ANALYSIS I & II & III

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Contents

In Analysis
there are no theorems
only proofs

These lecture notes take great inspiration from the lecture notes by Michael Struwe (Analysis III, German), as well as by Piotr Hajłasz (Analysis I). We will also follow the presentations in Evans-Gariepy [Evans and Gariepy, 2015] (measure theory), Grafakos [Grafakos, 2014] (Fourier Analysis) and wikipedia. Further sources are Piotr Hajlasz' Functional Analysis, Clasons [Clason, 2020] and everything available on the internet. Sometimes we follow those sources verbatim.

Pictures that were not taken from sources mentioned above (or wikipedia) are usually made with geogebra.

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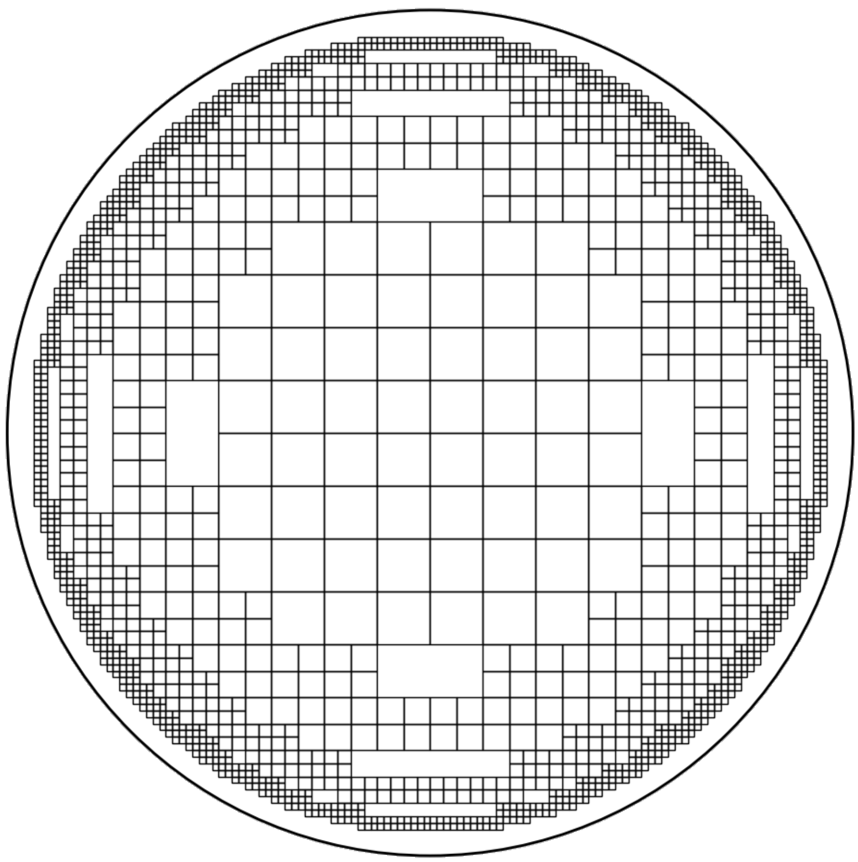
Index

Part 1. Analysis I: Measure Theory

1. Measures, σ-Algebras

What would be a reasonable notion of volume in $\mathbb{R}^n$? How do we determine the measure of the circle?



