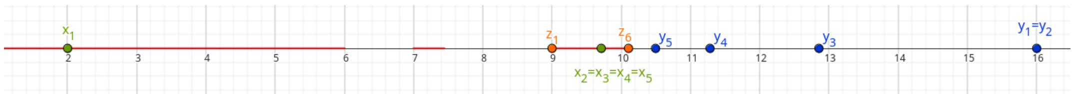


FIGURE 4.2. Construction in the proof of Theorem 4.33. The red line represents X. We see that both x_k and y_k move towards each other

We leave the uniqueness to the reader.

It remains to prove that the sequence $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ as above exists. Cf. Figure 4.2

We define these sequences inductively. Since $X \neq \emptyset$ there exists $x_1 \in X$. Since X is bounded, there exist $y_1 \in \mathbb{R}$ such that $y_1 > x$ for all $x \in X$, i.e. we have found x_1 and y_1 satisfying the assumptions of our series.

Now assume that for some $k \in \mathbb{N}$ the points x_{k-1}, y_{k-1} are already known – with $x_{k-1} \in X$ and y_{k-1} and upper bound of X.

Set

$$z := \frac{y_{k-1} - x_{k-1}}{2}$$

Clearly

$$x_{k-1} \leq z \leq y_{k-1}.$$

There are two possibilities

- If z is an upper bound of X we set $x_k := x_{k-1}$ and $y_k := z$. This satisfies the assumptions of our sequence, in particular we have

$$|x_k - y_k| = |x_{k-1} - z| = \frac{1}{2}|x_{k-1} - y_{k-1}|.$$

- If z is *not* an upper bound of X then there must be some element $x \in X$ such that $x > z \geq x_{k-1}$. We set $x_k := x$ and $y_k := y_{k-1}$. Since y_{k-1} was an upper bound of X, we still have $y_k = y_{k-1} \geq x_k$. Then

$$|x_k - y_k| \leq |z - y_k| = |z - y_{k-1}| = \frac{1}{2}|x_{k-1} - y_{k-1}|.$$

This defines the sequences $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ and we can conclude.

□

Exercise 4.34. *Find* $\sup A$ *and* $\inf A$ *where*

$$A = \left\{ \frac{x}{x+1} : x > 0 \right\}.$$

Proof. We first observe that $\frac{x}{x+1} < 1$ for all $x > 0$, so 1 is an upper bound. Assume there is a smaller upper bound $y < 1$, i.e. there is $y < 1$ such that

$$\frac{x}{x+1} \leq y \quad \forall x > 0.$$

Equivalence transformation says that since $0 < y < 1$ this is equivalent to

$$x \leq \frac{1}{1-y} \quad \forall x > 0.$$

But this is a contradiction, since we could choose $x := \frac{1}{1-y} + 1$ which is larger than $\frac{1}{1-y}$. So $y \geq 1$, i.e. $1 = \sup A$.

Similarly one shows $\inf A = 0$. □

Recall a characterization

Theorem 4.35. *Let M be an upper bound of $X \subset \mathbb{R}$. Then $M = \sup X$ if and only if for any $\varepsilon > 0$ there exists $x \in X$ such that $M - \varepsilon < x \leq M$.*

A similar statement holds for inf.

Exercise 4.36. *Prove Theorem 4.35 (using the definition of* sup *and* inf *from Definition 4.32)*

Exercise 4.37. *Find* $\sup A$ *and* $\inf A$*, where*

$$A = \left\{ \frac{n^2 + 2n - 3}{n+1} : \; n = 1, 2, 3, \ldots \right\}.$$

4.6. **Power function.**

Example 4.38. Prove that there exists $a \in \mathbb{R}$, $a > 0$, such that $a^2 = 2$. We denote this number by $\sqrt{2} := a$.

Proof. Consider

$$A := \{x \in \mathbb{R} : x^2 \leq 2\}.$$

This set is bounded, and thus $a := \sup A$ exists by Theorem 4.33. Clearly $a \geq 1 > 0$ (since a must be an upper bound and $1 \in A$).

We need to show that $a^2 = 2$. Let $\varepsilon \in (0, 1)$ then by Theorem 4.35 there exists $x \in A$ such that

$$a - \varepsilon < x.$$

Both sides are positive, so we square

$$(a - \varepsilon)^2 < x^2 \overset{x \in A}{\leq} 2.$$

Thus

$$(a - \varepsilon)^2 \leq 2 \quad \forall \varepsilon > 0 \tag{4.7}$$

That is

$$a^2 \leq \underbrace{a^2 + \varepsilon^2 - 2\varepsilon a}_{=(a-\varepsilon)^2 \leq 2} + 2\varepsilon a = (a - \varepsilon)^2 + 2\varepsilon a \overset{(4.7)}{\leq} 2 + 2\varepsilon a \quad \forall \varepsilon > 0.$$

This inequality we can reformulate

$$a^2 \leq 2 + 2\varepsilon a \quad \forall \varepsilon > 0$$
$$\overset{a>0}{\Leftrightarrow} \frac{a^2-2}{2a} \leq \varepsilon \quad \forall \varepsilon > 0$$

Assume now that $a^2 > 2$. Then the last statement is clearly false for $\varepsilon = \frac{a^2-2}{2a} > 0$. So we must have $a^2 \leq 2$.

On the other hand assume that $a^2<2$. Then there must be $\varepsilon > 0$ such that

$$(a+\varepsilon)^2 = \underbrace{a^2}_{<2} + \underbrace{\varepsilon^2 + 2a\varepsilon}_{\ll 1} < 2.$$

But then $a + \varepsilon \in A$ which is a contradiction to a being an upper bound of A. Thus we must have $a^2 \geq 2$

Consequently, $a^2 = 2$ which is what we wanted to show. □

Similarly one can show

Theorem 4.39. *For every real $x > 0$ and every integer $n > 0$ there is exactly one positive number $y \in \mathbb{R}$ such that $y^n = x$.*

We denote $y := \sqrt[n]{x} \equiv x^{\frac{1}{n}}$.

With this at hand we can define $x^{\frac{m}{n}} := \left(x^{\frac{1}{n}}\right)^m$ and $x^{-\frac{m}{n}} := \frac{1}{x^{\frac{m}{n}}}$.

That is, we can define x^q for any $x > 0$ and $q \in \mathbb{Q}$.

For $r \in \mathbb{R}$ and $x > 0$ we can define $x^r := \lim_{n\to\infty} x^{q_n}$ for an arbitary sequence of rational numbers q_n convergent to x and q_n monotone (if we ensure that q_n always has the same sign, then the sequence x^{q_n} is monotone, and since it is also bounded, it has a limit in $\mathbb{R}$)

So we can define powers x^r for two real numbers (if x is positive):

Definition 4.40. For $x, r \in \mathbb{R}$ we define x^r in the following way:

- If $x > 0$, $r > 0$ we set
$$x^r := \lim_{q_n \to r} x^{q_n},$$
where $q_n \in \mathbb{Q}$ is a nonnegative monotone increasing sequence of rational numbers that converges to r (and thus x^{q_n} is a bounded and monotone sequence)
- If $x > 0$ and $r < 0$ we set
$$x^r := \lim_{q_n \to r} \frac{1}{x^{q_n}}$$
- If $x > 0$ and $r = 0$ we set
$$x^0 = 1$$

- If $x = 0$ and $r > 0$
$$0^r = 0.$$
- 0^0 is not defined.
- x^r is not defined for $x < 0$.

Exercise 4.41. *Show the following:*

- *Assume $r > 1$ and $x < y$ show that $r^x < r^y$.*
- *Assume $r < 1$ and $x < y$ show that $r^x > r^y$.*
- *Show that $1^r = 1$*

In your proof be careful: limits can mess with strict inequalities! You get -1000 points if you make this mistake!

4.7. **More on Sequences in $\mathbb{R}$.** We recall (and defined above for metric spaces) the definition of the limit of a sequence (in $\mathbb{R}$ for simplicity).

A sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ converges to $x \in \mathbb{R}$ if
$$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) \in \mathbb{N}: \quad |x_n - x| < \varepsilon \quad \forall n \geq N.$$
In this case we write $\lim_{n\to\infty} x_n = x$.

Exercise 4.42. *Use the definition of the limit to prove that*
$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Solution. We need to prove that
$$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) \in \mathbb{N}: \quad \left|\frac{n}{n+1} - 1\right| < \varepsilon \quad \forall n \geq N.$$
So fix $\varepsilon > 0$. <u>How do we find N?</u> Well, we compute a bit around,
$$\begin{aligned} &\left|\frac{n}{n+1} - 1\right| < \varepsilon \\ \Leftrightarrow &\left|\frac{n-n-1}{n+1}\right| < \varepsilon \\ \Leftrightarrow &\frac{1}{n+1} < \varepsilon \\ \Leftrightarrow &\frac{1}{\varepsilon} - 1 < n \end{aligned}$$
So if we choose $N := \frac{1}{\varepsilon} - 1 + 1000$, then surely, for $n \geq N$ we have
$$\left|\frac{n}{n+1} - 1\right| < \varepsilon.$$
□

Theorem 4.43. *If $\lim_{n\to\infty} a_n = a \in \mathbb{R}$ and $\lim_{n\to\infty} b_n = b \in \mathbb{R}$ then*

- $\lim_{n\to\infty}(a_n + b_n) = a + b$
- $\lim_{n\to\infty}(a_n b_n) = ab$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ *provided that* $b \neq 0$.

