Contents

References 6
Index 7

Part 1. Analysis I: Measure Theory 10

1. Measures, \( \sigma \)-Algebras 10

1.1. Example: Hausdorff measure 13

1.2. Measurable sets 19

1.3. Construction of Measures: Carathéodory-Hahn Extension Theorem 25

1.4. Classes of Measures 30

1.5. More on the Lebesgue measure 42

1.6. Nonmeasurable sets 47

2. Measurable functions 50

3. Integration 56

3.1. \( L^p \)-spaces and Lebesgue dominated convergence theorem 63

3.2. Lebesgue integral vs Riemann integral 71

3.3. Theorems of Lusin and Egorov 73

3.4. Convergence in measure 78

3.5. \( L^p \)-convergence and weak \( L^p \) 81

3.6. Absolute continuity 83

3.7. Vitali’s convergence theorem 85
4. Product Measures, Multiple Integrals – Fubini’s theorem 88
4.1. Application: Interpolation between $L^p$-spaces – Marcienkiewicz interpolation theorem 92
4.2. Application: convolution 97
4.3. A first glimpse on Sobolev spaces 109
5. Differentiation of Radon measures - Radon-Nikodym Theorem on $\mathbb{R}^n$ 115
5.1. Preparations: Besicovitch Covering theorem 115
5.2. The Radon-Nikodym Theorem 116
5.3. Signed (pre-)measures 123
5.4. Lebesgue differentiation theorem 126
5.5. Riesz representation theorem 130
6. Transformation Rule 138
6.1. Area formula and integration on manifolds 141
7. Fourier Transform 143
7.1. Quick review of complex numbers 143
7.2. Some motivation 144
7.3. Precise Definition 145
7.4. Tempered Distributions and their Fourier transform 155
7.5. Real Fourier transform 163
7.6. Fourier transform for periodic functions 163
7.7. Application: Basel Problem 167
7.8. Application: Isoperimetric Problem - Hurwitz proof in 2D 167
7.9. Discrete Fourier Transform and periodicity 170

8. Normed Vector spaces 171
9. Linear operators, Dual space 174
10. Subspaces 176
11. Hahn-Banach Theorem
11.1. Separation theorems
12. The bidual and reflexivity
13. Weak Convergence & Reflexivity
13.1. Basic Properties of weak convergence
13.2. Weak convergence in $L^p$-spaces
13.3. Weak convergence in Sobolev space
13.4. More involved basic properties of weak convergence
13.5. Applications of weak compactness theorem - Theorem 13.1
13.6. Application: Direct Method of Calculus of Variations & Tonelli’s theorem
13.7. Proof of Theorem 13.1
13.8. Weak convergence and compactness for $L^1$ and Radon measures
13.9. Yet another definition of $L^p$ and $W^{1,p}$
14. Sobolev spaces
14.1. Approximation by smooth functions
14.2. Difference Quotients
14.3. Weak compactness in $W^{k,p}$
14.4. Extension Theorems
14.5. Traces
14.6. Embedding theorems
14.7. Fun inequalities: Ehrling’s lemma, Gagliardo-Nirenberg inequality, Hardy’s inequality
14.8. Rademacher’s theorem
14.9. Short excursion on degree and Brouwer Fixed Point theorem
14.10. Sobolev spaces for maps between manifolds and the $H=W$ problem
14.11. Fubini theorem for Sobolev spaces
15. BV
15.1. Sets of finite perimeter 263
15.2. Application: Isoperimetric Inequality 263
16. Fractional Sobolev spaces 264
16.1. Characterization of $W^{k,2}$ via Fourier transform 264
16.2. Fractional Sobolev spaces I – Bessel potential space $H^{s,p}$ 264
16.3. Fractional Sobolev spaces – Besov space $W^{s,p}$ 264

Part 3. Analysis III: Cool Tools from Functional Analysis 264
17. Fixed Point Theorems 264
18. Hilbert spaces 264
19. Compact operators 264
19.1. Fredholm Alternative 264
19.2. Spectrum 264
20. Semigroup theory 264
In Analysis
there are no theorems
only proofs
These lecture notes take great inspiration from the lecture notes by Michael Struwe (Analysis III, German), as well as by Piotr Hajlasz (Analysis I). We will also follow the presentations in Evans-Gariepy [Evans and Gariepy, 2015] (measure theory), Grafakos [Grafakos, 2014] (Fourier Analysis) and wikipedia. Sometimes we follow those sources verbatim.

Pictures that were not taken from above mentioned sources or wikipedia are usually made with geogebra.

References


Index

by density, 136, 202
by duality, 134
by reflexivity, 190

Calderon-Zygmund theory, 160
Campanato spaces, 241
Campanato’s theorem, 241
canonical embedding of $X^{**} \hookrightarrow X$, 187
Cantor set, 19
capacity, 246
Carathéodory–Hahn extension, 26
Caratheodory-function, 87
chain, 177
characteristic function, 52
coercive, 200, 202
compact, 203
compact support, 101
compactly contained, 105
complex conjugation, 143
concatenation, 59, 62
content, 12
convergence in measure, 78, 79
convergence in norm, 191
convex, 184
convolution, 97
counting measure, 13
dense, 70
density, 116
density point, 130
Dido’s problem, 167
differentiable with respect to $\mu$, 116
Dirac measure, 103
direct method, 200
direct method of the Calculus of Variations, 199
discrete Fourier transform, 170
distance, 171
distribution, 110
distributional derivative, 110
distributions on $\mathbb{R}^n$, 156
divergence, 254
dual space, 132, 174
duality, 181
dyadic cubes, 42

Eberlein–Smulian Theorem, 190
embedded, 176
energy method, 199
ANALYSIS I & II

Marcienkiewicz Interpolation Theorem, 94, 97
maximal, 177
measurable function, 50
measure, 11
measure space, 22
metric, 171
metric measure, 30
metric outer measure, 14
metric space, 171
metric topology, 172
Minkowski functional, 184
Minkowski-inequality, 64
mollification, 103
mollifier, 102
monotonicity, 11
Morrey Embedding theorem, 243
multiplier operator, 160
multiplier theorems, 160
mutually singular, 121

nearest point projection, 248
Newton potential, 161
non-measurable sets, 11
norm, 64, 172
normed space, 172

open ball, 172
open sets, 171
orientation, 138
outer measure, 11

pairing, 132
Paley-Wiener theorem, 154
Parseval's relation, 151
partial order, 177
partially ordered, 177
Plancherell identity, 151
Pontryagin dual group, 170
pre-Hilbert space, 173
pre-measure, 25
precise representative, 127
product measure, 89
pseudonorm, 64, 65
Rademacher's theorem, 244
Radon measure, 37
Radon-Nikodym Theorem, 116, 121
regular, 128
representative, 65
Riesz potential, 161

equivalent norms, 172
essential supremum, 63
Euler-Lagrange equation, 201
Fat Cantor set, 19
Fatou's lemma, 61
figure, 25
finite, 174
finite perimeter, 255
first variation, 201
Fourier inverse, 151
Fourier series, 164
Fourier transform, 144, 164
Fourier transform inversion, 151
Fourier-Laplace transform, 154
Fubini's theorem, 88, 89
functionals, 131
Hölder-inequality, 64
Hausdorff content, 13
Hausdorff dimension, 16
Hausdorff measure, 13
Heaviside function, 113
Heisenberg uncertainty principle, 145
Helmholtz decomposition, 181
Hilbert space, 173
Hodge decomposition, 181
induced metric, 172
inner Jordan content, 12
inner product, 173
inner product space, 173
interpolation, 94
inverse Fourier transform, 164
isoperimetric problem, 167
Jacobian, 141
Jensen, 64
Jordan content, 11
Laplace equation, 161
Lebesgue integral, 57
Lebesgue monotone convergence theorem, 58
Lebesgue outer measure, 12
Lebesgue point, 128
linear extension, 176, 177
linear space, 172, 173
linearly dependent, 173
lower semicontinuous, 55, 191
Lusin property, 45
scalar product, 173
scaling argument, 235
Schwarz classes, 145
Schwarz function, 145
Schwarz seminorms, 145
Schwarz’s theorem, 154
separation theorems, 184
separable, 183
Shor’s algorithm, 171
signed measures, 123
signed premeasure, 123
simple functions, 53
singular part, 122
Sobolev space, 209
Sobolev spaces, 109
step functions, 53, 55
strong convergence, 191
strongly converges, 191
sublinear, 176, 177
support, 101
support of \( f \), 70
Theorem of Eberlein-Smulian, 204
Tonelli’s theorem, 90
topological space, 51
topology, 51, 171
torus, 163
totally ordered, 177
Transfinite Induction, 177
translation, 148
trigonometric polynomial, 164
tubular neighborhood, 247
uniformly absolutely continuous integrals, 85
upper bound, 177
upper semicontinuous, 55
vector space, 172, 173
Vitali’s convergence theorem, 85
Vitali-, 47
weak \( L^p \)-space, 81
weakly closed, 198
weakly converges, 191
weakly* converges, 191
Young’s convolution inequality, 98
Part 1. Analysis I: Measure Theory

1. Measures, σ-Algebras

A measure is a way to measure (hence the name!) volumes. So for some set $X$ it should be a map

$$
\mu : 2^X \to [0, \infty]
$$

that to a subset $A \subset X$ assigns the volume $\mu(A)$. Here $2^X$ denotes the potential set, i.e. the collection of subsets of $X$.

What would we want from a volume in $\mathbb{R}^n$? Well it seems to be a reasonable assumption to axiomatically assume the following

- For any $A \subset \mathbb{R}^n$ we have $\mu(A) \in [0, \infty]$
- (Invariance under translation and rotation) For any set $A \subset \mathbb{R}^n$, any rotation $P \in O(n)$ and any vector $x \in \mathbb{R}^n$ we have $\mu(x + OA) = \mu(A)$ where we denote

$$
x + OA := \{x + Oa \in \mathbb{R}^n : a \in A\}
$$

- For any $A, B \subset \mathbb{R}^n$ disjoint we have $\mu(A \cup B) = \mu(A) + \mu(B)$

As reasonable as that sounds, there are two problems here:

- For $n \geq 3$ the only map $\mu : 2^{\mathbb{R}^n} \to [0, \infty]$ that satisfies our axiom is constant (Hausdorff, 1914)
- For $n = 1, 2$ there are indeed nonconstant maps $\mu : 2^{\mathbb{R}^n} \to [0, \infty]$ that satisfy the above axioms, however even if we fix $\mu([0,1]^n) := 1$ there is more than one possibility for such a $\mu$ (Banach 1923).
- the whole business about disjoint sets is really tricky, as illustrated by the Banach-Tarski-Paradoxon (1924):

  Let $n \geq 3$, $A$ and $B$ be bounded sets with $int(A)$ and $int(B) \neq \emptyset$. Then there exist finitely many $(x_i)_{i=1}^N \subset \mathbb{R}^n$, $(O_i)_{i=1}^N \subset O(n)$ and disjoint sets $(C_i)_{i=1}^N$ so that

$$
A = \bigcup_{i=1}^N C_i, \quad \text{and} \quad B = \bigcup_{i=1}^N (x_i + OC_i).
$$

That is we can deconstruct any set $A$ in $\mathbb{R}^n$ into disjoint sets, move them around (without any scaling!) and obtain another completely different set $B$ - see Figure 1.1.

This is crazy, so the axiomatic definition of a reasonable volume in $\mathbb{R}^n$ has failed, and we are back to square one.

So instead of defining a volume in $\mathbb{R}^n$ axiomatically, let us generally define what a reasonable notion of a volume should satisfy. Later we will then construct the Lebesgue measure that has most of the desired properties on $\mathbb{R}^n$. 
Figure 1.1. A ball can be decomposed into a finite number of disjoint sets and then reassembled into two balls identical to the original.

Clearly $\mu(\emptyset) = 0$ is a reasonable assumption. Ideally we would also like $\mu(A \cup B) = \mu(A) \cup \mu(B)$ – but this will be a surprisingly tricky, confusing, and paradox assumption, so let us settle for the following notion.