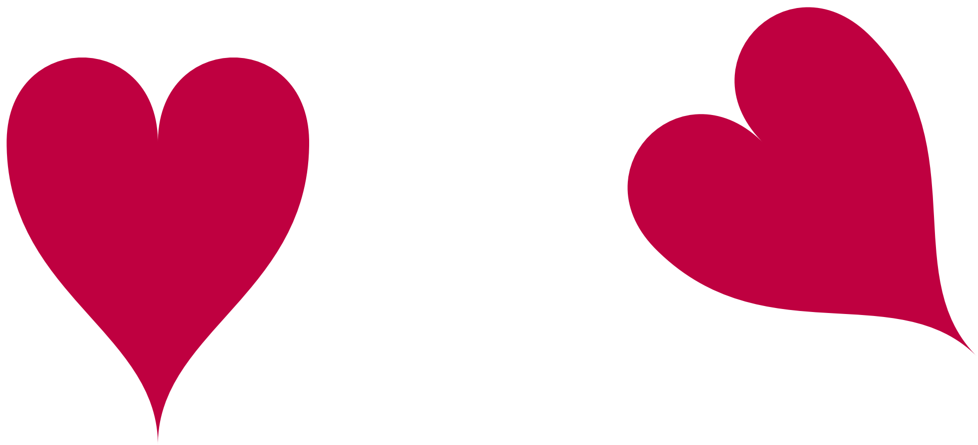
The basic, most natural idea is to somehow cut our set into pieces, of which we know the volume, and sum up. But what if the set is not the nice circle, but a point? Or a Cantor set? So how do we cut it into pieces?

Maybe we need to do this axiomatically...

1.1. First attempt, and the problems begin.

A measure is a way to measure (hence the name!) volumes. So for some set X it should be a map

$$\mu : 2^X \to [0, \infty]$$



A B

FIGURE 1.1. If μ is a reasonable volume (later: Lebesgue measure), the two sets A and B should have the same value, $\mu(A) = \mu(B)$.

that to a subset $A \subset X$ assigns the volume $\mu(A)$. Here 2^X denotes the *power set* of X, i.e. the collection of subsets of X.

$$2^X = \{A : \quad A \subset X\}.$$

It seems to be a reasonable assumption to axiomatically assume the following properties

Definition 1.1 (First attempt to define a volume for all sets)**.** We want[1] to find $\mu : 2^{\mathbb{R}^n} \to [0, \infty)$ such that

- For any $A \subset \mathbb{R}^n$ we have $\mu(A) \in [0, \infty]$
- (Invariance under translation and rotation)[2] For any set $A \subset \mathbb{R}^n$, any rotation $P \in O(n)$ and any vector $x \in \mathbb{R}^n$ we have $\mu(x + OA) = \mu(A)$ where we denote
$$x + OA := \{x + Oa \in \mathbb{R}^n : \quad a \in A\}$$
Cf. Figure 1.1.
- For any $A, B \subset \mathbb{R}^n$ disjoint we have $\mu(A \cup B) = \mu(A) + \mu(B)$

And then, if we moreover insist that $\mu([a_1, b_1] \times \ldots \times [a_n, b_n]) = \prod_{i=1}^n |b_i - a_i|$ then we should have defined a volume for all sets of $\mathbb{R}^n$... right?

As reasonable as that sounds, Definition 1.1 is sadly non-sensical. Indeed, Banach showed the following:

[1] μ is called a *Banach measure* (although it probably should be called a *Banach content*)
[2] i.e. invariance under congruence relation!

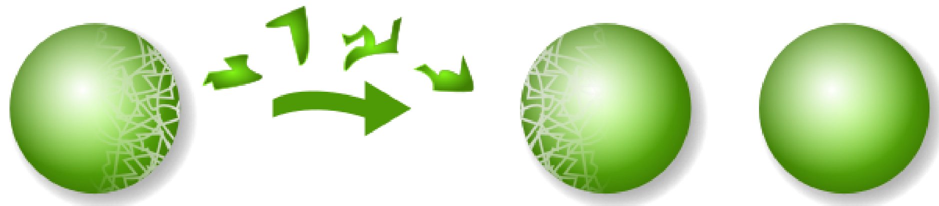


FIGURE 1.2. A ball can be decomposed into a finite number of disjoint sets and then reassembled into two balls identical to the original.