Exercise 4.44. *Prove Theorem 4.43 using the* ε*-definition of the limit.*

Example 4.45. Find the limit of

$$\lim_{n\to\infty} \frac{3n^2+5}{2n^2+n+7}.$$

Solution. In Calculus we argued as follows,

$$\frac{3n^2+5}{2n^2+n+7} = \frac{3+\frac{5}{n^2}}{2+\frac{1}{n}+\frac{7}{n^2}}$$

Since $\frac{5}{n^2}$, $\frac{1}{n}$ and $\frac{7}{n^2}$ all tend to zero as $n \to \infty$, the limit is

$$\lim_{n\to\infty} \frac{3n^2+5}{2n^2+n+7} = \lim_{n\to\infty} \frac{3+\frac{5}{n^2}}{2+\frac{1}{n}+\frac{7}{n^2}} = \lim_{n\to\infty} \frac{3+0}{2+0+0} = \frac{3}{2}.$$

That is kind of ok, but you must know how to write the formal proof (using several instances of Theorem 4.43)! □

Theorem 4.46 (Squeeze theorem)**.** *Let* $a_n \leq b_n \leq c_n$ *for all* $n \in \mathbb{N}$ *and assume that*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = g \in \mathbb{R}.$$

Then $\lim_{n\to\infty} b_n = g$.

Proof. This is a proof we have seen before, but again, we have to know how to prove it.

Let $\varepsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = g$ there must be N_1 and $N_2 \in \mathbb{N}$ such that

$$|a_n - g| < \varepsilon, \quad \forall n \geq N_1$$

$$|b_n - g| < \varepsilon, \quad \forall n \geq N_2$$

Let $N := \max\{N_1, N_2\}$. Then we have for all $n \geq N$

$$g - \varepsilon < a_n \leq b_n \leq c_n < g + \varepsilon,$$

and thus

$$|b_n - g| < \varepsilon \quad \forall n \geq N.$$

Since ε was arbitrary this implies that $\lim_{n\to\infty} b_n = g$. □

Example 4.47. For $a > 0$ we have $\lim_{n\to\infty} \sqrt[n]{a} = 1$.

Proof. For $a = 1$ the statement is trivial.

If $\underline{a \in (0,1]}$ then for $b := \frac{1}{a} \in [1,\infty)$ we have $\lim_{n\to\infty} \sqrt[n]{b} = 1$, and thus *if we know the case* $a > 1$ by the limit laws Theorem 4.43,

$$\sqrt[n]{a} = \frac{1}{\sqrt[n]{b}} \xrightarrow{n\to\infty} \frac{1}{1} = 1.$$

So, w.l.o.g. assume that $\underline{a > 1}$.

One way to prove this is brute force: Assume the statement is false. Note $n \mapsto \sqrt[n]{a}$ is monotone decreasing. Moreover $\sqrt[n]{a} \geq 1$. Thus, since by assumption $\lim_{n\to\infty} \sqrt[n]{a} = 1$ is false, there must be an $\varepsilon > 0$

$$\sqrt[n]{a} \geq 1 + \varepsilon \quad \text{for all } n \in \mathbb{N}$$

but then

$$a \geq (1+\varepsilon)^n \xrightarrow{n\to\infty} \infty$$

But this contradicts that a is finite, so we have $\lim_{n\to\infty} \sqrt[n]{a} = 1$.

We can also be more elegant: By Bernoulli's inequality, Proposition 3.3,

$$a = \left(1 + \sqrt[n]{a} - 1\right)^n \geq 1 + n\left(\sqrt[n]{a} - 1\right).$$

Thus

$$0 \overset{a\geq 1}{\leq} \sqrt[n]{a} - 1 \leq \underbrace{\frac{a-1}{n}}_{\xrightarrow{n\to\infty} 0}.$$

By the squeeze theorem, Theorem 4.46, $\sqrt[n]{a} - 1 \xrightarrow{n\to\infty} 0$ which is what we wanted to show (whenever $a \geq 1$).

□

Example 4.48. Find the limit $\lim_{n\to\infty} \sqrt[n]{3^n + 5^n}$

Proof. Observe that

$$\sqrt[n]{5^n + 3^n} = \sqrt[n]{5^n(1 + \left(\frac{3}{5}\right)^n} = 5\sqrt[n]{1 + \left(\frac{3}{5}\right)^n}.$$

Now by Example 4.47,

$$1 \xleftarrow{n\to\infty} \sqrt[n]{1} \leq \sqrt[n]{1 + \left(\frac{3}{5}\right)^n} \leq \sqrt[n]{2} \xrightarrow{n\to\infty} 1$$

So again by the squeeze theorem Theorem 4.46,

$$\lim_{n\to\infty} \sqrt[n]{1} \leq \sqrt[n]{1 + \left(\frac{3}{5}\right)^n} = 1$$

and thus

$$\lim_{n\to\infty} \sqrt[n]{5^n(1+\left(\frac{3}{5}\right)^n} = 5.$$

□

Example 4.49. If $p > 0$ then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

Solution. Given $\varepsilon > 0$ let $N \in \mathbb{N}$ with $N > (\frac{1}{\varepsilon})^p$. □

Example 4.50. Show that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. (No L'Hopital!)

Solution. If we do Bernoulli again, we don't get far:

$$n = \Big(1 + (\sqrt[n]{n} - 1)\Big)^n \geq 1 + n(\sqrt[n]{n} - 1)$$

merely implies $\sqrt[n]{n}$ is bounded.

We instead apply the binomial formula[16]

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

In particular, if a and b are nonnegative,

$$(a+b)^n \geq b^n + \binom{n}{2} a^{n-2} b^2 = b^n + \frac{n(n-1)}{2} a^{n-2} b^2.$$

Notice how similar this looks to the Bernoulli, but the asymptotics in n are different.

Then (observe $\sqrt[n]{n} - 1 \geq 0$))

$$n = \Big(1 + (\sqrt[n]{n} - 1)\Big)^n \geq 1 + \frac{n(n-1)}{2} (\sqrt[n]{n} - 1)^2$$

which implies

$$0 \xleftarrow{n\to\infty} \frac{2(n-1)}{(n-1)n} \geq (\sqrt[n]{n} - 1)^2.$$

By the squeeze theorem we conclude $(\sqrt[n]{n} - 1)^2 \xrightarrow{n\to\infty} 0$ which readily implies the claim. □

Lastly, we like to say what is meant by

$$\lim_{n\to\infty} x_n = +\infty$$

Definition 4.51. We say that the sequence $(x_n)_{n\in\mathbb{N}}$ *diverges* to $+\infty$ if

$$\forall M \exists N = N(M): \quad x_n \geq M \quad \forall n \geq N.$$

We write

$$\lim_{n\to\infty} a_n = +\infty.$$

[16] if you forgot it, prove it by induction!

Similarly we define divergence to $-\infty$ and

$$\lim_{n\to\infty} b_n = -\infty.$$

For simplicity we will write $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.

Observe that e.g. oscilating sequences x_n may not have limits in $\overline{\mathbb{R}}$.

Exercise 4.52. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$*

Is the following statement true[17]*?*

$$\text{If } \lim_{n\to\infty} x_n = \infty \text{ is not true, then } \lim_{n\to\infty} x_n < \infty$$

4.8. **Cesaro Mean.** Recall that $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.

Proposition 4.53 (Cesaro Mean (1))**.** *If $\lim_{n\to\infty} a_n = g \in \overline{\mathbb{R}}$ then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i = g.$$

Proposition 4.54 (Cesaro Mean (2))**.** *If $\lim_{n\to\infty} a_n = g \in \overline{\mathbb{R}}$, $g > 0$, then*

$$\lim_{n\to\infty} \sqrt[n]{a_1 \cdot \ldots a_n} = g.$$

Exercise 4.55. *Prove Proposition 4.54. Hint: Proof of Proposition 4.53 below.*

Proof of Proposition 4.53. The idea of Proposition 4.53 is relatively easy. "Eventually" $a_n \approx g$ (say for all $n \geq N$) meaning that for $n \gg N$),

$$\frac{1}{n} \sum_{i=1}^{n} a_i \approx \frac{1}{n} \sum_{i=1}^{N} a_i + \frac{1}{n}(n - N - 1)g$$

Since $\sum_{i=1}^{N} a_i$ is finite, $\frac{1}{n} \sum_{i=1}^{N} a_i \xrightarrow{n\to\infty} 0$. On the other hand $\frac{1}{n}(n - N - 1)g \xrightarrow{n\to\infty} g$.