Definition 1.1. Let $X$ be any set and $2^X$ the potential set of $X$. A map $\mu : 2^X \rightarrow [0, \infty]$ is a measure on $X$ if we have

1. $\mu(\emptyset) = 0$
2. $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$ whenever $A, A_k \subset X$, $k \in \mathbb{N}$ and $A \subset \bigcup_{k \in \mathbb{N}} A_k$

Remark 1.2. Condition (1) and (2) implies monotonicity,

$$\mu(A) \leq \mu(B) \quad \forall A \subset B.$$ (simply set $A_1 := B$ and $A_k := \emptyset$ for $k \geq 2$).

In particular we could equivalently replace (2) above by $\sigma$-subadditivity, namely

$$\mu(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \mu(A_k).$$

Remark 1.3. • A word of warning: we will use here the notion of an outer measure that is defined on all of $2^X$, not only on its $\sigma$-algebra of measurable sets.

• In particular we have $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for any set $A, B \subset X$. However, in general, we cannot hope for all disjoint sets $A$ and $B$ hope that $\mu(A \cup B) = \mu(A) + \mu(B)$ (see above), this will lead to the notion of non-measurable sets.

Example 1.4 (Jordan content). • The outer Jordan content $J^*(E)$ of a set $E \subset \mathbb{R}^n$ is defined as follows.

For a product of bounded cubes $C = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$ we set $\text{vol}(C) := (b_1 - a_1) \cdot (b_2 - a_2) \cdot \ldots \cdot (b_n - a_n)$.

$$J^*(E) := \inf \left\{ \sum_{i=1}^{N} \text{vol}(C_i) \mid \text{for some } N \in \mathbb{N}, \text{ and cubes } (C_i)_{i=1}^{N} \text{ such that } E \subset \bigcup_{i=1}^{N} C_i \right\}$$

Here we follow the convention that $\inf \emptyset = +\infty$.

$J^*(\cdot)$ is not a measure: take any enumeration of $\mathbb{Q} \cap [0,1] = \{q_1, \ldots, q_n, \ldots\}$. Set $A_k := \{q_k\}$ and $A := \bigcup_{k=1}^{\infty} A_k = [0,1] \cap \mathbb{Q}$. If $(C_i)_{i=1}^{\infty}$ is a finite cover of $[0,1] \cap \mathbb{Q}$.


then $\bigcup_{i} \overline{C}_{i} \supset [0, 1]$, so $J^{*}(A) = 1$. However $J^{*}(A_{k}) = 0$ for each $k$, we have $J^{*}(A) \leq \sum_{k=1}^{\infty} J^{*}(A_{k})$.

However $J^{*}$ satisfies finite additivity,

$$
\mu(A \cup B) \leq \mu(A) + \mu(B),
$$

i.e.

$$
\mu(A) \leq \sum_{k=1}^{N} \mu(A_{k}) \quad \text{whenever } A, A_{k} \subset X, \ k \in \{1, \ldots, N\}, \ N \in \mathbb{N}, \text{ and } A \subset \bigcup_{k \in \mathbb{N}} A_{k}.
$$

Such a map $J^{\varepsilon} : 2^{X} \to [0, \infty)$ is called a content.

• The countable version of the outer Jordan content, is called the Lebesgue outer measure

$$(1.1) \quad m^{*}(E) := \inf \left\{ \sum_{i=1}^{\infty} \operatorname{vol}(C_{i}) \right\} \text{ for some }, \text{ and cubes } (C_{i})_{i=1}^{\infty} \text{ such that } E \subset \bigcup_{i=1}^{\infty} C_{i}$$

It is again clear that $m^{*}(\emptyset) = 0$. Let now $A \subset \bigcup_{k=1}^{n} A_{k}$. We may assume that $m^{*}(A_{k}) < \infty$ otherwise there is nothing to show. Fix $\varepsilon > 0$. For each $k$ we can pick $(C_{k;i})_{i=1}^{\infty}$ such that $\bigcup_{i=1}^{\infty} C_{k;i} \supset A_{k}$ and

$$
\sum_{i=1}^{\infty} \operatorname{vol}(C_{k;i}) \leq m^{*}(A_{k}) + \frac{\varepsilon}{2^{k}}.
$$

Now $\bigcup_{k,i \in \mathbb{N}} C_{k;i} \supset A$ and thus

$$
m^{*}(A) \leq \sum_{k,i \in \mathbb{N}} \operatorname{vol}(C_{k;i}) \leq \sum_{k=1}^{\infty} m^{*}(A_{k}) + \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k}}
$$

That is, we have shown that for any $\varepsilon > 0$,

$$
m^{*}(A) \leq \sum_{k=1}^{\infty} m^{*}(A_{k}) + \varepsilon
$$

Taking $\varepsilon \to 0$ we conclude that $m^{*}(A) \leq \sum_{k=1}^{\infty} m^{*}(A_{k})$ – that is $m^{*}(A)$ is indeed a measure.

Later the Lebesgue measure $L^{n}$ will coincide with $m^{*}(A)$.

• The inner Jordan content,

$$
J_{*}(E) := \sup \left\{ \sum_{i=1}^{N} \operatorname{vol}(C_{i}) \right\} \text{ for some } N \in \mathbb{N}, \text{ and cubes } (C_{i})_{i=1}^{N} \text{ such that } \bigcup_{i=1}^{N} C_{i} \subset E
$$

Here we follow the convention that sup $\emptyset = 0$.

Still $J_{*}(\cdot)$ is not a measure. Take $A_{1} := [0, 1] \setminus \mathbb{Q}$ and for $i \geq 2$ we set $A_{i} = \{q_{i}\}$ for $\{q_{2}, \ldots, \}$ $= \mathbb{Q} \cap [0, 1]$ any enumeration of $\mathbb{Q} \cap [0, 1]$. Since $A_{1}$ has empty interior we have $J_{*}(A_{1}) = 0$. Similarly, $J_{*}(A_{i}) = 0$ for $i \geq 2$. However $A := \bigcup_{i=1}^{\infty} A_{i} = [0, 1]$ satisfies $J_{*}([0, 1]) = 1$. So we have $J_{*}(A) \not\leq \sum_{i=1}^{n} J_{*}(A_{i})$.

\[\text{Indeed, take } r \in [0, 1] \text{ then there exists } q_{k} \text{ converging to } r, \ q_{k} \text{ belongs infinitely often to the same interval, so } r \in \overline{U}_{i} \text{ for some } i.\]
• If we simply make the inner Jordan content countable, i.e. if we set
\[ \tilde{J}^*_s(E) := \sup \left\{ \sum_{i=1}^{\infty} \text{vol}(C_i) \mid \text{for cubes } (C_i)_{i=1}^{\infty} \text{ such that } \bigcup_{i=1}^{\infty} C_i \subset E \right\} \]
we run into the same problem as for \( J^*_s \), namely
\[ J^*_s([0,1] \setminus \mathbb{Q}) = 0. \]
So \( \tilde{J}^*_s(E) \) is still not a measure.

**Example 1.5** (Counting measure). Let \( X \) be any set. Then \#\( 2^X \rightarrow \mathbb{N} \cup \{0\} \) defined by
\[ \#A := \text{number of elements in } A, \]
is a measure, called the **counting measure**.

**Exercise 1.6.** Let \( X \) be a metric space and \( \mu : 2^X \rightarrow [0,\infty] \) a measure. Let \( A \subset X \) then the measure \( \mu_{\cap} A : 2^X \rightarrow [0,\infty] \) given by
\[ (\mu_{\cap} A)(B) := \mu(A \cap B) \]
is a measure.

1.1. **Example: Hausdorff measure.** Let \((X,d)\) be a metric space.

**Definition 1.7.** The \( s \)-dimensional **Hausdorff measure**, \( s > 0 \) is defined as follows.

Let \( \delta \in (0,\infty] \), then for any \( A \subset X \) we define
\[ \mathcal{H}^s_\delta(A) := \alpha(s) \inf \left\{ \sum_{k=1}^{\infty} r_k^s : A \subset \bigcup_{k=1}^{\infty} B(x_k, r_k), r_k \in (0,\delta) \right\}. \]

Here \( B(x_k, r_k) \) are open balls with radius \( r \) centered at \( x_k \), i.e.
\[ B(x_k, r_k) := \{ y \in X : d(x_k, y) < r_k \}. \]

Moreover\(^2\)
\[ \alpha(s) := \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2} + 1\right)}. \]
where \( \Gamma \) is the Gamma function.

Now observe that \( \delta \mapsto \mathcal{H}^s_\delta(A) \) is monotone decreasing. So we can write
\[ \mathcal{H}^s(A) := \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(A) \equiv \sup_{\delta > 0} \mathcal{H}^s_\delta(A) \in [0,\infty]. \]

Often one writes \( \mathcal{H}^0(A) := \#A \), the **counting measure**.

\( \mathcal{H}^s_\infty \) is called the **Hausdorff content**.

**Remark 1.8.** • Observe that while \( \mathcal{H}^s_\delta(A) < \infty \) whenever \( s > 0 \), \( \delta > 0 \) and \( A \) is any bounded set, as \( \delta \to 0 \) \( \mathcal{H}^s(A) \) will be infinite whenever \( s \) is smaller than the “dimension of \( A \)” (a notion we will define more carefully below).

\(^2\)Warning: Some authors set \( \alpha(s) := 1 \). The main reason to not do that is so that \( \mathcal{H}^n = \mathcal{L}^n \) in \( \mathbb{R}^n \)
Lemma 1.9. $\mathcal{H}^s$ is a measure in $\mathbb{R}^n$.

Proof. One can show similar to the argument for $m^*$ that $\mathcal{H}^s_\delta(\cdot)$ is a measure for each $\delta > 0$. We clearly have $\mathcal{H}^s(\emptyset) = 0$. Moreover, since $\mathcal{H}^s_\delta$ is a measure for any $\delta > 0$, we have for any $A \subset \bigcup_{k=1}^\infty A_k$,

$$\mathcal{H}^s(A) \leq \sum_{k=1}^\infty \mathcal{H}^s(A_k).$$

Taking the supremum over $\delta$ in this inequality we have $\sigma$-additivity for $\mathcal{H}^s$.

Exercise 1.10. Show that

$$\mathcal{H}^0_\delta(Q) \xrightarrow{\delta \to 0} \infty.$$

Lemma 1.11. $\mathcal{H}^s$ is a metric outer measure that means that if $A, B \subset (X, d)$ satisfy

$$d(A, B) := \inf_{a \in A, b \in B} d(a, b) > 0.$$

then

$$\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B).$$

Proof. This is relatively easy to see. Take $\delta < \frac{d(A, B)}{3}$, then since any covering $B(x, r)$ with $r < \delta$ cannot contain points of both $A$ or $B$ at the same time, we have that $\mathcal{H}^s_\delta$ is a metric outer measure. Taking $\delta \to 0^+$ we obtain that $\mathcal{H}^s$ is a metric outer measure.

Remark 1.12. One can, and we will in Corollary 1.79, show that the $n$-dimensional Hausdorff measure in $\mathbb{R}^n$ coincides with the Lebesgue measure $\mathcal{L}^n$, i.e.

$$\mathcal{L}^n(A) = \mathcal{H}^n(A).$$

Exercise 1.13. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz continuous that is

$$|f(x) - f(y)| \leq L|x - y|.$$

Then for any set $\Omega$,

$$\mathcal{H}^{\min\{n, m\}}(f(\Omega)) \leq L\mathcal{H}^{\min\{n, m\}}(\Omega).$$

Exercise 1.14. Show that

- $\mathcal{H}^1 = L^1$ in $\mathbb{R}$
- $\mathcal{H}^s(\lambda A) = \lambda^s\mathcal{H}^s(A)$ for all $\lambda > 0$, where $\lambda A = \{\lambda x : x \in A\}$.
- $\mathcal{H}^s(LA) = \mathcal{H}^s(A)$ whenever $L : \mathbb{R}^n \to \mathbb{R}^n$ is an affine isometry, i.e. if $Lx = Ax + b$ for $A \in O(n)$ and $b \in \mathbb{R}^n$ constant.