Theorem 1.2 ([Banach, 1923]). *For $n = 1, 2$ the map μ as in Definition 1.1 is not uniquely defined. For $n \geq 3$ there is no such map.*

The non-existence in dimension $n \geq 3$ is the famous *Banach-Tarski-Paradoxon* (1924):

Theorem 1.3 (Banach-Tarski). *Let $n \geq 3$, A and B be bounded sets with $int(A)$ and $int(B) \neq \emptyset$. Then there exist finitely many $(x_i)_{i=1}^N \subset \mathbb{R}^n$, $(O_i)_{i=1}^N \subset O(n)$ and pairwise disjoint sets $(C_i)_{i=1}^N$ so that $(x_i + O_iC_i)_{i=1}^N$ are pairwise disjoint and*

$$A = \bigcup_{i=1}^N C_i, \quad \text{and} \quad B = \bigcup_{i=1}^N (x_i + O_iC_i)\,. \tag{1.1}$$

That is we can deconstruct any set A in $\mathbb{R}^n$ into disjoint sets, move them around (without any scaling!) and obtain another completely different set B - see Figure 1.2.

The proof of Theorem 1.3 relies on group theory, see e.g. Terry Tao's notes.

Banach-Tarski destroys any hope for a (three-dimensional) reasonable notion of volume such as in Definition 1.1, indeed we have

Corollary 1.4. *Let $n = 3$. If we have μ as in Definition 1.1 and $\mathbb{B}^3$ is the unit ball in $\mathbb{R}^3$ then either $\mu(\mathbb{B}^3) = \infty$ or $\mu(\mathbb{B}^3) = 0$.*

Proof. Indeed denote $A := \mathbb{B}^3$ and $B := \mathbb{B}^3 \cup (5 + \mathbb{B}^3)$. Since $\mathbb{B}^3$ is the disjoint union of two balls (same size) and μ is translation invariant we have

$$\mu(B) = 2\mu(A). \tag{1.2}$$

But now apply Theorem 1.3: then we have

$$A = \bigcup_{i=1}^N C_i, \quad B = \bigcup_{i=1}^N (x_i + O_iC_i).$$

Each representation is disjoint, so we have

$$\mu(A) = \sum_{i=1}^N \mu(C_i) = \sum_{i=1}^N \mu(x_i + O_iC_i) = \mu(B) \overset{(1.2)}{=} 2\mu(A).$$

Thus we have $\mu(A) = 2\mu(A)$ which implies that either $\mu(A) = 0$ or $\mu(A) = \infty$. Since $A = \mathbb{B}^3$ we can conclude. □

So how do we fix this notion of measure?

We change our notion of a measure. For one (to deal with the non-uniqueness issue in dimension $n = 1, 2$) we need to assume an additional condition on our map μ: it should be monotone in the following way (it is important to allow ∞ many sets on the right-hand side!)

- $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$ whenever $A, A_k \subset X$, $k \in \mathbb{N}$ and $A \subset \bigcup_{k\in\mathbb{N}} A_k$

But it turns out that then even for $n = 1$ there are no reasonable maps μ

Theorem 1.5 (Vitali)**.** *Let* $\mu : 2^{\mathbb{R}} \to [0, \infty]$ *satisfy*

- $\mu(\emptyset) = 0$,
- *for any* $A, B \subset \mathbb{R}^n$ *disjoint we have* $\mu(A \cup B) = \mu(A) + \mu(B)$, *and*
- *(countable subadditivity)* $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$ *whenever* $A, A_k \subset X$, $k \in \mathbb{N}$ *and* $A \subset \bigcup_{k\in\mathbb{N}} A_k$.

If μ *is moreover translation invariant, i.e.*

$$\mu(x + A) = \mu(A) \quad \forall x \in \mathbb{R}, A \subset \mathbb{R}.$$

then $\mu([0, 1])$ *is either* 0 *or* ∞*.*

Proof. The idea of the proof is the construction of the so-called *Vitali-set* which relies on the *axiom of choice*.