For practice let us make this proof precise. For simplicity let us assume that $g \in \mathbb{R}$ ($g \in \{\pm\infty\}$ follows the same idea).

Since $\lim_{n\to\infty} a_n = g$ we have that $M := \sup\{|a_n|, n \in \mathbb{N}\} < \infty$ (Lemma 4.15).

[17] seen on the prelim

Let $\varepsilon > 0$ then there exists N_1 such that $|a_i - g| < \frac{1}{2}\varepsilon$ for all $i \geq N_1$. We then have for any $n \geq N_1$,

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{i=1}^{n} a_i - g\right| \\
=&\left|\frac{1}{n}\sum_{i=1}^{n} (a_i - g)\right| \\
\leq&\left|\frac{1}{n}\sum_{i=1}^{N_1-1} (a_i - g)\right| + \left|\frac{1}{n}\sum_{i=N_1}^{n} (a_i - g)\right| \\
\leq&\frac{N_1-1}{n}2M + \frac{1}{2}\varepsilon\frac{n+1-N_1}{n} \\
=&\frac{\varepsilon}{2} + \left(\frac{N_1-1}{n}2M + \frac{1}{2}\varepsilon\frac{1-N_1}{n}\right)
\end{aligned}$$

Now we can choose $N_2 \in \mathbb{N}$ (depending on M and ε) such that

$$\left(\frac{N_1-1}{n}2M + \frac{1}{2}\varepsilon\frac{1-N_1}{n}\right) < \frac{\varepsilon}{2} \quad \forall n \geq N_2.$$

So if we set $N := \max\{N_1, N_2\}$ we find

$$\left|\frac{1}{n}\sum_{i=1}^{n} a_i - g\right| < \varepsilon \quad \forall n \geq N.$$

□

Example 4.56. $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$ since

$$\sqrt[n]{n!} = \sqrt[n]{1 \cdot 2 \cdot \ldots \cdot n},$$

and $\lim_{n\to\infty} n = \infty$.

Exercise 4.57. *Show that* $\lim_{n\to\infty} \frac{1+\sqrt{2}+\sqrt[3]{3}+\ldots+\sqrt[n]{n}}{n} = 1$.

Theorem 4.58. *If $a_n > 0$ and all n and $\lim_{n\to\infty} a_{n+1}/a_n = a \in \bar{\mathbb{R}}$, then $\lim_{n\to\infty} \sqrt[n]{a_n} = a$.*

Proof. Since $a_{n+1}/a_n \to a$ we see that also the the following sequence converges to a

$$1,\ \frac{a_2}{a_1},\ \frac{a_3}{a_2}, \ldots, \frac{a_n}{a_{n-1}}, \ldots \longrightarrow a.$$

Therefore

$$\frac{\sqrt[n]{a_n}}{\sqrt[n]{a_1}} = \sqrt[n]{1 \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot \frac{a_n}{a_{n-1}}} \to a\,.$$

Since $\sqrt[n]{a_1} \to 1$, we conclude that $\sqrt[n]{a_n} \to a$ from Proposition 4.54. □

4.9. e and the exponential function.

Definition 4.59. The *Euler number e* is defined as

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

To make sense of this definition we first need to prove

Lemma 4.60. *The sequence $a_n = \left(1+\frac{1}{n}\right)^n$ is strictly increasing, $b_n = \left(1+\frac{1}{n}\right)^{n+1}$ is strictly decreasing. Both converge to the same limit.*

Proof. Since $1+\frac{1}{n} > 1$ we have

$$a_n \le b_n \quad \forall n \in \mathbb{N}.$$

We first show that a_n is strictly increasing by showing that $\frac{a_{n+1}}{a_n} > 1$. Indeed

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} = \frac{\left(\frac{n+2}{n+1}\right)^{n+1}}{\left(\frac{n+1}{n}\right)^n} \\
&= \left(\frac{(n+2)n}{(n+1)^2}\right)^n \frac{n+2}{n+1} \\
&= \left(\frac{n^2+2n}{n^2+2n+1}\right)^n \frac{n+2}{n+1} \\
&= \left(1-\frac{1}{n^2+2n+1}\right)^n \frac{n+2}{n+1} \\
&\ge \left(1-\frac{n}{n^2+2n+1}\right) \frac{n+2}{n+1} \qquad \text{(Bernoulli)} \\
&= \frac{n^3+3n^2+3n+2}{n^3+3n^2+3n+1} > 1.
\end{aligned}$$

Similarly we can show that the sequence b_n is decreasing (we leave it as an exercise). Since $a_n \le b_n$ for every n we have

$$2 = a_1 < a_2 < a_3 < \ldots < a_n < \ldots < b_n < b_{n-1} < \ldots < b_1 = 4.$$

Hence a_n is increasing and bounded from above, so convergent (Lemma 4.25 and completeness of $\mathbb{R}$). Also b_n is decreasing and bounded from above, so convergent. Clearly $\lim_{n\to\infty} a_n \in (2,4)$, so $\lim_{n\to\infty} a_n \neq 0$ and hence

$$\frac{\lim_{n\to\infty} b_n}{\lim_{n\to\infty} a_n} = \lim_{n\to\infty}\frac{b_n}{a_n} = \lim_{n\to\infty} 1+\frac{1}{n} = 1, \quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n.$$

□

One more proof that a_n is increasing. A clever application of the Arithmetic-Geometric mean inequality, Theorem 3.7, gives

$$\left(\left(1+\frac{1}{n}\right)^n \cdot 1\right)^{1/(n+1)} = \sqrt[n+1]{\left(1+\frac{1}{n}\right)\cdots\left(1+\frac{1}{n}\right)\cdot 1}$$
$$\leq \frac{\left(1+\frac{1}{n}\right)+\ldots+\left(1+\frac{1}{n}\right)+1}{n+1} = 1+\frac{1}{n+1}\,.$$

Hence

$$\left(1+\frac{1}{n}\right)^n \cdot 1 \leq \left(1+\frac{1}{n+1}\right)^{n+1}, \qquad \left(1+\frac{1}{n}\right)^n \leq \left(1+\frac{1}{n+1}\right)^{n+1}.$$

□

Remark 4.61. Since $\left(1+\frac{1}{n}\right)^n$ is increasing and $\left(1+\frac{1}{n}\right)^{n+1}$ is decreasing and e is their common limit, we have that

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$$

for every n. Taking n large we obtain lower and upper estimate for e. One can prove that[18]

$$e = 2.718281828\ldots$$

Definition 4.62 (Series)**.** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. We say that for $g \in \overline{\mathbb{R}}$

$$g = \sum_{n=1}^{\infty} a_n$$

if the *partial sum* $s_\ell := \sum_{n=1}^{\ell} a_n$ satisfies $\lim_{\ell\to\infty} s_\ell = g$.

Theorem 4.63. $e = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$.

Proof. Let

$$x_n = \left(1+\frac{1}{n}\right)^n, \quad y_n = 1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}\,.$$

[18]As a funny side-note, during the space race the Sowjet space program worked with e with precision up to 9 digits, which their engineers rememberd as 2.7 and twice Tolstoi (who was born in 1828). Side-note of a side-note, NASA supposedly only uses 15 digits of π for interplanetary travel (nowadays), https://www.jpl.nasa.gov/edu/news/2016/3/16/how-many-decimals-of-pi-do-we-really-need

Thus $(y_n)_{n\in\mathbb{N}}$ is the sequence of the partial sums of the above series. The binomial formula yields

$$
\begin{aligned}
x_n &= 1^n + \binom{n}{1}1^{n-1}\frac{1}{n} + \binom{n}{2}1^{n-2}\frac{1}{n^2} + \ldots + \binom{n}{n-1}1^1\frac{1}{n^{n-1}} + \binom{n}{n}1^0\frac{1}{n^n} \\
&= 1 + 1 + \frac{n(n-1)}{2!}\,\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\,\frac{1}{n^3} + \ldots \\
&+ \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\,\frac{1}{n^k} + \ldots + \frac{n(n-1)(n-2)\cdots 1}{n!}\,\frac{1}{n^n} \\
&= 1 + 1 + \frac{1}{2!}\left(\frac{n-1}{n}\right) + \frac{1}{3!}\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right) + \ldots + \frac{1}{n!}\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{1}{n}\right).
\end{aligned}
$$

Therefore $x_n \leq y_n$. On the other hand the coefficients at $1/k!$ converge to 1 as $n \to \infty$ so we should expect that the limit of x_n will be as large as that of y_n which together with the inequality $x_n \leq y_n$ should give equality of the limits. To turn this observation into a rigorous argument, fix k. Then for $n \geq k$ we have

$$
x_n \;\geq\; x_k = 1 + 1 + \frac{1}{2!}\left(\frac{n-1}{n}\right) + \ldots + \frac{1}{k!}\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-k+1}{n}\right).
$$