Exercise 1.15. Let $U \subset \mathbb{R}^n$ be any non-empty open set. Then $\mathcal{H}^s(U) = \infty$ for all $s < n$. 
Exercise 1.16 (translation and rotation invariant). Let $A \subset \mathbb{R}^n$ and $s \in (0, \infty)$. Show the following

1. If $p \in \mathbb{R}^n$ then $H^s(p + A) = H^s(A)$.
2. If $O \in O(n)$ (i.e. $O \in \mathbb{R}^{n \times n}$ and $O^t O = I$) then $H^s(OA) = H^s(A)$.
3. If $A \subset \mathbb{R}^\ell \times \{0\}$ for $0 < \ell < n$ and $\pi: (x_1, \ldots, x_n) := (x_1, \ldots, x_\ell)$ is the projection from $\mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}^{n-\ell}$ to $\mathbb{R}^\ell$, then $H^s_{\mathbb{R}^n}(A) = H^s_{\mathbb{R}^\ell}(\pi(A))$.

Lemma 1.17. Let $0 \leq s < t < \infty$.

1. If $H^t(A) < \infty$ then $H^s(A) = 0$
2. If $H^t(A) > 0$ then $H^s(A) = \infty$.

Proof. Indeed, whenever $r_k \leq \delta$ and $(B(x_k, r_k))_{k \in \mathbb{N}}$ cover $A$ we have

$$H^t_\delta(A) \leq \alpha(t) \sum_{k=1}^\infty r_k^\ell \leq \alpha(t) \delta^{t-s} \sum_{k=1}^\infty r_k^s.$$  

Taking the infimum over any such covering $(B(x_k, r_k))$ of $A$ we find

$$H^s_\delta(A) \leq \frac{\alpha(t)}{\alpha(s)} \delta^{t-s} H^s_\delta(A).$$

Taking $\lim_{\delta \to 0}$ on both sides we obtain

$$H^t(A) \leq \frac{\alpha(t)}{\alpha(s)} 0 \cdot H^s(A).$$

This implies that if $H^t(A) > 0$ then necessarily $H^s(A) = \infty$, and if $H^s(A) < \infty$ then $H^t(A) = 0$. \qed

Example 1.18. If $k \in \mathbb{N}$ it is conceivable that $H^k$ measures something of “dimension $k$”. For example assume that $C = [0, 1]^2 \times \{0\} \subset \mathbb{R}^3$ is a 2D-square of sidelength 1. We need $\approx \frac{1}{\delta^2}$ many balls to cover $C$. Then

$$H^s_\delta(C) \leq \alpha(s) \frac{1}{\delta^2} \delta^s.$$  

So if $s > 2$ we see that $H^s(C) \leq \lim_{\delta \to 0} \delta^{s-2} = 0$. That is $C$ has no $s$-volume for $s > 2$.

For $s = 2$ one can argue that covering uniformly by balls of radius $\delta$ is optimal and thus we have

$$0 < H^2(C) < \infty.$$  

In particular $H^2(C) = \infty$ for any $s < 2$.

(this argument is easy to generalize to a $\ell$-dimensional manifold in $\mathbb{R}^N$)

Indeed, with the Hausdorff measure we can define a dimension
**Definition 1.19.** The **Hausdorff dimension** is defined as

\[
\dim_H A := \inf \{ s \geq 0 : \mathcal{H}^s(A) = 0 \}.
\]

If \( \mathcal{H}^s(A) > 0 \) for all \( s > 0 \) then \( \dim_H(E) := \infty \).

**Lemma 1.20.** Let \( C \) be a set in a metric space and let \( s \geq 0 \)

1. If \( \mathcal{H}^s(C) = 0 \) then \( \dim_H(E) \leq s \).
2. If \( \mathcal{H}^s(C) > 0 \) then \( \dim_H(E) \geq s \).
3. If \( 0 < \mathcal{H}^s(C) < \infty \) then \( \dim_H(E) = s \).
4. If \( \mathcal{H}^\infty(C) > 0 \) and \( \mathcal{H}^s(C) < \infty \) then \( \dim_H(E) = s \).

**Proof.** This follows from Lemma 1.17 and the definition of Hausdorff measure.

(1) follows from the definition of the Hausdorff measure as infimum. Then \( \dim_H(E) \leq s \).
(2) If \( \mathcal{H}^s(C) > 0 \) then by Lemma 1.17 \( \mathcal{H}^t(C) = \infty \) for all \( t < s \). Again from the definition it is clear that \( \dim_H(E) \geq s \).
(3) This is a consequence of the two above statements.
(4) Follows from the statement before since \( \mathcal{H}^\infty(C) \leq \mathcal{H}^s(C) \)

\( \square \)

**Exercise 1.21** (Hausdorff dimension under Lipschitz and Hölder maps). Let \( (X, d_x) \) and \( (Y, d_Y) \) be two metric spaces and let \( f : X \to Y \). Assume that \( A \subset X \) has Hausdorff-dimension \( \dim_H(A) = s \).

1. If \( f \) is uniformly Lipschitz continuous, i.e. for some \( L > 0 \),
\[
d_Y(f(x), f(y)) \leq L d(x, y) \quad \forall x, y \in X
\]
then \( \dim_H(f(A)) \leq s \).
2. Give an example where \( \dim_H(A) < s \)
3. Assume \( f \) is uniformly Hölder continuous, i.e. for some \( L > 0 \) and \( \alpha > 0 \)
\[
d_Y(f(x), f(y)) \leq L d(x, y)^\alpha \quad \forall x, y \in X
\]
What can we say about the Hausdorff dimension of \( f(A) \subset Y \)?

Cf Exercise 1.13.

**Example 1.22.** The Cantorset is defined as follows.

\[
C_0 := [0, 1]
\]
Let \( C_0 := [0, 1] \). In the \( k \)-th step we construct \( C_k \) by removing of each interval the open middle interval of size \( 3^{-n} \). For example

\[
C_1 := [0, \frac{1}{3}] \cup [\frac{1}{3}, 1].
\]
See Figure 1.2.
Figure 1.2. The cantor set

Set $C := \bigcap_{k=1}^\infty C_k$. Observe that $C$ is closed and bounded, so compact.

**Lemma 1.23.** $\dim_H(C) = \frac{\log 2}{\log 3}$.

**Proof.** For each $k \in \mathbb{N}$ we have $C \subset C_k$. Observe that $C_k$ consists of $2^k$ disjoint intervals each of diameter $3^{-k}$ (i.e. radius $\frac{1}{2} 3^{-k}$). Thus for any $\delta > 0$ and for any $k \gg 1$ so that $\frac{1}{2} 3^{-k} < \delta$ we have

$$\mathcal{H}_\delta^s(C) \leq \alpha(s) \sum_{\ell=1}^{2^k} \left(\frac{1}{2} 3^{-k}\right)^s = 2^{-s} \left(\frac{2}{3}\right)^k \xrightarrow{k \to \infty} \alpha(s) \begin{cases} 2^{-s} & s = \frac{\log 2}{\log 3} \\ 0 & s > \frac{\log 2}{\log 3} \\ \infty & s > \frac{\log 2}{\log 3} \end{cases}$$

In particular we have

$$\mathcal{H}^s(C) = 0 \quad \forall s > \frac{\log 2}{\log 3}.$$

So from the definition of the Hausdorff dimension we get

$$\dim_H C \leq \frac{\log 2}{\log 3}.$$

Now we need to show the other direction. From now on set $s := \frac{\log 2}{\log 3}$. Let $(B(x_i, r_i))_{i=1}^\infty$ be any covering of $C$. We claim that

$$(1.2) \quad \sum_{i=1}^\infty r_i^s \geq \frac{1}{2^s 4}.$$  

Once we have (1.2) we are done, because (1.2) implies

$$\mathcal{H}^s(C) \geq \frac{1}{2^s 4}.$$  

In particular (recall that $s = \frac{\log 2}{\log 3}$) we have $\infty > \mathcal{H}^s(C) \geq \mathcal{H}^s_\infty(C) > 0$.

Let us make some notation. Denote by $A_k$ the intervals of $C_k$, i.e. $A_k$ consists of pairwise disjoint, closed intervals in $\mathbb{R}$ such that $C_k = \bigcup_{I \in A_k} I$. E.g.

$$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1], \quad A_1 = \{[0, \frac{1}{3}], [\frac{2}{3}, 1]\}.$$
Figure 1.3. If a ball intersects three intervals of $A_{K_i}$ its diameter is at least $5 \cdot 3^{-K_i}$.

Proof of (1.2) Since $C$ is compact, we may assume that there finitely many, w.l.o.g. the first $N$ balls $(B(x_i, r_i))_{i=1}^N$ already cover $C$. We may assume that each $r_i < \frac{1}{2}$, otherwise (1.2) is obvious.

Fix $i \in \{1, \ldots, N\}$.

Let $K_i \in \mathbb{N} \cup \{0\}$ so that $2r_i \in [3^{-K_i-1}, 3^{-K_i})$.

Now we consider the construction step $C_{K_i}$. Each ball $B(x_i, r_i)$ has nonempty intersection with at most 2 intervals of $C_{K_i}$. Indeed, otherwise its diameter would be at least $5 \cdot 3^{-K_i}$, see Figure 1.3.

But then $B(x_i, r_i)$ has nonempty intersection with at most $2 \cdot 2^{j-K_i}$ intervals of $C_j$ for any $j \geq K_i$. Since $s = \frac{\log 2}{\log 3}$ we have

$$2 \cdot 2^{j-K_i} = 2^{j+1} 2^{-K_i} = 2^{j+1} 3^{-K_i s} \leq 2^{j+1} 3^s (2r_i)^s = 2^{j+2} (2r_i)^s.$$ 

Set now $K := \max_{i=1}^N K_i$.

Then for any $i \in \{1, \ldots, N\}$ each of the balls $B(x_i, r_i)$ has nonempty intersection with at most $2^{K+2}(2r_i)^s$ many intervals of $A_K$.

So if we set $\Gamma_i$ to be the number of intervals in $A_K$ that intersect $B_{r_i}(x_i)$ we have $\Gamma_i \leq 2^{K+2}(2r_i)^s$ and thus

$$\sum_{i=1}^N \Gamma_i (3^{-K})^s \leq \sum_{i=1}^N (3^{-K})^s 2^{K+2}(2r_i)^s = 4 \cdot 2^s \sum_{i=1}^N (r_i)^s \tag{1.3}$$

Now for each $x \in C$ there is exactly one interval $I$ in $A_K$ such that $x \in I$. Since $(B_{r_i}(x_i))_{i=1}^N$ covers all of $C$ we have the following: for each interval $I$ in $A_K$ there exists some $i \in \{1, \ldots, N\}$ such that $B_{r_i}(x_i) \cap I \neq \emptyset$. That is,

$$\sum_{i=1}^N \Gamma_i \geq \text{number of intervals in } A_K = 2^K.$$
Thus,

\[(1.4) \sum_{i=1}^{N} \Gamma_i 3^{-Ks} \geq 2^{K} 3^{-Ks} = 1.\]

Together, (1.3) and (1.4) imply (1.2).

\[\Box\]

**Example 1.24.** The Smith–Volterra–Cantor set, aka fat cantor set is defined as follows.

Let \(C_0 := [0, 1]\). In the \(k\)-th step we construct \(C_k\) by removing of each interval the open middle interval of size \(a^n\). That is

\[
C_1 = [0, \frac{1-a}{2}] \cup \left[\frac{1+a}{2}, 1\right].
\]

\[
C_2 = [0, \frac{1-a}{4} - \frac{a^2}{2}] \cup \left[\frac{1-a}{4} + \frac{a^2}{2}, 1-a\right] \cup \left[\frac{1+a}{2}, \frac{1}{2} + \frac{1+a}{2} - \frac{a^2}{2}\right] \cup \left[\frac{1}{2} + \frac{1+a}{2} + \frac{a^2}{2}, 1\right].
\]

Cf. Figure 1.4.

Set \(C := \bigcap_{k=1}^{\infty} C_k\). For \(a = \frac{1}{3}\) this is the typical *Cantor set*. For \(a = \frac{1}{4}\) this is the *Fat Cantor set*.