Construction (Vitali) Define the equivalence relation

$$x \sim y \quad \Leftrightarrow x - y \in \mathbb{Q}$$

For $x \in \mathbb{R}$ denote by $[x]$ the set

$$[x] := \{y \in \mathbb{R} : \quad x - y \in \mathbb{Q}\}.$$

Let $V \subset \mathbb{R}$ be a set such that for each class $[x]$ there exists exactly one element $y \in V \cap [x]$. The set V exists by the *axiom of choice*: if we set

$$X := \{[x] \subset \mathbb{R} : \quad x \in \mathbb{R}\}$$

then the axiom of choice says there exists a choice function $f : X \to \mathbb{R}$ such that $f([x]) \in [x]$ for all $[x] \in X$. Then $V := f(X)$.

Without loss of generality, $V \subset [0, 1]$. Indeed if we can adapt the choice function f above such that

$$\tilde{f}([x]) := f([x]) - k,$$

where $k \in \mathbb{Z}$ is chosen such that $f([x]) \in [k, k + 1)$.

Now if $q_1, q_2 \in \mathbb{Q}$, $q_1 \neq q_2$ then

$$(q_1 + V) \cap (q_2 + V) = \emptyset$$

Indeed if for $v_1, v_2 \in V$

$$\begin{aligned} & q_1 + v_1 = q_2 + v_2 \\ \Leftrightarrow & v_2 - v_1 = q_1 - q_2 \in \mathbb{Q} \setminus \{0\} \\ \Rightarrow & [v_2] = [v_1] \end{aligned}$$

Since V contains exactly one representative of $[v_2] = [v_1]$ we conclude that $v_1 = v_2$ which implies $q_1 = q_2$.

Take now

$$q_1, q_2, q_3, \ldots \subset \mathbb{Q} \cap [-1, 1]$$

and enumeration of $\mathbb{Q} \cap [-1, 1]$ and set

$$V_k := q_k + V.$$

Then $V_k \cap V_j = 0$ for $k \neq j$. Moreover by translation invariance we have

$$\mu(V_k) = \mu(V).$$

Notice that we also have

$$[0, 1) \subset \bigcup_k V_k \tag{1.3}$$

Indeed, any $x \in [0, 1)$ belongs to some equivalence class $[x] = [v]$ for some $v \in V$. That is for some $v \in V$ we have $x - v \in \mathbb{Q}$. Since $x \in [0, 1)$ and $v \in [0, 1]$ we have $x - v \in [-1, 1)$, thus there exists k such that $q_k = x - v$, i.e. $x \in V_k$.

Set $S := \bigcup_k V_k$.

We have for any $N \in \mathbb{N}$ using finite additivitiy of measurable sets, Theorem 1.51,

$$\mu(S) = \mu(\bigcup_{k=1}^{N} V_k + \bigcup_{k=N+1}^{\infty} V_k) = \mu(\bigcup_{k=1}^{N} V_k) + \mu(\bigcup_{k=N+1}^{\infty} V_k) \geq \mu(\bigcup_{k=1}^{N} V_k) = N\mu(V_k) = N\mu(V).$$

On the other hand $V_k \subset [-1, 2]$, so

$$\mu(S) = \mu([-1, 2]) - \mu(S \setminus [-1, 2]) \leq \mu([-1, 2]) \leq 3\mu([0, 1]).$$

The last inequality is by translation invariance.

So

$$N\mu(V) \leq 3\mu([0, 1]) \quad \forall N \in \mathbb{N}$$

If $\mu([0, 1]) < \infty$ we find $\mu(V) = 0$ (otherwise we let $N \to \infty$ to arrive at a contradiction).

On the other hand, by countable subadditivity

$$\mu([0, 1)) \overset{(1.3)}{\leq} \sum_{k \in \mathbb{N}} \mu(V_k) = \sum_{k \in \mathbb{N}} \mu(V)$$

So if $\mu(V) = 0$ we must have $\mu([0,1)) = 0$.

So either $\mu([0,1)) = 0$ (which by translation invariance and monotonicity means $\mu([0,1]) = 0$), or $\mu([0,1]) = \infty$. □

This is all crazy. We have to fix this.