With that fixed k, letting $n \to \infty$ on both sides of the above inequality yields

$$
e = \lim_{n\to\infty} x_n \geq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{k!} = y_k.
$$

This and the inequality $x_n \leq y_n$ gives $e \leftarrow x_n \leq y_n \leq e$ and hence

$$
\sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n\to\infty} y_n = e.
$$

□

Theorem 4.64. *e is irrational.*

Proof. Let

$$
x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}.
$$

In particular observe that $x_n n! \in \mathbb{N}$. Then

$$\begin{aligned} e - x_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \ldots \\ &= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+2)(n+3)(n+4)} + \ldots\right) \\ &< \frac{1}{(n+1)!}\Big(\underbrace{1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \ldots}_{\text{geometric series}}\Big) \\ &= \frac{1}{(n+1)!}\,\frac{1}{1-\frac{1}{n+1}} = \frac{1}{n!\,n}\,. \end{aligned}$$

Hence

$$0 < e - x_n < \frac{1}{n!\,n}\,.$$

Suppose that e is a rational number i.e., $e = p/q$ for some $p, q \in \mathbb{N}$. Then

$$0 < e - x_q < \frac{1}{q!\,q}\,,$$

$$0 < \underbrace{eq!}_{\text{integer}} - \underbrace{x_q q!}_{\text{integer}} < \frac{1}{q}\,.$$

Since there are no integers between 0 and $1/q$ we arrived a contradiction. This proves that e cannot be a rational number. □

For any $x \in \mathbb{R}$ the value e^x is well-defined, and we have $e^x > 0$, cf. Exercise 4.41.

Definition 4.65. It is relatively easy[19] to show that for any $r \in (0, \infty)$ there exists exactly one $x \in \mathbb{R}$ such that $e^x = r$.

The *natural logarithm* is defined by

$$\ln r = \log r = \log_e r = x \quad \text{if } e^x = r.$$

Observe that differently than in high school, $\log x$ is with base e instead of 10.

We will assume all the logarithm rules from calculus (that we can derive from power laws).

It is not clear at this point why the base e is more important than any other base. It will be transparent later when we will study derivatives, but even now the following result shows a nice and important inequality that is true for the natural logarithm.

Lemma 4.66. $\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$ *for* $n = 1, 2, 3, \ldots$.

[19]this is an easy consequence of the intermediate value theorem, Theorem 8.12 , so we don't do this here

Proof. The inequality

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$$

implies

$$n\ln\left(1+\frac{1}{n}\right) < 1 < (n+1)\ln\left(1+\frac{1}{n}\right) .$$

The left inequality gives

$$\ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$$

and the right inequality gives

$$\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) .$$

□

Theorem 4.67. *The sequence*

$$a_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln n$$

is convergent to a finite limit

$$\gamma := \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln n\right) .$$

Remark 4.68. The limit $\gamma = 0.5772156649\ldots$ is called the Euler constant. It is not known if γ is rational or not.

Proof. We will prove that the sequence is decreasing. To this end it suffices to show that $a_{n+1}-a_n<0$. We have

$$\begin{aligned} a_{n+1}-a_n &= \left(1+\frac{1}{2}+\ldots+\frac{1}{n}+\frac{1}{n+1}\right)-\ln(n+1)-\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)+\ln n \\ &= \frac{1}{n+1}-\ln(n+1)+\ln n \\ &= \frac{1}{n+1}-\ln\left(\frac{n+1}{n}\right) \\ &= \frac{1}{n+1}-\ln\left(1+\frac{1}{n}\right) < 0 , \end{aligned}$$

where the last inequality follows from Lemma 4.66. Therefore the sequence (a_n) is decreasing. Applying the lemma one more time we have

$$1 > \ln(1+1),\ \frac{1}{2} > \ln\left(1+\frac{1}{2}\right), \ldots, \frac{1}{n} > \ln\left(1+\frac{1}{n}\right) ,$$

and hence

$$\begin{aligned} a_n &= 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n \\ &> \ln(1+1) + \ln\left(1 + \frac{1}{2}\right) + \ldots + \ln\left(1 + \frac{1}{n}\right) - \ln n \\ &= \ln 2 + \ln\frac{3}{2} + \ldots + \ln\frac{n+1}{n} - \ln n \\ &= \ln\left(2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdots \frac{n+1}{n}\right) - \ln n \\ &= \ln(n+1) - \ln n > 0. \end{aligned}$$

Thus the sequence is decreasing and bounded from below by 0. Hence it is convergent, as it is a Cauchy sequence in a complete metric space Lemma 4.25. □

As a corollary we obtain another proof that

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

Indeed, since the sequence of partial sums

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

is increasing it suffices to show that it is not convergent. Suppose it is convergent. Since the sequence in Theorem 4.67 is also convergent, the difference of two sequences i.e., the sequence $s_n - a_n = \ln n$ is also convergent, but it is not, since $\lim_{n\to\infty} \ln n = +\infty$.

Exercise 4.69. *Find the limit* $\lim\limits_{n\to\infty} \dfrac{n}{e^{1+\frac{1}{2}+\cdots+\frac{1}{n}}}$.

4.10. **Examples.**

Example 4.70. Prove that the sequence $\sqrt[n]{n}$ is decreasing starting from $n = 3$.

Solution. We have

$$n^{1/n} > (n+1)^{1/(n+1)} \iff n^{n+1} > (n+1)^n \iff n > \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n.$$

The last inequality is true for $n \geq 3$, because $n \geq 3 > e > (1 + 1/n)^n$ and hence the first inequality is true for $n \geq 3$ as equivalent. □

Example 4.71. Find the following limits

(1) $\lim_{n\to\infty} \left(\frac{n!}{n^n e^{-n}}\right)^{1/n}$,
(2) $\lim_{n\to\infty} \left(\frac{(n!)^3}{n^{3n} e^{-n}}\right)^{1/n}$.

Solution. (1) Let $a_n = \frac{n!}{n^n e^{-n}}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}e^{-(n+1)}}\frac{n^n e^{-n}}{n!} = \frac{n!(n+1)}{(n+1)(n+1)^n e^{-n}e^{-1}}\frac{n^n e^{-n}}{n!}$$
$$= \frac{n^n e}{(n+1)^n} = \frac{e}{\left(1+\frac{1}{n}\right)^n} \to 1$$

and hence Theorem 4.58 gives

$$\sqrt[n]{a_n} = \left(\frac{n!}{n^n e^{-n}}\right)^{1/n} \to 1.$$

(2) Let $a_n = \frac{(n!)^3}{n^{3n}e^{-n}}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{\big((n+1)!\big)^3}{\big((n+1)^{n+1}\big)^3 e^{-(n+1)}}\frac{n^{3n}e^{-n}}{(n!)^3} = \frac{(n!)^3(n+1)^3}{(n+1)^3(n+1)^{3n}e^{-n}e^{-1}}\frac{n^{3n}e^{-n}}{(n!)^3}$$
$$= \frac{n^{3n}e}{(n+1)^{3n}} = \frac{e}{\left(\left(1+\frac{1}{n}\right)^n\right)^3} \to \frac{e}{e^3} = e^{-2}$$

and hence Theorem 4.58 gives

$$\sqrt[n]{a_n} = \left(\frac{(n!)^3}{n^{3n}e^{-n}}\right)^{1/n} \to e^{-2}$$

□

4.11. **subsequences in $\mathbb{R}^n$ – Bolzano Weierstrass.** Let (X,d) be a metric space.

If $(x_n)_{n\in\mathbb{N}}$ is a sequence and

$$n_1 < n_2 < n_3 < \dots$$

are positive integers then the sequence

$$(y_k)_{k\in\mathbb{N}}, \quad y_k := x_{n_k}$$

is called a *subsequence*.

It is easy to prove

Lemma 4.72. *Let (X,d) be any metric space. For any $(x_k)_{k\in\mathbb{N}} \subset X$ sequence and any $x \in X$ the following are equivalent*

- $\lim_{k\to\infty} x_k = x$
- $\lim_{k\to\infty} x_{k_i} = x$ *for all subsequences $(x_{k_i})_{i\in\mathbb{N}}$ of $(x_k)_{k\in\mathbb{N}}$.*

- *Let $((n_{i;\ell})_{\ell\in\mathbb{N}})_{i\in I} \subset \mathbb{N}$ be a collection of strictly increasing sequence in $\mathbb{N}$ which cover all but finitely many elements of $\mathbb{N}$. I.e. assume that for any $i \in I$, $(n_{i;\ell})_{\ell\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$ and that*
$$\left(\bigcup_{i\in I,\ell\in\mathbb{N}} n_{i;\ell}\right) \backslash \mathbb{N} \textit{is a finite set.}$$
and $(x_{n_{i;\ell}})_{\ell\in\mathbb{N}}$ is convergent to x for each $i \in I$.