**Exercise 1.25.** The fat Cantor above set has positive \(\mathcal{H}^1\)-measure.

1.2. **Measurable sets.** As we have discussed, our definition of measure does not include the “natural” condition that \(\mu(B) = \mu(B \cap A) + \mu(B \setminus A)\) for all \(A, B \subset X\) – because this “natural” condition leads to incompatibility such as the Banach-Tarski Paradoxon.

So we will denote the class of sets \(A \subset 2^X\) where we have the above “natural” condition as the \(\sigma\)-algebra of measurable sets.

**Definition 1.26** (Carathéodory). Let \(\mu\) be a measure on \(X\).

\(A \subset X\) is called \(\mu\)-measurable if

\[\mu(B) = \mu(A \cap B) + \mu(B \setminus A)\] for any \(B \subset X\)

**Remark 1.27.** By additivity of the measure, measurability is equivalent to

\[\mu(B) \geq \mu(A \cap B) + \mu(B \setminus A)\] for any \(B \subset X\)

**Exercise 1.28.** Let \(X \neq \emptyset\) be any set
• and assume $\mu(\emptyset) = 0$ and $\mu(A) = 1$ for any $A \neq \emptyset$. Then $A$ is $\mu$-measurable if and only if $A = \emptyset$ or $A = X$.
• If $\nu = \#$ the counting measure then any set $A$ is $\nu$-measurable.

Clearly, whatever choice of measure we have, $\emptyset$ and $X$ are measurable sets. We also have

$$(1.5) \quad (A_i)_{i=1}^N \text{ are measurable} \Rightarrow \bigcup_{i=1}^N A_i \text{ is measurable}$$

**Proof of (1.5).** We proof this by induction. Clearly this holds for $N = 1$. So to conclude (1.5) we only need to show:

If $A_1, A_2$ are $\mu$-measurable, then so is $A_1 \cup A_2$.

So assume $A_1$ and $A_2$ are $\mu$-measurable and $B \subset X$.

$$\mu(B) = \mu(B \setminus A_1) + \mu(B \cap A_1)$$
$$= \mu((B \setminus A_1) \cap A_2) + \mu((B \setminus A_1) \setminus A_2)$$
$$+ \mu((B \cap A_1) \cap A_2) + \mu((B \cap A_1) \setminus A_2)$$
$$\geq \mu(B \setminus (A_1 \cup A_2)) + \mu(B \cap (A_1 \cup A_2))$$

In the last step we have used that

$$\mu((B \setminus A_1) \cap A_2) + \mu((B \cap A_1) \cap A_2) + \mu((B \cap A_1) \setminus A_2) \geq \mu(B \setminus (A_1 \cup A_2)),$$

by sublinearity and the fact that

$$B \setminus (A_1 \cup A_2) = ((B \setminus A_1) \cap A_2) \cup ((B \cap A_1) \cap A_2) \cup ((B \cap A_1) \setminus A_2).$$

By Remark 1.27 we have that $(A_1 \cup A_2)$ is also measurable.  

We have much more than that:

**Lemma 1.29.** Let $X$ be a set and $\mu$ be a measure on $X$.

The collection $\mathcal{A} \subset 2^X$ of $\mu$-measurable functions
$$\mathcal{A} := \{ A \subset X : \text{ A is } \mu\text{-measurable} \}$$

is a $\sigma$-algebra, that is

1. $X \in \mathcal{A}$
2. $A \in \mathcal{A}$ implies that $X \setminus A \in \mathcal{A}$
3. If $(A_i)_{i=1}^\infty \subset \mathcal{A}$ then $\bigcup_{i=1}^\infty A_i \in \mathcal{A}$.\(^3\)

In particular

\(^3\)This is the $\sigma$ in $\sigma$-algebra, $\sigma$ means for countably many. If we only had for any $N \in \mathbb{N}$: $(A_i)_{i=1}^N \subset \mathcal{A}$ then $\bigcup_{i=1}^N A_i \in \mathcal{A}$, $\mathcal{A}$ would be merely an Algebra (no $\sigma$!)
• $\emptyset \in A$

• if $(A_i)_{i=1}^{\infty} \subset A$ then $\cap_{i=1}^{\infty} A_i \in A$

Proof. 

(1) For any $B \subset X$: since $B \cap X = B$ and $B \setminus X = \emptyset$ we have

$$\mu(B) = \mu(B) + \mu(\emptyset) = \mu(B \cap X) + \mu(B \setminus X).$$

(2) Assume that $A \in A$. Set $\tilde{A} := X \setminus A$. For any $B \subset X$ we have

$$\tilde{A} \cap B = (X \setminus A) \cap B = B \setminus A,$$

and

$$B \setminus \tilde{A} = B \setminus (X \setminus A) = B \cap A.$$ 

Since $A$ is measurable we then have

$$\mu(B \cap \tilde{A}) + \mu(B \setminus \tilde{A}) = \mu(B \setminus A) + \mu(B \cap A) = \mu(B).$$

(3) Let $(A_i)_{i \in \mathbb{N}} \subset A$. Set $A := \bigcup_{i=1}^{\infty} A_i$.

Without loss of generality we have that $A_i \cap A_j = \emptyset$ for $i \neq j$. Indeed, otherwise we set $\tilde{A}_1 := A_1$ and $\tilde{A}_k := A_k \setminus \bigcup_{i=1}^{k-1} A_i$. By the previously proven properties and (1.5) each $\tilde{A}_k$ belongs to $A$ and we have $A = \bigcup_{k=1}^{\infty} \tilde{A}_k$ – so we could work with $\tilde{A}_k$ instead of $A_k$.

We have by measurability of each $A_k$ and since $A_N$ and $\bigcup_{k=1}^{N-1} A_k$ are disjoint,

$$\mu(B \cap \bigcup_{k=1}^{N} A_k) = \mu(B \cap \left( \bigcup_{k=1}^{N} A_k \right) \cap A_N) + \mu(B \cap \left( \bigcup_{k=1}^{N} A_k \right) \setminus A_N)$$

$$= \mu(B \cap A_N) + \mu(B \cap \bigcup_{k=1}^{N-1} A_k).$$

Repeating this computation $N - 1$ times we obtain

$$\mu(B \cap \bigcup_{k=1}^{N} A_k) = \sum_{k=1}^{N} \mu(B \cap A_k).$$

By (1.5) and the monotonicity of $\mu$, Remark 1.2, we then have

$$\mu(B) = \mu(B \cap \bigcup_{k=1}^{N} A_k) + \mu(B \setminus \bigcup_{k=1}^{N} A_k) \geq \sum_{k=1}^{N} \mu(B \cap A_k) + \mu(B \setminus \bigcup_{k=1}^{\infty} A_k).$$

This holds for any $N$, so we obtain

$$\mu(B) \geq \sum_{k=1}^{\infty} \mu(B \cap A_k) + \mu(B \setminus \bigcup_{k=1}^{\infty} A_k).$$

By the $\sigma$-subadditivity of $\mu$ we then have

$$\mu(B) \geq \mu(B \cap \bigcup_{k=1}^{\infty} A_k) + \mu(B \setminus \bigcup_{k=1}^{\infty} A_k).$$

In view of Remark 1.27 this implies measurability of $\bigcup_{k=1}^{\infty} A_k$. 
Definition 1.30. Let $\mathcal{C} \subset 2^X$ any nonempty family of subsets of $X$, then

$$\sigma(\mathcal{C})$$

denotes the $\sigma$-Algebra generated by $\mathcal{C}$, namely the smallest $\sigma$-algebra containing $\mathcal{C}$.

Exercise 1.31. • $\{\emptyset, X\}$ is a $\sigma$-algebra of $X$
• $2^X$ is a $\sigma$-algebra of $X$
• Let $(X, d)$ be a metric space. Denote $\mathcal{O} \subset 2^X$ the family of all open sets. Let $\mathcal{F}$ be the family of $\sigma$-Algebras that contain all open sets. That is, $\mathcal{A} \subset 2^X$ belongs to $\mathcal{F}$ if and only if $\mathcal{A}$ is a $\sigma$-Algebra, and any open set $O \in \mathcal{O}$ belongs to $\mathcal{A}$, i.e. $O \in \mathcal{A}$.

Define $B := \bigcap\{\mathcal{A} : \mathcal{A} \in \mathcal{F}\}$.

Show that (a) $\mathcal{F}$ is nonempty, (b) $B$ is a $\sigma$-algebra and (c) $B$ is the smallest $\sigma$-Algebra containing all open sets, i.e. show that $B = \sigma(\mathcal{O})$.

$B$ is called the Borel $\sigma$-Algebra and a set $B \in B$ is called a Borel set.

Definition 1.32. If $\mu : 2^X \to [0, \infty]$ is a measure on $X$, and $\Sigma$ is the $\sigma$-algebra of $\mu$-measurable sets, then once calls $(X, \Sigma, \mu)$ a measure space.

Some author choose to define measures only on their $\sigma$-algebra $\Sigma$ of measurable sets, and call our definition of a measure an outer measure.

There is a reason for restricting $\mu$ only to act on measurable sets – a measure $\mu$ acts in a very intuitive way on its measurable sets!

Theorem 1.33. Let $(X, \Sigma, \mu)$ be a measure space.

Let $(A_k)_{k \in \mathbb{N}} \subset \Sigma$ (i.e. each $A_k$ is measurable). Then we have

1. If $A_k \cap A_\ell = \emptyset$ for $k \neq \ell$ we have

$$\mu(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$$

(\sigma\text{-Additivity})

2. If $A_1 \subset A_2 \subset \ldots \subset A_k \subset \ldots$ then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k \to \infty} \mu(A_k).$$

3. If $\mu(A_1) < \infty$ and $A_1 \supset A_2 \supset \ldots \supset A_k \supset \ldots$ then

$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k \to \infty} \mu(A_k).$$
Proof. (1) Above in (1.6) we computed (take $B = X$) that for finitely many pairwise disjoint sets

$$\mu(\bigcup_{k=1}^{N} A_k) = \sum_{k=1}^{N} \mu(A_k).$$

By monotonicity

$$\mu(\bigcup_{k=1}^{\infty} A_k) \geq \mu(\bigcup_{k=1}^{N} A_k) = \sum_{k=1}^{N} \mu(A_k)$$

Taking $N \to \infty$ we find

$$\mu(\bigcup_{k=1}^{\infty} A_k) \geq \sum_{k=1}^{\infty} \mu(A_k).$$

By $\sigma$-subadditivity,

$$\mu(\bigcup_{k=1}^{\infty} A_k) \geq \sum_{k=1}^{\infty} \mu(A_k) \geq \mu(\bigcup_{k=1}^{\infty} A_k).$$

The number on left- and right-hand side are the same so we have

$$\sum_{k=1}^{\infty} \mu(A_k) = \mu(\bigcup_{k=1}^{\infty} A_k).$$

(2) Let $\tilde{A}_1 := A_1$ and $\tilde{A}_k := A_k \setminus A_{k-1}$. Then $(\tilde{A}_k)_{k=1}^{\infty}$ is pairwise disjoint, and each $\tilde{A}_k$ is measurable. So

$$\mu(\bigcup_{k=1}^{\infty} A_k) = \mu(\bigcup_{k=1}^{\infty} \tilde{A}_k) = \sum_{k=1}^{\infty} \mu(\tilde{A}_k)$$

$$= \lim_{N \to \infty} \sum_{k=1}^{N} \mu(\tilde{A}_k)$$

$$= \lim_{N \to \infty} \mu(\bigcup_{k=1}^{N} \tilde{A}_k)$$

$$= \lim_{N \to \infty} \mu(A_N).$$

(3) Set $\tilde{A}_k := A_1 \setminus A_k$, $k \in \mathbb{N}$. Then $\emptyset = \tilde{A}_1 \subset \tilde{A}_2 \subset \ldots$. Moreover we have

$$\mu(A_1) = \mu(\tilde{A}_k) + \mu(A_k), \quad k \in \mathbb{N}. $$
By the above argument (observe that \( \mu(A_k) \leq \mu(A_1) < \infty \))

\[
\mu(A_1) - \lim_{k \to \infty} \mu(A_k) = \lim_{k \to \infty} \mu(\hat{A}_k)
\]

\[
= \mu\left( \bigcup_{k=1}^{\infty} \hat{A}_k \right)
\]

\[
= \mu(A_1 \setminus \bigcap_{k=1}^{\infty} A_k)
\]

\[
= \mu(A_1) - \mu(\bigcap_{k=1}^{\infty} A_k).
\]

Since \( \mu(A_1) < \infty \) we can conclude.

\[\square\]

**Example 1.34.**

- There is no way we can assume (or should hope for) that for uncountable unions we have (even sub-)additivity: For example \( \mathbb{R} = \bigcup_{x \in \mathbb{R}} \{x\} \). The Lebesgue measure would satisfy \( \mathcal{L}^1\{x\} = 0 \) for all \( x \), so

\[
\mathcal{L}^1(\mathbb{R}) \neq \lim_{k \to \infty} \mathcal{L}^1\{x\} = 0.
\]

- The assumption \( \mu(A_1) < \infty \) in Theorem 1.33(3) is necessary.