Option 1: Not only Vitali, but also Banach-Tarski's argument relies on the axiom of choice: we could *become constructivists* and just abandon the axiom of choice. (some people, Brouwer for one, did)

Option 2: We relax the notion of a measure. We accept, that the assumptions in Theorem 1.5 are too strong (because there is no reasonable object that satisfies them). The problem is the assumption

$$\mu(A \cup B) = \mu(A) + \mu(B) \quad \text{whenever } A \cap B = \emptyset.$$

It is just not a good assumption. The solution is require the above not for *all sets A, B*, but only for a subclass of sets, the measurable sets. So the big conceptual achievement obtained in the 1920s is the acceptance of "*non-measurable sets*", i.e. sets whose "volume" just doesn't make any sense. As we shall see, the sets in Theorem 1.3 are examples of non-measurable sets (for the Lebesgue measure), and the existence of non-measurable sets (for the Lebesgue measure) is closely tied to the axiom of choice, again see Theorem 1.105).

So, *let us start again*. We have to lower our expectiations on a reasonable volume.

Alfred Tarski (14 January 1901–26 October 1983) was a Polish-born American mathematician and logician whose work on model theory, metamathematics, and the semantic theory of truth reshaped modern logic. Educated at the University of Warsaw (Ph.D., 1924), he taught in Poland until 1939, then joined the University of California, Berkeley in 1942 and remained there until his death. His landmark contributions include Tarski's undefinability theorem, Convention T for truth, and foundational results in model theory and algebraic logic: To illustrate Convention T, he offered the famous schema **"'Snow is white' is true if and only if snow is white"**. He was born Alfred Teitelbaum to a Polish-Jewish family and in 1923 he and his brother formally adopted the surname "Tarski". On the eve of World War II, he travelled to the United States (August 1939) to give lectures at Harvard and Chicago – just weeks before the German and Soviet invasions of Poland – and never returned.

Giuseppe Vitali (26 August 1875–29 February 1932) was an Italian mathematician whose pioneering work in measure theory gave us the first example of a non-Lebesgue-measurable set, now known as the Vitali set. Born in Ravenna to Domenico Vitali and Zenobia Casadio, he studied at the Scuola Normale Superiore in Pisa, graduating in 1899, and then earned a teaching diploma. From 1904 until 1923 he taught in various Italian secondary schools; it was during this period—as a high-school teacher rather than a university academic—that he published his landmark 1905 paper *Sul problema della misura dei gruppi di punti di una retta*, giving rise to the Vitali set via a local printer, Tipografia Gamberini e Parmeggiani. After 1923 Vitali held professorships at the Universities of Modena (1923–1925), Padua (1925–1930), and Bologna (from 1930), and he proved the Vitali covering theorem along with key convergence theorems for measurable and holomorphic functions. In the mid-1920s he developed a paralysis in his writing arm that left him unable to write by hand, yet remarkably about half of his research output appeared in the last four years of his life. An active member of the Italian Socialist Party until its forced dissolution by Mussolini's regime in 1922, he endured increasing social isolation under fascist rule. On 29 February 1932, shortly after delivering a lecture at the University of Bologna, Vitali collapsed in conversation with his colleague Ettore Bortolotti and died suddenly. Despite the deep paradox his non-measurable set embodied, Vitali was known among friends for his wry amusement at how a simple choice-based construction could upend the very notion of "length." He would quip that in mathematics, as in life, sometimes **"the smallest freedom can lead to the most astonishing contradictions."**

1.2. **The real definition: (outer) measures.** Instead of defining a volume in $\mathbb{R}^n$ axiomatically, let us generally define what a reasonable notion of a volume should satisfy. Later we will then construct the *Lebesgue measure* that has most of the desired properties on $\mathbb{R}^n$ (and coincides with the notion of volumes for easy shapes).