It is a special property of $\mathbb{R}$ (more generally finite dimensional spaces) that bounded sequences have convergent subsequences (we will later say that bounded sets in $\mathbb{R}$, $\mathbb{R}^n$ etc. are *precompact*). This is called the *Bolzano-Weierstrass theorem*

Theorem 4.73 (Bolzano-Weierstrass)**.** *Every bounded sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^n$ has a convergent subsequence.*

It is worth recalling the proof.

Proof. Since $(x_n)_{n\in\mathbb{N}}$ is bounded, there exists $M > 0$ such that $(x_n)_{n\in\mathbb{N}} \subset [-M, M]^n =: C_1$.

For simplicity assume that $M = 1$. We can divide the cube C_1 into 2^n cubes of sidelength 1. Each of these cubes can be subdivided into 2^n cubes of sidelenght 2^{-1}, and so on[20]. This division into small cubes is called *dyadic decomposition*.

Since x_n is an infinte sequence, for each subsequence and each $\kappa \in \mathbb{N}$, infinitely elements of that subsequence must belong to one of the cubes of sidelength 2^{2-k}.

We can construct a subsequence now as follows:

Let $n_1 := 1$ and $C_1 = [-1, 1]^n$. In the $i + 1$st step, fix any dyadic cube C_{i+1} of sidelength 2^{1-i}, which is contained in C_i such that there are infinitely many sequence elements $n > n_i$ in C_{i+1}. Take $n_{i+1} > n_i$ so that $x_{n_{i+1}} \in C_{i+1}$.

We thus obtain a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ such that
$$x_{n_k} \subset C_i \quad \forall k \geq i.$$
In particular, because of the sidelenght of C_i being 2^{2-i},
$$|x_{n_k} - x_{n_\ell}| < \sqrt{n}2^{2-i} \quad \forall k, \ell \geq i.$$
In particular x_{n_k} is a Cauchy sequence, and since $\mathbb{R}^n$ is complete x_{n_k} converges. □

Exercise 4.74. *Let $(x_n)_{n=1}^\infty$ be a sequence of points in $\mathbb{R}^3$ such that $|x_{n+1}-x_n| \leq 1/(n^2+n)$, $n \geq 1$. Show that $(x_n)_n$ converges.*

[20]Here the dimension comes into play: we can split the interval $[a, b]$ into two intervals of half the diameter. In $\mathbb{R}^2$ we can split the square $[a, b]^2$ into four intervals of half the diameter. In infinite dimensions we would have to split a bounded set into infinitely many sets of smaller diameter – so this argument would fail there

Exercise 4.75. *Give an example of a sequence* $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ *such that*

$$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon): \quad |x_n - x_{n+1}| < \varepsilon$$

but $(x_n)_{n\in\mathbb{N}}$ *is not convergent.*

Exercise 4.76. *Assume we have a sequence* $(x_n)_{n\in\mathbb{N}} \subset (X,d)$ *where* (X,d) *is a metric space. Assume for some* $\lambda \in (0,1)$ *we have*

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_{n-2}) \quad \forall n \geq 3.$$

Show that $(x_n)_{n\in\mathbb{N}}$ *is a Cauchy sequence.*

Hint: You can freely use the formula (for $\lambda \neq 1$*)*

$$1 + \lambda + \lambda^2 + \cdots + \lambda^n = \frac{1-\lambda^{n+1}}{1-\lambda}.$$

Exercise 4.77. *Let* $f: \mathbb{R} \to \mathbb{R}$ *be continuous*[21] *and assume that for some* $\lambda \in (0,1)$ *we have*

$$|f(x) - f(y)| \leq \lambda |x-y| \quad \text{for all } x, y \in \mathbb{R}$$

Show that there exists $\bar{x}$ *such that* $f(\bar{x}) = \bar{x}$*.*

Hint: Take any $x \in \mathbb{R}$*. Set* $x_n := f(x_n)$*. Then use Exercise 4.76. Then think about what happens to*

$$x_n = f(x_n) \quad \text{as } n \to \infty.$$

Do not use a Fixed point theorem. Prove the Fixed Point theorem!

4.12. **The upper and the lower limits.** Sequences can be subdivided into subsequences. The *limit superior*, lim sup is the largest possible limit (or $+\infty$) of any subsequence, the *limit inferior*, lim inf is the smallest possible limit of any subsequence. More precisely,

Definition 4.78. Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence.

- If $(x_n)_{n\in\mathbb{N}}$ is bounded from above we set
$$\limsup_{n\to\infty} x_n := \inf_{n\in\mathbb{N}} \sup_{k\geq n} x_k.$$
(observe that this number exists since $(x_n)_{n\in\mathbb{N}}$ is bounded from above)
- If $(x_n)_{n\in\mathbb{N}}$ is not bounded from above we set $\limsup_{n\to\infty} x_n := +\infty$
- If $(x_n)_{n\in\mathbb{N}}$ is bounded from below we set
$$\liminf_{n\to\infty} x_n := \sup_{n\in\mathbb{N}} \inf_{k\geq n} x_k.$$
(observe that this number exists since $(x_n)_{n\in\mathbb{N}}$ is bounded from below)
- If $(x_n)_{n\in\mathbb{N}}$ is not bounded from below we set $\liminf_{n\to\infty} x_n := -\infty$

[21] Lets all agree that we already know what this means: if $\lim_{n\to\infty} x_n = x$ for some sequence $(x_n)_{n\in\mathbb{N}}$ then $\lim_{n\to\infty} f(x_n) = f(x)$

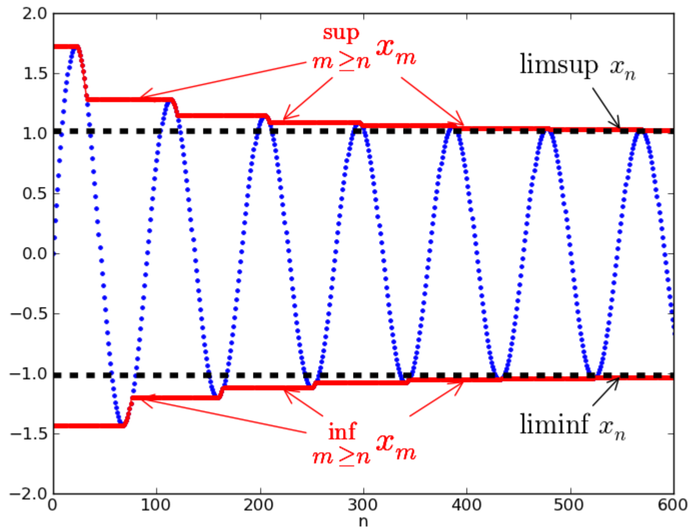


FIGURE 4.3. An illustration of limit superior and limit inferior. The sequence x_n is shown in blue. The two red curves approach the limit superior and limit inferior of x_n, shown as dashed black lines. In this case, the sequence accumulates around the two limits. The superior limit is the larger of the two, and the inferior limit is the smaller of the two. The inferior and superior limits agree if and only if the sequence is convergent (i.e., when there is a single limit). (text and image: Eigenjohnson, Wikipedia)

Cf. Figure 4.3.

Example 4.79. Let

$$x_n := \begin{cases} \frac{1}{n} & n \text{ even} \\ -n & n \text{ odd} \end{cases}$$

then

$$\limsup_{n\to\infty} x_n = 0$$

and

$$\liminf_{n\to\infty} x_n = -\infty.$$

To coincide limsup and liminf with our intuition observe

Lemma 4.80. *Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence.*

(1) Set $a_n := \sup_{k\geq n} x_k$, then, if the right-hand side exists,

$$\limsup_{n\to\infty} x = \lim_{n\to\infty} a_n$$

(2) *Set* $b_n := \inf_{k \geq n} x_k$, *then, if the right-hand side exists,*

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} b_n$$

(3) *Let* $(x_{n_i})_{i\in\mathbb{N}}$ *be any convergent subsequence. Then*

$$\liminf_{n\to\infty} x_n \leq \lim_{i\to\infty} x_{n_i} \leq \limsup_{n\to\infty} x_n.$$

(4) *If* $\limsup_{n\to\infty} x_n \in (-\infty,\infty)$ *then there exists a convergent subsequence* $(x_{n_i})_{i\in\mathbb{N}}$ *with*

$$\lim_{i\to\infty} x_{n_i} = \limsup_{n\to\infty} x_n.$$

(5) *If* $\liminf_{n\to\infty} x_n \in (-\infty,\infty)$ *then there exists a convergent subsequence* $(x_{n_i})_{i\in\mathbb{N}}$ *with*

$$\lim_{i\to\infty} x_{n_i} = \liminf_{n\to\infty} x_n.$$

(6) *If* $\limsup_{n\to\infty} x_n = \infty$ *then there exists a subsequence* $(x_{n_i})_{i\in\mathbb{N}}$ *with* $\lim_{i\to\infty} x_{n_i} = \infty$. *If* $\limsup_{n\to\infty} x_n = -\infty$ *then all subsequences* $(x_{n_i})_{i\in\mathbb{N}}$ *satisfy* $\lim_{i\to\infty} x_{n_i} = -\infty$.