Let \( X = \mathbb{N} \) and \( \mu \) the counting measure. Set for \( k \in \mathbb{N} \)

\[
A_k = \{k, k+1, k+2, \ldots\}.
\]

Then \( A_k \supset A_{k+1} \), but \( \mu(A_k) = \infty \) for all \( k \in \mathbb{N} \). However \( \bigcap_{k \in \mathbb{N}} A_k = \emptyset \), so

\[
0 = \mu(\bigcap_{k \in \mathbb{N}} A_k) \neq \lim_{k \to \infty} \mu(A_k) = \infty.
\]

Lastly, from Advanced Calculus we are aware of sets of measure zero (there it was that Riemann-integrable functions are continuous outside a set of Lebesgue-measure zero)

**Definition 1.35** (Zero sets). A set \( N \subset X \) is called a set of \( \mu \)-measure zero if \( \mu(N) = 0 \). We also say \( N \) is a \( \mu \)-zeroset.

A property \( P(x) \) holds \( \mu \text{-a.e.} \) in \( X \) if \( P(x) \) holds in \( X \setminus N \) where \( N \) is a \( \mu \)-zeroset.

**Theorem 1.36.** Let \( N \subset X \) be a \( \mu \)-zero set. Then \( N \) is measurable.

**Proof.** Let \( B \subset X \), by monotonicity we have \( \mu(B \cap N) \leq \mu(N) = 0 \). Moreover \( \mu(B \setminus N) \leq \mu(B) \). So we have

\[
\mu(B \cap N) + \mu(B \setminus N) \leq \mu(N) + \mu(B) = \mu(B).
\]

This implies measurability. \[\square\]

**Exercise 1.37.** Let \( X \) be a set and \( \mu \) be a measure on \( X \).

1. Let \( (N_i)_{i \in \mathbb{N}} \) be sets of \( \mu \)-measure zero. Show that \( \bigcup_{i \in \mathbb{N}} N_i \) is a set of \( \mu \)-measure zero.
(2) Let $N$ be a $\mu$-zeroset. Show that any $A \subset N$ is a $\mu$-zeroset.

(3) Show that if a property $P(x)$ holds for $\mu$-a.e. $x$ and a property $Q(y)$ holds for $\mu$-a.e. $y$ then $Q(x)$ and $P(x)$ hold (simultaneously) for a.e. $x$.

1.3. Construction of Measures: Carathéodory-Hahn Extension Theorem. While we already have constructed in (1.1) the Lebesgue (outer) measure, we are still interested in finding a more axiomatic approach to construct the Lebesgue measure.

The idea is to define a pre-measure on some sets (like cubes!) and build a measure out of that.

**Definition 1.38 (Algebra).** Let $X$ be a set and $\mathcal{A} \subset 2^X$. $\mathcal{A}$ is called an algebra if

1. $X \in \mathcal{A}$
2. $A \in \mathcal{A}$ implies that $X \setminus A \in \mathcal{A}$
3. If $A_1, A_2 \in \mathcal{A}$ then $A_1 \cup A_2 \in \mathcal{A}$.

**Definition 1.39 (Pre-measure).** Let $X$ be a set, $\mathcal{A} \subset 2^X$ an algebra (not necessarily a $\sigma$-algebra!). A map $\lambda : \mathcal{A} \to [0, \infty]$ is a pre-measure, if

1. $\lambda(\emptyset) = 0$
2. $\lambda(A) = \sum_{k=1}^{\infty} \lambda(A_k)$ for any $A \in \mathcal{A}$ such that $A = \bigcup_{k=1}^{\infty} A_k$ for some pairwise disjoint $(A_k)_{k=1}^{\infty} \subset \mathcal{A}$.

A pre-measure is called $\sigma$-finite, if $X = \bigcup_{k=1}^{\infty} S_k$ with $S_k \in \mathcal{A}$ and $\lambda(S_k) < \infty$ for each $k$.

**Exercise 1.40.** Let $\lambda : \mathcal{A} \to [0, \infty]$ be a premeasure

1. Assume $A \subset B$ with $A, B \in \mathcal{A}$ then $\lambda(A) \leq \lambda(B)$.

**Example 1.41.** An interval is the set $(a, b)$ or $[a, b]$ or $(a, b]$ or $[a, b)$, where $-\infty < a \leq b < \infty$ (we allow $\pm \infty$ if the set is open).

A block $Q$ in $\mathbb{R}^n$ is the cartesian product of $n$ intervals $Q = I_1 \times I_2 \times \ldots \times I_n$.

A figure is made out of finitely many blocks

$$\mathcal{A} := \left\{ A \subset \mathbb{R}^n : A = \bigcup_{i=1}^{N} Q_i \quad \text{some } N \in \mathbb{N}, Q_i \text{ blocks with pairwise disjoint interior} \right\}.$$ 

Exercise: $\mathcal{A}$ is an Algebra

The volume of a block $Q = I_1 \times \ldots \times I_n$ is given by the $n$-dimensional volume, i.e.

$$\text{vol}(Q) = \prod_{i=1}^{n} |I_i|,$$

where as usual, $|[a, b]| = |(a, b)| = |[a, b)| = |(a, b]| := |b - a|.$

Indeed, vol defines now a premeasure on $\mathcal{A}$:
Whenever \( A \in \mathcal{A} \), i.e. \( A = \bigcup_{i=1}^{N} Q_i \) for pairwise disjoint blocks \( Q_i \) then
\[
\text{vol}(A) := \sum_{i=1}^{N} \text{vol}(Q_i).
\]

One can check that this is independent of the specific choice of \( Q_i \). I.e. if also \( A = \bigcup_{j=1}^{N} \tilde{Q}_j \) for another combination of pairwise disjoint blocks \( \tilde{Q}_j \) then
\[
\sum_{i=1}^{N} \text{vol}(Q_i) = \sum_{j=1}^{\tilde{N}} \text{vol}(\tilde{Q}_j).
\]

Indeed, now vol defines a pre-measure on \( \mathcal{A} \). vol is also \( \sigma \)-finite, simply take \( S_k := [-k, k]^n \).

The important thing is that a premeasure extends (more or less uniquely) into a real measure. This is called Carathéodory–Hahn extension.

**Theorem 1.42** (Carathéodory–Hahn extension). Let \( X \) be a set, and \( \mathcal{A} \subset 2^X \) an algebra with a premeasure \( \lambda : \mathcal{A} \to [0, \infty] \).

For \( A \subset X \) the Carathéodory–Hahn extension \( \mu : 2^X \to [0, \infty] \) is defined as
\[
\mu(A) := \inf \left\{ \sum_{k=1}^{\infty} \lambda(A_k) : A \subset \bigcup_{k=1}^{\infty} A_k, \ A_k \in \mathcal{A} \right\}
\]

Then

1. \( \mu : 2^X \to [0, \infty] \) is a measure on \( X \),
2. \( \mu(A) = \lambda(A) \) for all \( A \in \mathcal{A} \),
3. Any \( A \in \mathcal{A} \) is \( \mu \)-measurable.

Compare this to the definition of the Lebesgue measure, (1.1).

**Proof of Theorem 1.42.** (1) Clearly \( \mu : 2^X \to [0, \infty] \) is well-defined (observe that \( X \in \mathcal{A} \), so that \( \mu(B) \leq \mu(X) \) for all \( B \subset X \), and \( \mu(\emptyset) = 0 \).

Now let \( B \subset \bigcup_{k=1}^{\infty} B_k \). Take any \( \varepsilon > 0 \). By definition of the infimum, for each \( B_k \) there exist some \( (A_{k;\ell})_{\ell=1}^{\infty} \subset \mathcal{A} \) such that \( B_k \subset \bigcup_{\ell=1}^{\infty} A_{k;\ell} \) and
\[
\sum_{\ell=1}^{\infty} \lambda(A_{k;\ell}) \leq \mu(B_k) + 2^{-k}\varepsilon.
\]
Clearly, \( A \subset \bigcup_{k=1}^\infty A_k \) so we have
\[
\mu(A) \leq \sum_{k=1}^\infty \sum_{\ell=1}^\infty \lambda(A_k; \ell)
\leq \sum_{k=1}^\infty \left( \mu(B_k) + 2^{-k} \varepsilon \right)
\leq \left( \sum_{k=1}^\infty \mu(B_k) \right) + \varepsilon.
\]
This holds for any \( \varepsilon > 0 \), so letting \( \varepsilon \to 0 \) we find
\[
\mu(A) \leq \left( \sum_{k=1}^\infty \mu(B_k) \right) + \varepsilon.
\]
Thus, \( \mu \) is \( \sigma \)-subadditive, and thus \( \mu \) is a measure.

(2) For \( A \in \mathcal{A} \) we clearly have \( \mu(A) \leq \lambda(A) \).

Now we show \( \lambda(A) \leq \mu(A) \). We may assume that \( \mu(A) < \infty \) otherwise there is nothing to show.

Take any \( (A_k)_{k=1}^\infty \subset \mathcal{A} \) such that \( \bigcup_{k=1}^\infty A_k \supset A \). By the usual argument we may assume (without loosing that \( A_k \in \mathcal{A} \)) that \( A_k \cap A_j = \emptyset \) for all \( k \neq j \). Set \( \tilde{A}_k := A_k \cap A \), \( k \in \mathbb{N} \). Then \( (\tilde{A}_k)_{k=1}^\infty \subset \mathcal{A} \) are pairwise disjoint, so since \( \lambda \) is premeasure
\[
\lambda(A) = \sum_{k=1}^\infty \lambda(\tilde{A}_k) \leq \sum_{k=1}^\infty \lambda(A_k).
\]
Here we also used Exercise 1.40 (monotonicity of \( \lambda \)) and \( \tilde{A}_k \subset A_k \).

Taking the infimum over all covers \( (A_k)_{k=1}^\infty \) as above we obtain that
\[
\lambda(A) \leq \mu(A),
\]
as claimed.

(3) Let \( A \in \mathcal{A} \) and \( B \subset X \).

Fix \( \varepsilon > 0 \) arbitrary. By the definition of \( \mu \), there exist \( (B_k)_{k=1}^\infty \subset \mathcal{A} \) such that \( B \subset \bigcup_{k=1}^\infty B_k \) and
\[
\sum_{k=1}^\infty \mu(B_k) \geq \mu(B) \geq \sum_{k=1}^\infty \mu(B_k) - \varepsilon.
\]
Since \( B_k \in \mathcal{A} \) we have \( \mu(B_k) = \lambda(B_k) \). Since \( A, B_k \in \mathcal{A} \) and \( \mathcal{A} \) is an algebra we have \( B \cap A \) and \( B_k \setminus A \in \mathcal{A} \). These are disjoint sets, and since \( \lambda \) is a pre-measure we have
\[
\mu(B_k) = \lambda(B_k) = \lambda(B_k \cap A) + \lambda(B_k \setminus A).
\]
So we have
\[
\mu(B) \geq \sum_{k=1}^\infty \lambda(B_k \cap A) + \sum_{k=1}^\infty \lambda(B_k \setminus A) - \varepsilon.
\]
Now observe that $B \cap A \subset \bigcup_{k \in \mathbb{N}} B_k \cap A$ and $B \setminus A \subset \bigcup_{k=1}^{\infty} B_k \setminus A$ (and both coverings belong to $\mathcal{A}$). By the definition of $\mu$ we thus find

$$\sum_{k=1}^{\infty} \lambda(B_k \cap A) \geq \mu(B \cap A)$$

and

$$\sum_{k=1}^{\infty} \lambda(B_k \setminus A) \geq \mu(B \setminus A)$$

Together we arrive at

$$\mu(B) \geq \mu(B \cap A) + \mu(B \setminus A).$$

That is, $A$ is $\mu$-measurable.

\[ \square \]

**Definition 1.43** (Lebesgue measure). The Lebesgue measure $\mathcal{L}^n$ in $\mathbb{R}^n$ is the Caratheodory-Hahn extension of $\lambda$ in Example 1.41. Compare this with (1.1).