Clearly $\mu(\emptyset) = 0$ is a reasonable assumption. Ideally we would also like $\mu(A \cup B) = \mu(A) + \mu(B)$ – but as we have seen this is tricky, confusing, and paradox assumption (the above examples are a warning). We settle for the following notion

Definition 1.6. Let X be any set and 2^X the potential set of X. A map $\mu : 2^X \to [0, \infty]$ is a *measure* on X if we have

(1) $\mu(\emptyset) = 0$
(2) $\mu(A) \le \sum_{k=1}^{\infty} \mu(A_k)$ whenever $A, A_k \subset X$, $k \in \mathbb{N}$ and $A \subset \bigcup_{k\in\mathbb{N}} A_k$ (*σ-subadditivity*)

Exercise 1.7. *Show that condition (1) and (2) of Definition 1.6 imply* monotonicity, *i.e. we have*

$$\mu(A) \le \mu(B) \quad \forall A \subset B.$$

Exercise 1.8. *Let X be any set and 2^X the potential set of X. Show that a map $\mu : 2^X \to [0,\infty]$ is a measure on X (in the sense of Definition 1.6) if and only if we have*

(1) $\mu(\emptyset) = 0$
(2) $\mu(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \mu(A_k)$ *for all* $A_k \subset X$

Remark 1.9.
- A word of warning: we will use here the notion of an *outer measure* that is defined on all of 2^X, not just on some σ-Algebra. Other textbooks might use a different notion of measure, only defined on its σ-algebra of measurable sets (see below).
- Definition 1.6 implies in particular that $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for any set $A, B \subset X$. However, in general, we do not assume (at all) $\mu(A \cup B) = \mu(A) + \mu(B)$ for disjoint sets A and B. Again, this will lead to the notion of *measurable sets* and *non-measurable sets*.

Example 1.10 (Jordan content)**.**
- The outer *Jordan content* $J_*(E)$ of a set $E \subset \mathbb{R}^n$ is defined as follows.

  For a product of bounded cubes $C = [a_1, b_1) \times [a_2, b_2) \times \ldots \times [a_n, b_n)$ we set

$$\mathrm{vol}(C) := (b_1 - a_1) \cdot (b_2 - a_2) \cdot \ldots (b_n - a_n).$$

$$J^*(E) := \inf \left\{ \sum_{i=1}^{N} \mathrm{vol}(C_i) \quad \text{for some } N \in \mathbb{N}, \text{ and cubes } (C_i)_{i=1}^N \text{ such that } E \subset \bigcup_{i=1}^{N} C_i \right\}$$

  Here we follow the convention that $\inf \emptyset = +\infty$.

  $J^*(\cdot)$ is *not* a measure – take any enumeratotion of $\mathbb{Q} \cap [0,1] = \{q_1, \ldots, q_n, \ldots\}$. Set $A_k := \{q_k\}$ and $A := \bigcup_{k=1}^{\infty} A_k = [0,1] \cap \mathbb{Q}$. If $(C_i)_{i=1}^N$ is a finite cover of $[0,1] \cap \mathbb{Q}$ then[3] $\bigcup_i \overline{C_i} \supset [0,1]$, so $J^*(A) = 1$. However $J^*(A_k) = 0$ for each k, we violate the subadditivity assumption of measures, Definition 1.6,

$$J^*(A) \not\leq \sum_{k=1}^{\infty} J^*(A_k).$$

  However J^* satisfies finite subadditivity,

$$J^*(A \cup B) \leq J^*(A) + J^*(B),$$

  so, by induction,

$$J^*(A) \leq \sum_{k=1}^{N} J^*(A_k) \quad \text{whenever } A, A_k \subset X,\ k \in \{1, \ldots, N\},\ N \in \mathbb{N}, \text{ and } A \subset \bigcup_{k \in \mathbb{N}} A_k.$$

  Such a map $J^* : 2^X \to [0,\infty)$ is called a *content*.