(7) *If* $\liminf_{n\to\infty} x_n = -\infty$ *then there exists a subsequence* $(x_{n_i})_{i\in\mathbb{N}}$ *with* $\lim_{i\to\infty} x_{n_i} = -\infty$. *If* $\liminf_{n\to\infty} x_n = +\infty$ *then all subsequences* $(x_{n_i})_{i\in\mathbb{N}}$ *satisfy* $\lim_{i\to\infty} x_{n_i} = +\infty$.

Proof. (1) If $(a_n)_{n\in\mathbb{N}}$ is not bounded from above, $(x_n)_{n\in\mathbb{N}}$ is not bounded from above, and so $\lim_{n\to\infty} a_n = \limsup_{n\to\infty} x_n = \infty$.

If a_n is bounded from above then it is a monotonce decreasing, bounded, sequence. Thus we have convergence, by Definition 4.16 and the completeness of $\mathbb{R}$, we find that a_n is convergent and

$$\lim_{n\to\infty} a_n = \inf_n \sup_{k\geq n} x_k = \limsup_{n\to\infty} x_n.$$

(2) exercise! (almost the same argument as as above)

(3) We only show

$$\lim_{i\to\infty} x_{n_i} \leq \limsup_{n\to\infty} x_n. \tag{4.8}$$

The other inequality follows the same way.

If $\limsup_n x_n = \infty$ then (4.8) is trivially satisfied. So let us assume $\limsup_n x_n < \infty$. Then

$$x_{n_i} \leq \sup_{k\geq n_i} x_k =: a_i \quad \forall i \in \mathbb{N}.$$

We observe that $(a_i)_{i\in\mathbb{N}}$ is a monotone increasing sequence. Since $\limsup_n x_n < \infty$ we have that a_i is bounded from above. So by Definition 4.16 and completeness of $\mathbb{R}$, a_i is convergent and

$$\lim_{i\to\infty} a_i = \inf_i \sup_{k\geq n_i} x_k \leq \inf_n \sup_{k\geq n} x_k = \limsup_{n\to\infty} x_n.$$

By monotonicity of the limit,

$$\lim_{i\to\infty} x_{n_i} \leq \lim_{i\to\infty} a_i = \limsup_{n\to\infty} x_n.$$

(4) Set

$$a_n := \sup_{k\geq n} x_k.$$

Since $\limsup_{n\to\infty} x_n < \infty$, by the definition of supremum as lowest upper bound, for any $n \in \mathbb{N}$ there must be a number $K = K(n) \geq n$ such that

$$a_n - \frac{1}{n} \leq x_K \leq a_n.$$

Now we build our subsequence as follows. $n_1 := K(1)$, $n_2 := K(n_1 + 1)$, $n_i := K(n_{i-1} + 1)$. This is an strictly increasing sequence, and we have $n_i \geq i$, so that (together with the monotonicity we can ensure that

$$a_{n_{i-1}+1} - \frac{1}{i} \leq x_{n_i} \leq a_{n_{i-1}+1}.$$

Observe that $(a_{n_{i-1}+1})_i$ is a subsequence of the convergent sequence a_n, and as such convergent itself. By the squeeze theorem, Theorem 4.46, we have that

$$\lim_{i\to\infty} x_{n_i} = \lim_{i\to\infty} a_{n_{i-1}+1} = \limsup_{n\to\infty} x_n.$$

(5) same as above

(6) If $\limsup_{n\to\infty} x_n = \infty$ then $\inf_{n\in\mathbb{N}} a_n = \infty$ where $a_n = \sup_{k\geq n} x_k$. That means that for any $M \in \mathbb{N}$ and for any $n \in \mathbb{N}$ there exists $k = k(n) \geq n$ with $x_k > M$. From this we can build a subsequence. Take $x_{n_1} := x_{k(1)}$, $x_{n_2} := x_{k(k(1)+1)}$ etc. This subsequence goes to infinity.

Assume now that $\limsup_{n\to\infty} x_n = -\infty$ and take $(x_{n_i})_{i\in\mathbb{N}}$ any subsequence.

Then $\inf_{n\in\mathbb{N}} a_n = -\infty$ where $a_n = \sup_{k\geq n} x_k$. That is, for any $M > 0$ there must be some $N \in \mathbb{N}$ such that $a_N < -M$. But since $a_N = \sup_{k\geq N} x_k$, this implies $x_k \leq -M$ for *all* $k \geq N$. That is, for all $M > 0$ we have that $x_n < -M$ for all but finitely many $n \in \mathbb{N}$. In particular, for all $M > 0$ we have that $x_{n_i} < -M$ for all but finitely many $i \in \mathbb{N}$. This means that $\lim_{i\to\infty} x_{n_i} = -\infty$.

(7) analogue argument to above.

□

Lemma 4.81. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$*

(1) $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$

(2) For any subsequence (x_{n_i}),

$$\liminf_{n\to\infty} x_n \leq \liminf_{i\to\infty} x_{n_i} \leq \limsup_{i\to\infty} x_{n_i} \leq \limsup_{n\to\infty} x_n$$

(3) If $\lim_{n\to\infty} x_n = x$ *then* $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$.

(4) $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$ *then* $\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.

Proof. (1) obvious from the definition

(2) Obvious from the definition of lim sup, and monotonicity of the supremum/infimum.

(3) From Lemma 4.80 we have that there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ such that

$$\lim_{i\to\infty} x_{n_i} = \limsup_{n\to\infty} x_n.$$

On the other hand, since x_n converges, so does any of its subsequences, so

$$\lim_{i\to\infty} x_{n_i} = \lim_{n\to\infty} x_n.$$

Together we find

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} x_n.$$

The same argument works for the lim inf.

(4) Let $a_n := \inf_{k\geq n} x_k$ and $b_n := \sup_{k\geq n} x_k$. Then

$$a_n \leq x_n \leq b_n \quad \forall n \in \mathbb{N}.$$

Since by assumption and Lemma 4.80,

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \limsup_{n\to\infty} x_n$$

We conclude by the squeeze theorem, Theorem 4.46 that

$$\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \limsup_{n\to\infty} x_n$$

□

5. Series

Recall from Definition 4.16 the notion of *Cauchy sequence*. Also recall that $\mathbb{R}$ is defined as the metric completion of $(\mathbb{Q}, |\cdot|)$ (Definition 4.20), and thus Cauchy-sequence and converging sequence are the same (in $\mathbb{R}$, not in $\mathbb{Q}$).

Recall

Definition 5.1. We say that for a sequence $(a_i)_{i\in\mathbb{N}}$ and $g \in \overline{\mathbb{R}}$ the series is convergent/divergent to g,

$$\sum_{i=1}^{\infty} a_i = g \in \overline{\mathbb{R}},$$

iff the sequence (partial sum)

$$s_n := \sum_{i=1}^{n} a_i \xrightarrow{n\to\infty} g.$$

We say that the series $\sum_{i=1}^{\infty} a_i$ is an *absolutely convergent series* if $\sum_{i=1}^{\infty} |a_i| < \infty$, i.e. the sequence

$$t_n := \sum_{i=1}^{n} |a_i| \xrightarrow{n\to\infty} \tilde{g} < \infty$$

Equivalently (since (t_n) is monotone increasing sequence in $\mathbb{R}$) one could require that t_n is a bounded sequence, $\sup_n t_n < \infty$).

Since $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ if and only if it is convergent, one obtains the *Cauchy criterion* for sequences

Theorem 5.2. *The series $\sum_{i=1}^{\infty} a_i$ is convergent if and only if*

$$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) \in \mathbb{N} \quad \forall n \geq N : \forall m \in \mathbb{N} \quad \left|\sum_{i=n}^{n+m} a_i\right| < \varepsilon.$$

Corollary 5.3. *If*[22] $\sum_{i=1}^{\infty} a_i \in \mathbb{R}$ *then* $\lim_{n\to\infty} a_i = 0$.

Proof. It follows directly from Theorem 5.2, but one can also argue directly. For $s_n = \sum_{i=1}^{n} a_i$ we have

$$a_n = s_n - s_{n+1}.$$

Since by assumption

$$\mathbb{R} \ni \sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n+1}$$

we get from the limit rules, Theorem 4.43, that $\lim_{n\to\infty} a_n = 0$. □

Theorem 5.4 (Comparison test)**.** *(1) Suppose there is $N \in \mathbb{N}$ such that $|a_n| \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.*

(2) Suppose there is $N \in \mathbb{N}$ such that $a_n \geq b_n \geq 0$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges (i.e. goes to $+\infty$), then $\sum_{n=1}^{\infty} a_n$ diverges (to $+\infty$).