If the pre-measure is *additionally $\sigma$-finite*, then the Carathéodory-Hahn extension $\mu$ is essentially unique (on the sets we care about: the measurable sets).

**Theorem 1.44** (Uniqueness). Let $\lambda : \mathcal{A} \rightarrow [0, \infty]$ as in Theorem 1.42 be additionally $\sigma$-finite and denote by $\mu$ the Carathéodory-Hahn-extension.

Whenever $\tilde{\mu} : 2^X \rightarrow [0, \infty]$ is another measure such that

$$\tilde{\mu}(A) = \lambda(A) \quad \text{for all } A \in \mathcal{A},$$

then indeed

$$\tilde{\mu}(A) = \mu(A) \quad \text{for all } \mu\text{-measurable } A$$

**Proof.**

(1) We have $\tilde{\mu}(A) \leq \mu(A)$ for all $A \subset X$. Indeed, let $A \subset \bigcup_{k \in \mathbb{N}} A_k$ for $A_k \in \mathcal{A}$. Then by $\sigma$-subadditivity of $\mu$,

$$\tilde{\mu}(A) \leq \sum_{k=1}^{\infty} \tilde{\mu}(A_k) = \sum_{k=1}^{\infty} \lambda(A_k).$$

Taking the infimum over all such covers $(A_k)_{k=1}^{\infty} \subset \mathcal{A}$ of $A$ we obtain

$$\tilde{\mu}(A) \leq \mu(A) \quad \forall A \subset X$$

(1.7)

(2) Let $\Sigma$ be the $\sigma$-algebra of $\mu$-measurable sets.

We now show: $\tilde{\mu}(A) = \mu(A)$ for all $A \subset \Sigma$ such that there is $S \in \mathcal{A}$ with $\lambda(S) < \infty$, and $S \supset A$.

So fix such $A$ and $S$.

Then

$$\tilde{\mu}(S \setminus A) \leq \tilde{\mu}(S) \overset{S \in \mathcal{A}}{=} \lambda(S) < \infty.$$
Consequently, since \( A \in \Sigma \) and \( S \in \mathcal{A} \),
\[
\tilde{\mu}(A) + \tilde{\mu}(S \setminus A) \overset{(1.7)}{\leq} \mu(A) + \mu(S \setminus A) A \in \Sigma \quad \tilde{\mu}(S) \leq \tilde{\mu}(A) + \tilde{\mu}(S \setminus A).
\]
So we have equality everywhere, which leads to
\[
\tilde{\mu}(A) + \tilde{\mu}(S \setminus A) = \mu(A) + \mu(S \setminus A)
\]
With (1) we conclude
\[
\tilde{\mu}(A) + \tilde{\mu}(S \setminus A) \geq \mu(A) + \mu(S \setminus A)
\]
and thus \( \tilde{\mu}(A) \geq \mu(A) \) (here we use that \( \tilde{\mu}(S \setminus A) < \infty \)). Again with (1) we have shown that
\[
(1.8) \quad \mu(A) = \tilde{\mu}(A) \quad \forall A \in \Sigma : \text{s.t.} \exists S \in \mathcal{A} : A \subset S, \lambda(S) < \infty.
\]
(3) \( \tilde{\mu}(A) = \mu(A) \) for all \( A \subset \Sigma \)
In comparison to (2) we need to remove the restriction \( A \subset S \) for some \( \lambda(S) < \infty \).
We write \( X = \bigcup_{k=1}^{\infty} S_k \) with \( (S_k)_{k \in \mathbb{N}} \) pairwise disjoint, \( S_k \in \mathcal{A} \), \( \lambda(S_k) < \infty \) for all \( k \in \mathbb{N} \).
Set \( A_k := A \cap S_k \), which are pairwise disjoint sets that all belong to \( \Sigma \). From (1.8) we have for any \( m \in \mathbb{N} \)
\[
\tilde{\mu}(\bigcup_{k=1}^{m} A_k) = \mu(\bigcup_{k=1}^{m} A_k).
\]
Thus, by monotonicity of \( \tilde{\mu} \), and since each \( A_k \in \Sigma \), we have
\[
\tilde{\mu}(A) \geq \limsup_{m \to \infty} \tilde{\mu}(\bigcup_{k=1}^{m} A_k) = \limsup_{m \to \infty} \mu(\bigcup_{k=1}^{m} A_k) A_k \in \Sigma \overset{\sum_{k=1}^{\infty} \mu(A_k).}
\]
From Theorem 1.33 (1) we obtain
\[
\sum_{k=1}^{\infty} \mu(A_k) = \mu(\bigcup_{k=1}^{\infty} A_k).
\]
Thus we have shown for any \( A \in \Sigma \),
\[
\tilde{\mu}(A) \geq \mu(A),
\]
and we have equality in view of (1.7).

\[\square\]

**Remark 1.45.** Observe that in Theorem 1.44 it is **not** said that any \( \mu \)-measurable \( A \) was also \( \tilde{\mu} \)-measurable.

**Example 1.46.** In general \( \mu \) and \( \tilde{\mu} \) in Theorem 1.44 might be different for non-measurable sets.
Let \( X = [0, 1], \mathcal{A} = \{\emptyset, X\} \), and set
\[
\lambda(\emptyset) := 0, \quad \lambda([0, 1]) := 1.
\]
Then $\lambda : A \to \mathbb{R}$ is a pre-measure.

One can check that the Caratheodory-Hahn extension of $\lambda$ is given by

$$
\mu(A) := \begin{cases} 
0 & A = \emptyset \\
1 & A \neq \emptyset.
\end{cases}
$$

If on the other hand we consider $\tilde{\mu}$ the Lebesgue measure on $[0,1]$, i.e. the Caratheodory-Hahn extension of $\tilde{\lambda} : A \to [0, \infty]$ where $\tilde{A}$ are the figures from Example 1.41 and $\tilde{\lambda}$ is the volume as defined. Then $\tilde{\mu}$ coincides with $\mu$ on $A$ but not in $\tilde{A}$, because, e.g.

$$
\frac{1}{2} = \lambda([0, 1/2]) = \tilde{\mu}([0, 1/2]) \neq \mu([0, 1/2]) = 1.
$$

1.4. Classes of Measures.

**Definition 1.47.** Let $X$ be a metric space and $\mu$ be a measure on $X$. $\mu$ is called a **metric measure** if

$$
\mu(E \cup F) = \mu(E) + \mu(F)
$$

whenever $\text{dist}(E, F) > 0$

**Exercise 1.48.** The Hausdorff measure $\mathcal{H}^s$ on a metric space $X$ is metric measures for any $s \geq 0$.

**Exercise 1.49.** The Lebesgue measure on $\mathbb{R}^n$ is a metric measure.

**Definition 1.50.** Let $X$ be a metric space and $\mu$ be a measure on $X$.

- Let $\mathcal{B} \subset 2^X$ be the smallest $\sigma$-algebra that contains all open sets of $\mathbb{R}^n$, that is $^4$

$$
\mathcal{B} := \bigcap \{ \mathcal{A} \subset 2^X : \text{\mathcal{A} is \sigma-algebra, all open sets belong to \mathcal{A}} \}.
$$

(Cf. Exercise 1.31). Any set $A \in \mathcal{B}$ is called a **Borel set** and $\mathcal{B}$ is called the **Borel \sigma-algebra**.
- A measure $\mu$ on $X$ for which (at least) all Borel-sets are $\mu$-measurable is called a **Borel measure**.

**Exercise 1.51.** If $f : X \to Y$ is homeomorphism then $f(A)$ is Borel if and only if $A$ is Borel

**Theorem 1.52.** If $\mu$ is a metric measure on a metric space $X$ then $\mu$ is a Borel measure, i.e. all open sets are $\mu$-measurable. In particular Lebesgue and Hausdorff measure are Borel measures.

---

$^4$It is an easy exercise to show that this indeed defines a $\sigma$-algebra. Observe in particular that $2^X$ is a $\sigma$-algebra which contains all open sets so the right-hand side is not an intersection of empty sets.
Proof. It suffice to show that all open sets in $X$ are $\mu$-measurable, since then the set of measurable sets (which is a $\sigma$-algebra) must contain the Borel sets $\mathcal{B}$.

Let $G \subset X$ be open and $A \subset X$ arbitrary. We need to show

$$\mu(A) \geq \mu(A \cap G) + \mu(A \backslash G).$$

If $\mu(A) = \infty$ this is obvious, so from now on assume $\mu(A) < \infty$.

For $k \in \mathbb{N}$ define

$$G_k := \{ x \in G : \text{dist} (x, X \backslash G) > \frac{1}{k} \}$$

Then $\text{dist} (G_k, X \backslash G) \geq \frac{1}{k} > 0$.

We have by monotonicity,

$$\mu(A \cap G) + \mu(A \backslash G) \leq \mu(A \cap G_k) + \mu(A \backslash G) + \mu(A \cap (G \setminus G_k)).$$

Since $\mu$ is a metric measure and $\text{dist} (A \cap G_k, A \backslash G) > 0$ we conclude

$$\mu(A \cap G) + \mu(A \backslash G) \leq \mu(A) - \mu(A \cap (G \setminus G_k)).$$

So the statement is proven once we show

$$\lim_{k \to \infty} \mu(A \cap (G \setminus G_k)) = 0. \quad (1.9)$$

To see (1.9) let

$$D_k := G_{k+1} \backslash G_k = \{ x \in G : \text{dist} (x, X \backslash G) \in (\frac{1}{k+1}, \frac{1}{k}] \}.$$ 

We then have (here we use that $G$ is open)

$$G \setminus G_k = \bigcup_{i=k}^{\infty} D_i \quad \text{so} \quad \mu(A \cap (G \setminus G_k)) \leq \sum_{i=k}^{\infty} \mu(A \cap D_i). \quad (1.10)$$

and whenever $i + 2 \leq j$ we have

$$\text{dist} (D_i, D_j) \geq \frac{1}{i+1} - \frac{1}{j} > 0.$$

Since $\mu$ is a metric measure we can sum up even and odd $D_i$'s i.e.

$$\sum_{i=1}^{k} \mu(A \cap D_{2i-1}) = \mu(A \cap \bigcup_{i=1}^{k} D_{2i-1}) \leq \mu(A)$$

and

$$\sum_{i=1}^{k} \mu(A \cap D_{2i}) = \mu(A \cap \bigcup_{i=1}^{k} D_{2i}) \leq \mu(A).$$

In particular we have

$$\sum_{i=1}^{\infty} \mu(A \cap D_i) \leq 2\mu(A) < \infty.$$
From (1.10),
\[ \mu(A \cap (G \setminus G_k)) \leq \sum_{i=k}^{\infty} \mu(A \cap D_i) \]
and since the series on the right-hand side converges
\[ \mu(A \cap (G \setminus G_k)) \xrightarrow{k \to \infty} 0. \]
That is (1.9) is established and we can conclude. \(\square\)

**Theorem 1.53.** Let \( X \) be a metric space and \( \mu \) any Borel measure. Suppose that \( X \) is a union of countably many open sets of finite measure. Then for all Borel sets \( E \subset X \)
\[ \mu(E) = \inf_{U \supset E; U \text{ open}} \mu(U) = \sup_{C \subset E; C \text{ closed}} \mu(C). \]

The first part of Theorem 1.53 is a direct consequence of

**Proposition 1.54.** Let \( X \) be a metric space and \( \mu \) any Borel measure. Suppose that \( X \) is a union of countably many open sets of finite measure. If we define \( \tilde{\mu} : 2^X \to [0, \infty] \) as
\[ (1.11) \quad \tilde{\mu}(E) := \inf_{U \supset E; U \text{ open}} \mu(U) \quad E \subset X \]
then \( \tilde{\mu} \) is a metric measure and we have
\[ \tilde{\mu}(E) = \mu(E) \quad \forall \text{Borel sets } E \subset X. \]

The Hausdorff measure \( \mathcal{H}^s \) (which is metric and thus Borel) shows that we cannot skip the assumption that \( X \) finite union of countably many open sets of finite measure.