[3]Indeed, take $r \in [0,1]$ then there exists q_k converging to r, q_k belongs infinitely often to the same interval, so $r \in \overline{C_i}$ for some i

- The countable version of the outer Jordan content, is called the *Lebesgue outer measure*

$$m^*(E) := \inf \left\{ \sum_{i=1}^{\infty} \mathrm{vol}(C_i) \quad \text{for some , and cubes } (C_i)_{i=1}^{\infty} \text{ such that } E \subset \bigcup_{i=1}^{\infty} C_i \right\} \tag{1.4}$$

It is again clear that $m^*(\emptyset) = 0$. Let now $A \subset \bigcup_{k=1}^{n} A_k$. We may assume that $m^*(A_k) < \infty$ otherwise there is nothing to show. Fix $\varepsilon > 0$. For each k we can pick $(C_{k;i})_{i=1}^{\infty}$ such that $\bigcup_{i=1}^{\infty} C_{k;i} \supset A_k$ and

$$\sum_{i=1}^{\infty} \mathrm{vol}(C_{k,i}) \leq m^*(A_k) + \frac{\varepsilon}{2^k}.$$

Now $\bigcup_{k,i\in\mathbb{N}} C_{k,i} \supset A$ and thus (since (1.4) allows for infinite covers)

$$m^*(A) \leq \sum_{k,i\in\mathbb{N}} \mathrm{vol}(C_{k,i}) \leq \sum_{k=1}^{\infty} m^*(A_k) + \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k}$$

That is, we have shown that for any $\varepsilon > 0$,

$$m^*(A) \leq \sum_{k=1}^{\infty} m^*(A_k) + \varepsilon$$

Taking $\varepsilon \to 0$ we conclude that $m^*(A) \leq \sum_{k=1}^{\infty} m^*(A_k)$ – that is $m^*(A)$ is indeed a measure.

Later the Lebesgue measure $\mathscr{L}^n$ will coincide with $m^*(A)$.

Henri Léon Lebesgue (28 June 1875–26 July 1941) was a French mathematician whose work laid the foundations of modern analysis. Born in Beauvais, Oise, to a typesetter father and schoolteacher mother, Lebesgue showed early mathematical talent. He entered the École Normale Supérieure in Paris in 1894, graduating in 1897, then pursued graduate studies at the Sorbonne under Émile Borel. In 1902 he completed his doctoral thesis *Intégrale, longueur, aire* ("Integral, Length, Area"), which introduced the concepts of measure and integration that bear his name. After earning his doctorate, Lebesgue lectured at the University of Rennes (1902–1906) and the University of Poitiers (1906–1910). In 1910 he joined the Sorbonne as *maitre de conférences*, became full professor in 1919, and in 1921 was appointed to the Collège de France, where he remained until his death. Lebesgue's theory of measure and the **Lebesgue integral** generalized the classical Riemann integral, providing a rigorous framework for convergence theorems and underpinning modern real analysis, probability, and functional analysis. He was elected to the Académie des Sciences in 1922 and awarded the Poncelet Prize in 1914. He died in Paris on 26 July 1941. Lebesgue once quipped, "Réduites à des théories générales, les mathématiques seraient une belle forme sans contenu." ("Reduced to general theories, mathematics would be a beautiful form without content.")

- The *inner Jordan content*,

$$J_*(E) := \sup \left\{ \sum_{i=1}^{N} \mathrm{vol}(C_i) \quad \text{for some } N \in \mathbb{N}, \text{ and cubes } (C_i)_{i=1}^N \text{ such that } \bigcup_{i=1}^{N} C_i \subset E \right\}$$

  Here we follow the convention that $\sup \emptyset = 0$.

  Still $J_*(\cdot)$ is not a measure. Take $A_1 := [0,1]\backslash\mathbb{Q}$ and for $i \geq 2$ we set $A_i = \{q_i\}$ for $\{q_2, \ldots, \} = \mathbb{Q} \cap [0,1]$ any enumeration of $\mathbb{Q} \cap [0,1]$. Since A_1 has empty interior we have $J_*(A_1) = 0$. Similarly, $J_*(A_i) = 0$ for $i \geq 2$. However $A := \bigcup_{i=1}^{\infty} A_i = [0,1]$ satisfies $J_*([0,1]) = 1$. So we have $J_*(A) \not\leq \sum_{i=1}^{n} J_*(A_i)$.
- If we simply make the innter Jordan content countable, i.e. if we set