Proof. (1) We establish the Cauchy Criterion in Theorem 5.2 for $\sum_{n=1}^{\infty} a_n$. Let $\varepsilon > 0$ then since $\sum_{n=1}^{\infty} b_n$ is convergent in view of Theorem 5.2 there must be $N_1 \in \mathbb{N}$ (w.l.o.g. $N_1 \geq N$ where N is from the assumption) such that

$$\forall m \in \mathbb{N} : \left|\sum_{n=N_1}^{N_1+m} b_i\right| < \varepsilon$$

Observe that by assumption

$$\left|\sum_{n=N_1}^{N_1+m} a_i\right| \leq \sum_{n=N_1}^{N_1+m} b_i = |\sum_{n=N_1}^{N_1+m} b_i|,$$

so we get the Cauchy criterion for $\sum a_i$.

(2) obvious.

□

Corollary 5.5. *If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.*

[22] i.e. $\sum_{i=1}^{\infty} a_i$ is convergent to a number in $\mathbb{R}$

Proof. Use $b_n := |a_n|$ in Theorem 5.4(1). $\square$

Theorem 5.6 (Alternating sequences)**.** *Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence with the following properties*

(1) $|a_1| \geq |a_2| \geq |a_3| \geq \dots$
(2) $a_1 \geq 0$, $a_2 \leq 0$, $a_3 \geq 0$, *etc. (alternating)*
(3) $\lim_{n\to\infty} c_n = 0$

Then $\sum_{n=1}^{\infty} a_n$ is convergent.[23]

Proof. Let

$$s_\ell := \sum_{n=1}^{\ell} a_n.$$

Since a_n alternate and their absolute value decrease s_ℓ we observe the following (prove by induction!)

$$s_2 \leq s_4 \leq s_6 \ldots \leq s_7 \leq s_5 \leq s_3 \leq s_1$$

So $(s_{2k})_{k\in\mathbb{N}}$ and $(s_{2k-1})_{k\in\mathbb{N}}$ are montone, bounded sequences, and thus convergent. Moreover,

$$|s_{2k+1} - s_{2k}| = |a_{2k+1}| \xrightarrow{k\to\infty} 0,$$

we find that s_{2k+1} and s_{2k} must converge to the same limit $g \in \mathbb{R}$. But then by Lemma 4.72 s_k converges. $\square$

Theorem 5.7. *If $|q| < 1$ then*

$$\sum_{n=0}^{\infty} q^n = 1 + q + q^2 + \ldots = \frac{1}{1-q}.$$

If $|q| \geq 1$

$$\sum_{n=0}^{\infty} q^n \textit{does not converge.}$$

Proof. If $|q| \geq 1$ then q^n does not converge to zero as $n \to \infty$, so the series cannot converge by Corollary 5.3.

So let $|q| < 1$. By induction one proves

$$\sum_{n=0}^{\ell} q^n = \frac{1-q^{\ell+1}}{1-q}.$$

Now observe that

$$\frac{1-q^{\ell+1}}{1-q} \xrightarrow{\ell\to\infty} = \frac{1}{1-q}.$$

[23]but maybe not absolutely convergent. The sequence $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ is thus convergent, but it is certainly not absolutely convergent.

□

Theorem 5.8 (Cauchy Condensation test)**.** *Suppose $a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.*

Proof. This is a consequence of the comparison test and a little bit of combinatorics.

If $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges then $\sum_{n=1}^{\infty} a_n$ converges:

Let

$$s_n := \sum_{i=1}^{n} a_i$$

and

$$t_k := \sum_{\ell=0}^{k} 2^\ell a_\ell.$$

For $n < 2^k$ we have because of the nonegativity and monotonicy of the sequence (a_n),

$$\begin{aligned} s_n \leq& a_1 + \underbrace{(a_2 + a_3)}_{\leq 2a_2} + \underbrace{(a_4 + a_5 + a_6 + a_7)}_{\leq 4a_4} + \underbrace{(a_8 + \ldots + a_{15})}_{\leq 8a_8} + \ldots + \underbrace{(a_{2^k} + \ldots + a_{2^{k+1}-1})}_{\leq 2^k a_{2^k}} \\ \leq& t_k \end{aligned}$$

So if t_k is bounded then s_n is bounded, and since all sequence elements are nonnegative this is equivalent to saying that if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges:

We argue with a similar idea. For $n > 2^k$ we have

$$\begin{aligned} s_n \geq& a_1 + a_2 + \underbrace{(a_3 + a_4)}_{\geq 2a_4} + \underbrace{(a_5 + \ldots + a_8)}_{\geq 4a_8} + \ldots + \underbrace{(a_{2^{k-1}+1} + \ldots + a_{2^k})}_{\geq 2^{k-1} a_{2^k}} \\ \geq& \frac{1}{2} t_k \end{aligned}$$

We argue as above: if $\sum_{n=1}^{\infty} a_n$ converges, then $(s_n)_{n\in\mathbb{N}}$ is bounded, so the inequality implies that $(t_k)_k$ is bounded. Since all sequence elements are nonnegative this is equivalent to saying $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges. □

Theorem 5.8 allows to prove elegantly the following theorem (which, of course, we could also prove by an integral comparison test).

Theorem 5.9. *$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p >$ and diverges for $p \leq 1$.*

Proof. If $p \leq 1$ then

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \geq \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

so we have divergence for $p \leq 1$, Theorem 5.4.

Assume now $p > 1$ and set $a_n := \frac{1}{n^p}$. Then $2^n a_{2^n} = 2^n \frac{1}{(2^n)^p} = (2^{1-p})^n$. Observe that then

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = \sum_{n=0}^{\infty} 2^n a_{2^n} = \sum_{n=0}^{\infty} (2^{1-p})^n.$$

This is the geometric series with $q = 2^{1-p} \in (0,1)$, so it is convergent. By Theorem 5.8 we obtain that $\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty$. □

Theorem 5.10. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ *converges for* $p > 1$ *and diverges for* $0 < p \leq 1$.

Proof. Let $a_n = 1/(n(\log n)^p)$. Then

$$2^n a_{2^n} = 2^n \frac{1}{2^n(\log 2^n)^p} = \left(\frac{1}{\log 2}\right)^p \frac{1}{n^p}\,.$$

Hence the previous result yields that

$$\sum_{n=1}^{\infty} 2^n a_{2^n} = \left(\frac{1}{\log 2}\right)^p \sum_{n=2}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$ and the theorem follows from the Cauchy condensation test. □

Theorem 5.11 (Ratio Test (d'Alembert Test))**.**
- **(a)** *If* $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, *then the series* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*
- **(b)** *If* $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Theorem 5.12 (Root Test (Cauchy Test))**.**
- **(a)** *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, *then the series* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*
- **(b)** *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Remark 5.13. If

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \quad \text{or} \quad \lim_{n\to\infty} \sqrt[n]{|a_n|} = 1,$$

then we cannot conclude convergence or divergence of the series. For example, if $a_n = 1/n$, then the above limits are equal 1 and the series $\sum_{n=1}^{\infty} a_n$ diverges. If $a_n = 1/n^2$, then still the above limits are equal 1, but this time the series $\sum_{n=1}^{\infty} a_n$ converges.

Exercise 5.14. *Provide an example of a convergent series* $a_1 + a_2 + a_3 + \ldots$, *where* $a_n > 0$, $n = 1, 2, 3, \ldots$ *such that the limit* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ *does not exist.*

Exercise 5.15. *Let* $a_1, a_2, a_3, \ldots > 0$. *Prove that if*

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1\,,$$

then the series $a_1 + a_2 + a_3 + \ldots$ *converges.*

Exercise 5.16. *Prove that there is a sequence of positive integers* $n_1 < n_2 < n_3 < \ldots$ *such that the sequence* $a_k = \sin n_k$ *converges.*

We will prove the d'Alembert test only; the proof for the Cauchy test is similar and left as an exercise.

Proof of Theorem 5.11. If $\lim_{n\to\infty} |a_{n+1}/a_n| < 1$, then there is $0 < q < 1$ and n_0 such that

$$\left|\frac{a_{n+1}}{a_n}\right| < q \quad \text{for } n \geq n_0.$$

For $n \geq n_0$ we have

$$|a_{n+1}| < q|a_n| < q^2|a_{n-1}| < \ldots < q^{n+1-n_0}|a_{n_0}|,$$
$$|a_{n+1}| < \left(q^{-n_0}|a_{n_0}|\right) q^{n+1}$$

Replacing $n+1$ by n in this formula we have

$$|a_n| < \left(q^{-n_0}|a_{n_0}|\right) q^n \quad \text{for } n > n_0.$$

Since the series

$$\sum_{n=1}^{\infty} \left(q^{-n_0}|a_{n_0}|\right) q^n = \left(q^{-n_0}|a_{n_0}|\right) \sum_{n=1}^{\infty} q^n$$

converges, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely by the comparison test.