Indeed if \( s < n \) for any nonempty open set \( \mathcal{H}^s(U) = \infty \), so (1.11) is certainly false.

**Proof of Proposition 1.54.** It is easy to see that \( \tilde{\mu} \) is a measure (exercise).

We will show, it is a metric measure: let \( E, F \subset X \) such that \( \delta := \text{dist}(E, F) > 0 \). Set
\[ V_E := \bigcup_{x \in E} B(x, \frac{1}{3}\delta), \]
and
\[ V_F := \bigcup_{x \in F} B(x, \frac{1}{3}\delta). \]

\( V_E \) and \( V_F \) are then disjoint and open. Fix \( \varepsilon > 0 \), let \( U \supset E \cup F \) be open such that
\[ \mu(U) \leq \tilde{\mu}(E \cup F) + \varepsilon. \]

Since \( V_E \cap U \) and \( V_F \cap U \) are open they are \( \mu \)-measurable and since they are moreover disjoint
\[ \mu((V_E \cap U) \cup (V_F \cap U)) = \mu(V_E \cap U) + \mu(V_F \cap U). \]

Since \( E = E \cap V_E \subset U \cap V_E \) and \( F = F \cap V_F \subset U \cap V_F \) we have
\[ \tilde{\mu}(E) + \tilde{\mu}(F) \leq \mu(V_E \cap U) + \mu(V_F \cap U) = \mu((V_E \cap U) \cup (V_F \cap U)) \leq \mu(U) \leq \tilde{\mu}(E \cup F) + \varepsilon. \]
Taking $\varepsilon \to 0$ we obtain that $\tilde{\mu}(E) + \tilde{\mu}(F) \leq \tilde{\mu}(E \cup F)$ which shows that $\tilde{\mu}$ is metric.
Consequently, in view of Theorem 1.52, $\tilde{\mu}$ is Borel.

It remains to show $\tilde{\mu}(E) = \mu(E)$ for all Borel sets. Clearly,

$$\tilde{\mu}(E) \geq \mu(E) \quad \forall E \subset X.$$ 
and

$$\tilde{\mu}(E) = \mu(E) \quad \forall E \subset X \text{ open.}$$

By assumption we can write $X = \bigcup_{k=1}^{\infty} X_k$ with $X_k$ open and $\mu(X_k) < \infty$. We also may assume that $X_k \subset X_{k+1}$.

Then we have for all set $E \subset X$

$$\mu(X_n \setminus E) \leq \tilde{\mu}(X_n \setminus E)$$
and

$$\mu(X_n \cap E) \leq \tilde{\mu}(X_n \setminus E).$$

Now if $E$ is Borel, we have

$$(1.12) \quad \mu(X_n \setminus E) = \tilde{\mu}(X_n \setminus E) \quad \text{and} \quad \mu(X_n \cap E) = \tilde{\mu}(X_n \setminus E).$$

Indeed, if not, either $\mu(X_n \setminus E) < \tilde{\mu}(X_n \setminus E)$ or $\mu(X_n \cap E) < \tilde{\mu}(X_n \setminus E)$, which leads to

$$\mu(X_n) = \mu(E \cap X_n) + \mu(X_n \setminus E_n) < \tilde{\mu}(E \cap X_n) + \tilde{\mu}(X_n \setminus E_n) = \tilde{\mu}(X_n) = \mu(X_n)$$
the second to last equation is the $\tilde{\mu}$-measurability of $E$ (since it is a Borel set and $\tilde{\mu}$ is a Borel measure), the last equation uses that $X_n$ is open. The above estimate is impossible (since $\mu(X_n), \tilde{\mu}(X_n) < \infty$), so $(1.12)$ is established.

So in particular we have for any Borel set $E$, $\mu(X_n \cap E) = \tilde{\mu}(X_n \cap E)$. In view of Theorem 1.33 (recall: both $\mu$ and $\tilde{\mu}$ are Borel!)

$$\tilde{\mu}(E) = \lim_{n \to \infty} \tilde{\mu}(X_n \cap E) = \lim_{n \to \infty} \mu(X_n \cap E) = \mu(E).$$
That is, we have shown for any Borel set $E$,

$$(1.13) \quad \mu(E) = \tilde{\mu}(E) := \inf_{U \supset E; U \text{ open}} \mu(U).$$

\[ \square \]

**Proof of Theorem 1.53 last part.** Having from Proposition 1.54 (1.13), it remains to prove that for Borel sets $E$

$$\mu(E) = \sup_{C \subset E; C \text{ closed}} \mu(C).$$

We apply (1.13) to $X_n \setminus E$ and find an open set $U_n$ such that

$$X_n \setminus E \subset U_n$$
and

$$\mu(U_n \setminus (X_n \setminus E)) = \mu(U_n) - \mu(X_n \setminus E) < \frac{\varepsilon}{2^n}.$$
The set $U := \bigcup_{n=1}^{\infty} U_n$ is open and $C = X \setminus U \subset E$ is closed. Now it suffices to observe that

$$E \setminus C = E \cap \bigcup_{n=1}^{\infty} U_n \subset \bigcup_{n=1}^{\infty} G_n \setminus (U_n \setminus E)$$

and hence $\mu(E \setminus C) = \mu(E) - \mu(C) < \varepsilon$. Thus, $\mu(C) \leq \mu(E) \leq \mu(C) + \varepsilon$. Taking $\varepsilon \to 0$ the proof is complete.

\[ \square \]

**Corollary 1.55.** Let $X$ be a metric space and $\mu, \nu$ Borel measure. Suppose that $X$ is a union of countably many open sets of finite measure. If $\nu$ and $\mu$ coincide for open sets, namely if

$$\nu(U) = \mu(U) \quad \forall \text{open set } U \subset X,$$

then

$$\nu(E) = \mu(E) \quad \forall \text{Borel sets } E.$$

**Proof.** Twice applying Theorem 1.52, we find that for any Borel set $E$

$$\mu(E) = \inf_{U \supseteq E, U \text{ open}} \mu(U) = \inf_{U \supseteq E, U \text{ open}} \nu(U) = \nu(E).$$

\[ \square \]

On $\mathbb{R}^n$ we can simplify Corollary 1.55

**Theorem 1.56** (Borel measures on $\mathbb{R}^n$ that coincide on rectangles). Let $\mu$ and $\nu$ be two finite Borel measures on $\mathbb{R}^n$ such that

$$\mu(R) = \nu(R)$$

for all closed rectangles $R$ of the form

$$R := \{ x \in \mathbb{R}^n : \ a_i \leq x_i \leq b_i, \ i = 1, \ldots, n \}$$

where $-\infty \leq a_i \leq b_i \leq \infty$, $i = 1, \ldots, n$.

Then

$$\mu(B) = \nu(B)$$

for all Borel sets $B \subset \mathbb{R}^n$

For the proof of Theorem 1.56 we need a essentially combinatorial observation, called the $\pi$-$\lambda$ Theorem. $\pi$ and $\lambda$-systems are families of sets which are invariant under less operations than the $\sigma$-Algebras.

**Definition 1.57.** (1) A nonempty family $\mathcal{P} \subset 2^X$ is called a $\pi$-system if it is closed under (finitely many) intersections, i.e.

$$A, B \in \mathcal{P} \implies A \cap B \in \mathcal{P}.$$

(2) A family of subsets $\mathcal{L} \subset 2^X$ is called a $\lambda$-system if

- $X \in \mathcal{L}$
- $A, B \in \mathcal{L}$ and $B \subset A$ implies $A \setminus B \in \mathcal{L}$
• if $A_k \in \mathcal{L}$ and $A_k \subset A_{k+1}$ for $k = 1, \ldots$, then
$$\bigcup_{k=1}^{\infty} A_k \in \mathcal{L}.$$  

**Exercise 1.58.** Show that if $\mathcal{P}$ is a $\lambda$-system and a $\pi$-system, then it is a $\sigma$-Algebra.

Clearly any $\sigma$-Algebra is also a $\pi$-system and a $\lambda$-system.

**Theorem 1.59** ($\pi$-$\lambda$ Theorem). If $\mathcal{P}$ is a $\pi$-system and $\mathcal{L}$ is a $\lambda$-system with
$$\mathcal{P} \subset \mathcal{L}$$
then$^5$
$$\sigma(\mathcal{P}) \subset \mathcal{L}$$

**Proof.** Define $\mathcal{S}$ to be the intersection of all $\lambda$-systems $\mathcal{L}'$ containing $\mathcal{P}$
$$\mathcal{S} := \bigcap_{\mathcal{L}' \supset \mathcal{P}} \mathcal{L}' ,$$
We first claim that $\mathcal{S}$ is a $\pi$-system.
Indeed let $A, B \in \mathcal{S}$. We must show $A \cap B \in \mathcal{S}$. Define
$$A := \{ C \subset X : A \cap C \in \mathcal{S} \}.$$  
Since $\mathcal{S}$ is a $\lambda$-system, it follows that $A$ is a $\lambda$-system. Therefore $\mathcal{S} \subset A$. But then since $B \in \mathcal{S}$ we see that $A \cap B \in \mathcal{S}$.

Next we claim that $\mathcal{S}$ is a $\sigma$-algebra
Indeed, we only need to show that $\mathcal{S}$ is a $\lambda$-system (cf. Exercise 1.58).
Since $X \in \mathcal{S}$, $\emptyset = X \setminus X \in \mathcal{S}$. Also $A \in \mathcal{S}$ implies $X \setminus A \in \mathcal{S}$. So $\mathcal{S}$ is closed under complements and under finite intersections, and thus under finite unions.
If $A_1, A_2 \in \mathcal{S}$ then $B_n := \bigcup_{k=1}^{n} A_k \in \mathcal{S}$. Since $\mathcal{S}$ is a $\lambda$-system we conclude that $\bigcup_{k=1}^{\infty} A_k = \bigcup_{n=1}^{\infty} B_n \in \mathcal{S}$. Thus $\mathcal{S}$ is a $\sigma$-algebra.
Since $\mathcal{S} \supset \mathcal{P}$ is a $\sigma$-algebra it follows that
$$\sigma(\mathcal{P}) \subset \mathcal{S} \subset \mathcal{L}.$$  

**Proof of Theorem 1.56.** Let
$$\mathcal{P} := \{ A \subset \mathbb{R}^n : \text{ for some } n \in \mathbb{N}, A = \bigcap_{k=1}^{n} R_k \text{ where } (R_k)_{k=1}^{n} \text{ are a rectangles} \},$$

$^5$Recall that $\sigma(\mathcal{P})$ is the $\sigma$-Algebra generated by $\mathcal{P}$
which a $\pi$-system. We also set
\[ L := \{ B \subset \mathbb{R}^n : B \text{ Borel: } \mu(B) = \nu(B) \}. \]

While it is not so clear that $L$ is a $\sigma$-Algebra, one can check it is a $\lambda$-system. Also $P \subset L$ by assumption (observe that each $R_k$ is $\mu$ and $\sigma$-measurable).

By Theorem 1.59 we have $\sigma(P) \subset L$. Since $\sigma(P)$ contains the Borel sets (any open set can be written as countable union of rectangles, see Lemma 1.75 below), any Borel set $B$ satisfies $B \in L$ and thus $\mu(B) = \nu(B)$, and we can conclude. \hfill $\Box$

**Definition 1.60** (Borel Regular measure). A Borel measure $\mu$ is **Borel regular**, if for any $A \subset \mathbb{R}^n$ there exists some Borel set $B \supset A$ such that $\mu(A) = \mu(B)$.