If $\lim_{n\to\infty} |a_{n+1}/a_n| > 1$, there are n_0 and $q > 1$ such that $|a_{n+1}/a_n| > q$ for $n \geq n_0$ and it easily follows that a_n does not converge to zero. Hence the series $\sum_{n=1}^{\infty} a_n$ diverges (see Theorem 5.4). □

Example 5.17. For every $x \in \mathbb{R}$ the series $\sum_{n=0}^{\infty} x^n/n!$ converges absolutely. It is obvious if $x = 0$, so we can assume that $x \neq 0$. If $a_n = x^n/n!$, then $|a_{n+1}/a_n| = |x|/(n+1) \to 0$, so the absolute convergence follows from the d'Alembert test.

Example 5.18. Investigate convergence of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{(n+1)n}$.

Solution. Let $a_n = \left(\frac{n}{n+1}\right)^{(n+1)n}$. Then

$$\sqrt[n]{a_n} = \left(\frac{n}{n+1}\right)^{n+1} = \frac{1}{\left(\frac{n+1}{n}\right)^{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \frac{1}{1+\frac{1}{n}} \to \frac{1}{e} < 1$$

and hence the series converges. □

Theorem 5.19. *Assume that* $a_n > 0$, $b_n > 0$ *and*

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n} \quad \text{for all } n \geq n_0.$$

If the series $\sum_{n=1}^{\infty} b_n$ *converges the series* $\sum_{n=1}^{\infty} a_n$ *converges, too.*

Remark 5.20. If $\lim_{n\to\infty} b_{n+1}/b_n < 1$, then convergence of the series $\sum_{n=1}^{\infty} a_n$ follows from the d'Alambert test. However, if

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 1$$

and we know that the series $\sum_{n=1}^{\infty} b_n$ converges, we still can conclude convergence of $\sum_{n=1}^{\infty} a_n$ even if d'Alambert's test does not apply. We will see examples after we prove the theorem.

Proof of Theorem 5.19. Let $c_n = a_n/b_n$. Then

$$c_{n+1} = \frac{a_{n+1}}{b_{n+1}} \leq \frac{a_n}{b_n} = c_n \quad \text{for } n \geq n_0,$$

so c_n is decreasing starting from $n = n_0$. Hence c_n is bounded, say $c_n \leq M$ for all n. Therefore

$$a_n = c_n b_n \leq M b_n$$

and convergence of the series $\sum_{n=1}^{\infty} M b_n = M \sum_{n=1}^{\infty} b_n$ implies convergence of $\sum_{n=1}^{\infty} a_n$. □

Now we will show two applications of the above result.

Example 5.21. Investigate convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^{n-2}}{e^n n!}$.

Solution. Let $a_n = \dfrac{n^{n-2}}{e^n n!}$. Then

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(n+1)^{n-1}}{e^{n+1}(n+1)!}\,\frac{e^n n!}{n^{n-2}} = \frac{(n+1)^{n-2}(n+1)e^n n!}{e\, e^n (n+1) n! n^{n-2}} \\ &= \frac{\left(1+\frac{1}{n}\right)^{n-2}}{e} = \underbrace{\frac{\left(1+\frac{1}{n}\right)^{n}}{e}}_{<1} \left(1+\frac{1}{n}\right)^{-2} \\ &< \left(\frac{n}{n+1}\right)^2 = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\,. \end{aligned}$$

Hence

$$\frac{a_{n+1}}{a_n} < \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\,.$$

Since the series $\sum_{n=1}^{\infty} 1/n^2$ converges, the series $\sum_{n=1}^{\infty} a_n$ converges, too. □

Example 5.22. Investigate convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^{n}}{e^n n!}$.

Solution. Let $a_n = \dfrac{n^n}{e^n n!}$. Then

$$
\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(n+1)^{n+1}}{e^{n+1}(n+1)!}\,\frac{e^n n!}{n^n} = \frac{(n+1)^n(n+1)e^n n!}{e^n e n!(n+1)n^n} \\
&= \frac{\left(1+\frac{1}{n}\right)^n}{e} > \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n}\right)^{n+1}} = \frac{1}{1+\frac{1}{n}} \\
&= \frac{n}{n+1} = \frac{\frac{1}{n+1}}{\frac{1}{n}}\,.
\end{aligned}
$$

Hence

$$\frac{\frac{1}{n+1}}{\frac{1}{n}} \leq \frac{a_{n+1}}{a_n}\,.$$

Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges. The the theorem would give convergence of the series $\sum_{n=1}^{\infty} 1/n$ which is a contradiction. Therefore $\sum_{n=1}^{\infty} a_n$ diverges. □

5.1. **Multiplication of Series.** Formally we would like to multiply two series as follows

$$(a_1 + a_2 + a_3 + \ldots)(b_1 + b_2 + b_3 + \ldots) = a_1b_1 + (a_1b_2 + a_2b_1) + (a_1b_3 + a_2b_2 + a_3b_1) + \ldots$$

In the first group a_1b_1 we collect all terms with indices that add up to 2. In the second group $a_1b_2 + a_2b_1$ we collect terms with indices that add up to 3. Then terms with indices that add up to 4 and so on. Since we deal with infinite sums we have to rigorously investigate when the above formula is correct. We have

Theorem 5.23 (Cauchy Multiplication Formula)**.** *If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely and the series $\sum_{n=1}^{\infty} b_n$ converges, then*

$$\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right) = \sum_{n=1}^{\infty} c_n\,,$$

where

$$
\begin{gathered}
c_1 = a_1b_1 \\
c_2 = a_1b_2 + a_2b_1 \\
\cdots \\
c_n = a_1b_n + a_2b_{n-1} + \ldots + a_nb_1 \\
\cdots
\end{gathered}
$$

Proof. See [Hajłasz, 2020]. □

Exercise 5.24. *Use the Cauchy multiplication formula to find the sum of the series*

$$\sum_{n=1}^{\infty} nx^{n-1}, \quad |x| < 1.$$

Solution. The series $\sum_{n=0}^{\infty} x^n$ converges absolutely for $|x| < 1$. Hence

$$\begin{aligned}
\left(\frac{1}{1-x}\right)^2 &= \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} x^n\right) = (1+x+x^2+\ldots)(1+x+x^2+\ldots) \\
&= 1 + (1\cdot x + x\cdot 1) + (1\cdot x^2 + x\cdot x + x^2\cdot 1) + (1\cdot x^3 + x\cdot x^2 + x^2\cdot x + x^3\cdot 1) + \ldots \\
&+ (1\cdot x^n + x\cdot x^{n-1} + x^2\cdot x^{n-2} + \ldots + x^n\cdot 1) + \ldots \\
&= 1 + 2x + 3x^2 + 4x^3 + \ldots + (n+1)x^n + \ldots \\
&= \sum_{n=0}^{\infty}(n+1)x^n = \sum_{n=1}^{\infty} nx^{n-1}.
\end{aligned}$$

Thus

$$\sum_{n=1}^{\infty} nx^{n-1} = \left(\frac{1}{1-x}\right)^2.$$

□

5.2. **Changing the order in a series.** Let $\sum_{n=1}^{\infty} a_n$ be a series and let $\varphi : \mathbb{N} \to \mathbb{N}$ be a bijection. Then a series $\sum_{n=1}^{\infty} a_{\varphi(n)}$ is obtained from $\sum_{n=1}^{\infty} a_n$ by rearrangement of the elements: we add exactly the same numbers, but in a different order.

Theorem 5.25. *If $\sum_{n=1}^{\infty} a_n$ converges absolutely and $\varphi : \mathbb{N} \to \mathbb{N}$ is a bijection, then $\sum_{n=1}^{\infty} a_{\varphi(n)}$ converges and*

$$\sum_{n=1}^{\infty} a_{\varphi(n)} = \sum_{n=1}^{\infty} a_n.$$

Proof. Suppose that $\sum_{n=1}^{\infty} a_n$ converges absolutely and let $\varphi : \mathbb{N} \to \mathbb{N}$ be a bijection. Then from the Cauchy condition for series we have

$$\forall \varepsilon > 0\ \exists n_0\ \forall m \quad |a_{n_0}| + |a_{n_0+1}| + \ldots + |a_{n_0+m}| < \varepsilon. \tag{5.1}$$

Denote partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{\varphi(n)}$ by s_n and t_n respectively. Choose p so large that

$$\{1, 2, \ldots, n_0 - 1\} \subset \{\varphi(1), \varphi(2), \ldots, \varphi(p)\}.$$

If $n > p$, then the numbers $a_1, a_2, \ldots, a_{n_0-1}$ will cancel out in the difference of partial sums

$$s_n - t_n = (a_1 + a_2 + \ldots + a_{n_0-1} + a_{n_0} + \ldots + a_n) - (a_{\varphi(1)} + a_{\varphi(2)} + \ldots + a_{\varphi(p)} + a_{\varphi(p+1)} + \ldots + a_{\varphi(n)}).$$

The remaining terms will be a_i's with $i \geq n_0$ and signs + or −. The + sign will be associated with terms in the partial sum s_n and the − sign will be associated with the terms in the partial sum t_n. No index i will be repeated twice as elements with the same index will show once with sign + and once with sign − so they will cancel out. Therefore