**Corollary 1.61.** Let $X$ be a metric space and $\mu$ any Borel measure. Suppose that $X$ is a union of countably many open sets of finite measure. If we define $\tilde{\mu} : 2^X \to [0, \infty]$ as
\[ \tilde{\mu}(E) := \inf_{U \supset E; U \text{ open}} \mu(U) \quad E \subset X \]
then $\tilde{\mu}$ is a metric, Borel-regular measure that coincides with $\mu$ on Borel sets.

**Proof.** In view of Proposition 1.54 we only need to show that $\tilde{\mu}$ is Borel-regular.

Indeed, let $E \subset X$ with $\mu(E) < \infty$ (otherwise $\mu(E) = \mu(X) = \infty$) be arbitrary. Let $(U_k)_{k=1}^\infty$ be open sets so that $U_k \supset E$ and
\[ \mu(U_k) - \frac{1}{k} \leq \tilde{\mu}(E). \]

Set $U := \cap_{k=1}^\infty U_k$. Then we have $E \subset U$ and thus $\tilde{\mu}(E) \leq \tilde{\mu}(U)$. On the other hand we have
\[ \tilde{\mu}(U) = \tilde{\mu}(\bigcap_{k=1}^\infty U_k) \leq \mu(U_k) \leq \tilde{\mu}(E) + \frac{1}{k}. \]
This holds for any $k \in \mathbb{N}$ so letting $k \to \infty$ we find
\[ \tilde{\mu}(U) \leq \tilde{\mu}(E). \]
So $\tilde{\mu}(U) = \tilde{\mu}(E)$, and since as an intersection of countably many open sets $U$ is a Borel-set, we can conclude. \hfill $\Box$

**Example 1.62.** The Lebesgue measure $\mathcal{L}^n$ (as defined in Definition 1.43) is Borel regular.

**Proof.** In view of Corollary 1.61 it suffices to show that
\[ \mathcal{L}^n(A) = \inf \{ \mathcal{L}^n(G) : G \subset \mathbb{R}^n \text{ open and } G \supset A \}. \]

$\leq$ is obvious by monotonicity. For $\geq$ assume $\mathcal{L}^n(A) < \infty$, let $\varepsilon > 0$ and take blocks $(Q_i)_{i=1}^\infty$ such that
\[ \sum_{i=1}^\infty \text{vol}(Q_i) \leq \mathcal{L}^n(A) + \varepsilon. \]
To each block $Q_i$ we can choose an open block $Q_o \supset Q_i$ such that

$$\text{vol}(Q_o) \leq \text{vol}(Q_i) + 2^{-i} \varepsilon.$$ 

Set $G := \bigcup_{i=1}^{\infty} (Q_i)^o \supset A$. Then

$$\mathcal{L}^n(G) \leq \sum_{i=1}^{\infty} \text{vol}(Q_o) \leq \sum_{i=1}^{\infty} \text{vol}(Q_i) + \sum_{i=1}^{\infty} 2^{-i} \varepsilon \leq \mathcal{L}^n(A) + 2 \varepsilon.$$

That is we have shown

$$\mathcal{L}^n(A) + 2 \varepsilon \geq \inf \{ \mathcal{L}^n(G) : G \subset \mathbb{R}^n \text{ open and } G \supset A \}.$$ 

which holds for any $\varepsilon > 0$, letting $\varepsilon \to 0$ we conclude.

**Example 1.63.** Let $X$ be a metric space. For any $s \geq 0$, $\mathcal{H}^s(X)$ is Borel-regular.

**Proof.** In this case we cannot use Corollary 1.61, since we cannot assume that we can cover $X$ with countably many finite measure sets.

If $s = 0$ the claim is easy. Take any $E \subset X$. If $\mathcal{H}^0(E) = \infty$, then $\mathcal{H}^0(E) = \mathcal{H}^0(X) = \infty$. If $\mathcal{H}^0(E) < \infty$ then $E$ contains finitely many points, thus $E$ is closed and thus a Borel set.

Now let $s > 0$ and $E \subset X$. If $\mathcal{H}^s(E) = \infty$ then again $\mathcal{H}^s(E) = \mathcal{H}^s(X) = \infty$ and we conclude. So assume $\mathcal{H}^s(E) < \infty$. In particular $\mathcal{H}^s_\delta(E) < \infty$ for all $\delta > 0$.

So for each $\ell$ there exists a covering of $E$ by open balls $(B(r_{k;\ell}))_{k=1}^{\infty}$ with radius $r_{k;\ell} \leq \frac{1}{\ell}$ such that

$$\mathcal{H}^s_\frac{1}{\ell} \left( \bigcup_{k=1}^{\infty} B(r_{k;\ell}) \right) \leq \mathcal{H}^s_\frac{1}{2} (E) + \frac{1}{\ell} \leq \mathcal{H}^s(E) + \frac{1}{\ell}.$$ 

In the last step we used again $\mathcal{H}^s(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A)$.

Then $G := \bigcap_{\ell=1}^{\infty} \bigcup_{k=1}^{\infty} B(r_{k;\ell})$ is a Borel set, and we have

$$\mathcal{H}^s(G) = \lim_{\ell \to \infty} \mathcal{H}^s_\frac{1}{\ell} (G) \leq \limsup_{\ell \to \infty} \mathcal{H}^s_\frac{1}{\ell} \left( \bigcup_{k=1}^{\infty} B(r_{k;\ell}) \right) \leq \mathcal{H}^s(E) + 0$$

Since on the other hand $G \supset E$ we have

$$\mathcal{H}^s(G) = \mathcal{H}^s(E).$$

We can conclude.

**Definition 1.64.** Let $X$ be a metric space and $\mu : 2^X \to [0, \infty]$ a Borel regular measure. $\mu$ is called a Radon measure if $\mu(K) < \infty$ for all compact sets.

**Example 1.65.**

- $\mathcal{L}^n$ is a Radon measure on $\mathbb{R}^n$
- $\mathcal{H}^s$ is not a Radon measure in $\mathbb{R}^n$ whenever $s < n$. 
Example 1.66. Let $X$ is a locally compact and separable metric space and $\mu$ is a Borel measure such that $\mu(K) < \infty$ for all $K$ compact.

Then $X$ is the union of countably many open sets of finite measure and thus $\mu$ coincides with a Radon measure $\tilde{\mu}$ on all Borel sets; cf. Corollary 1.61.

Exercise 1.67. Let $X$ be a metric space and $\mu$ any Radon measure. Suppose that $X$ is a union of countably many open sets of finite $\mu$-measure. Let $A \subset X$ be $\mu$-measurable. Show that $\mu\ll A$ is a Radon measure.

Theorem 1.68. Let $X$ be a metric space and $\mu$ any Borel-regular measure. Suppose that $X$ is a union of countably many open sets of finite $\mu$-measure.

(1) If $\mu$ is Borel-regular, then for any $A \subset X$ we have

$$\mu(A) = \inf_{A \subset G, G \text{ open}} \mu(G)$$

(2) If $X = \mathbb{R}^n$ and $\mu$ is a Radon measure, for any $\mu$-measurable $A \subset X$ we have

$$\mu(A) = \sup_{F \subset A, F \text{ compact}} \mu(F).$$

Proof. (1) Let $A \subset X$. By monotonicity we always have

$$\mu(A) \leq \inf_{A \subset G, G \text{ open}} \mu(G).$$

For the other inequality, since $\mu$ is by assumption Borel-regular we find a Borel set $E \supset A$ such that $\mu(A) = \mu(E)$. By Theorem 1.53 we have

$$\mu(A) = \mu(E) = \inf_{E \subset G, G \text{ open}} \mu(G) \geq \inf_{A \subset G, G \text{ open}} \mu(G)$$

(2) Follows from part (1), see Exercise 1.69.

Exercise 1.69. Show Theorem 1.68 (2).

Use the argument as in the proof of Theorem 1.53 last part and Theorem 1.68(1). Observe we need measurability of $A$ because we need to use that $\mu(U \setminus (X_n \setminus A)) = \mu(U) - \mu(X_n \setminus A)$, which is the measurability of $X_n \setminus A$.

Theorem 1.70. Let $X$ be a metric space $\mu$ a Radon measure on $X$, and assume that $X$ is a union of countably many open sets of finite $\mu$-measure. Then the following are equivalent for $A \subset X$.

(1) $A$ is $\mu$-measurable

(2) For every $\varepsilon > 0$ there is an open set $G \subset X$ such that $A \subset G$ and $\mu(G \setminus A) < \varepsilon$

---

6Separable means, a countable set is dense. Locally compact means that every point has a neighborhood whose closure is compact. $\mathbb{R}^n$ is locally compact and separable, but also $\mathbb{R}^n$ 0 is locally compact and seperable
(3) There is a $G_δ$-set, namely a set $H = \bigcap_{i=1}^{\infty} G_i$ for some open sets $G_i \subset \mathbb{R}^n$, such that $A \subset H$ and $\mu(H \setminus A) = 0$

(4) For every $\varepsilon > 0$ there is a closed set $F$ such that $F \subset A$ and $\mu(A \setminus F) < \varepsilon$

(5) There is a $F_α$-set, namely a set $M = \bigcup_{i=1}^{\infty} F_i$ for some closed set $F_i \subset \mathbb{R}^n$, such that $M \subset A$ and $\mu(A \setminus M) = 0$

(6) For every $\varepsilon > 0$ there is an open set $G$ and a closed set $F$ such that $F \subset A \subset G$ and $\mu(G \setminus F) < \varepsilon$.

(7) $A = B \cup N$ where $B$ is a Borel set and $N$ is a $\mu$-zero-set.

**Proof.** By assumption we have $X = \bigcup_{k=1}^{\infty} X_k$ with $X_k$ open and $\mu(X_k) < \infty$.

(1) $\Rightarrow$ (2): Let $\varepsilon > 0$. Set $A_k := A \cap X_k$ then $\mu(A_k) < \infty$ and by Theorem 1.68 there exists for each $k$ an open set $G_k \supset A_k$ such that

$$\mu(G_k) \leq \mu(A_k) + \frac{\varepsilon}{2^k}.$$ 

Since $A_k$ is measurable and $G_k \supset A_k$ we have

$$\mu(G_k \setminus A_k) = \mu(G_k) - \mu(A_k) \leq \frac{\varepsilon}{2^k}.$$ 

Hence $A \subset G := \bigcup_{k=1}^{\infty} G_k$ and we have

$$\mu(G \setminus A) \leq \sum_{k=1}^{\infty} \mu(G_k \setminus A) \leq \sum_{k=1}^{\infty} \mu(G_k \setminus A_k) \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$ 

Letting $\varepsilon \to 0$ we conclude. (2) $\Rightarrow$ (3). We set $H := \bigcap_{i=1}^{\infty} U_i$ where $U_i$ are open sets such that $A \subset U_i$ and $\mu(U_i \setminus A) < \frac{1}{i}$. Then $A \subset H$ and

$$\mu(H \setminus A) \leq \mu(U_i \setminus A) \leq \frac{1}{i} \xrightarrow{i \to \infty} 0.$$ 

(3) $\Rightarrow$ (1) Let $A \subset H$ for some $G_δ$-set $H$ with $\mu(H \setminus A) = 0$. Then for any $B \subset X$ we have by monotonicity

$$\mu(B \cap A) \leq \mu(B \cap H)$$

and since $B \cap H \subset (B \cap A) \cup (H \setminus A)$,

$$\mu(B \cap H) \leq \mu(B \cap A) + \mu(H \setminus A) = \mu(B \cap A).$$

thus

$$\mu(B \cap H) = \mu(B \cap A).$$

Similarly,

$$\mu(B \setminus H) \leq \mu(B \setminus A)$$

and since $B \setminus A \subset (B \setminus H) \cup (H \setminus A)$,

$$\mu(B \setminus A) \leq \mu(B \setminus H) + \mu(H \setminus A) = \mu(B \setminus H).$$

thus

$$\mu(B \setminus H) = \mu(B \setminus A).$$