$$\tilde{J}_*(E) := \sup \left\{ \sum_{i=1}^{\infty} \mathrm{vol}(C_i) \quad \text{for cubes } (C_i)_{i=1}^N \text{ such that } \bigcup_{i=1}^{\infty} C_i \subset E \right\}$$

  we run into the same problem as for J_*, namely $J_*([0,1]\backslash\mathbb{Q}) = 0$. So $\tilde{J}_*(E)$ is still not a measure.

Count von Count is a fictional character from the children's television show Sesame Street, known for his love of counting just about anything. Modeled as a friendly parody of a vampire in the style of Bela Lugosi's Dracula, the Count first appeared in 1972 and quickly became one of the most recognizable and beloved Muppet characters on the show.
With his monocle, goatee, cape, and thick Eastern European accent, the Count adds a spooky but comical flair to the task of learning numbers. His main role on Sesame Street is to teach children how to count, often breaking into maniacal laughter ("Ah ah ah!") after finishing a sequence. This laugh is typically accompanied by thunder and lightning, which appears magically even on sunny days.
Created by Norman Stiles and voiced originally by Jerry Nelson (and later by Matt Vogel), the Count was designed to make early math engaging and memorable for children. His obsession with counting is presented as humorous and charming rather than scary, turning a potentially intimidating subject into fun.
The idea that vampires compulsively count things comes from real folklore in parts of Eastern Europe, which may have inspired the character's shtick. According to legend, one way to distract a vampire was to scatter grains or seeds–because the vampire would feel compelled to count them all. He is not at all related to the counting measure of Exercise 1.11

Exercise 1.11 (Counting measure)**.** *Let X be any set. Show that $\# : 2^X \to \mathbb{N} \cup \{0\}$ defined by*

$$\#A := \text{number of elements in } A,$$

is a measure. It is called the counting measure.

Exercise 1.12 (Dirac measure)**.** *Let X be any set and $a \in X$. Show that $\delta_a : 2^X \to \{0, 1\}$ defined by*

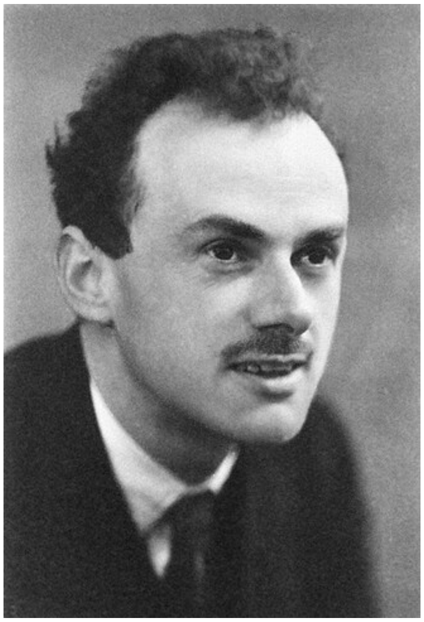
$$\delta_a A = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{if } a \notin A \end{cases}$$

is a measure. It is called the Dirac measure.

Exercise 1.13. *Let X be a metric space and $\mu : 2^X \to [0, \infty]$ a measure. Let $A \subset X$ and define $\mu \llcorner A$: $2^X \to [0, \infty]$ by*

$$(\mu \llcorner A)(B) := \mu(A \cap B).$$

Show that $\mu \llcorner A$ is a measure.



Paul Adrien Maurice Dirac (1902–1984) was a British theoretical physicist and mathematician, widely regarded as one of the founding figures of quantum mechanics and quantum field theory. Born in Bristol, England, he studied electrical engineering and then mathematics, eventually earning a PhD in theoretical physics under Ralph Fowler at Cambridge. Dirac's work was marked by extraordinary mathematical elegance and depth, often anticipating entire fields of later development.

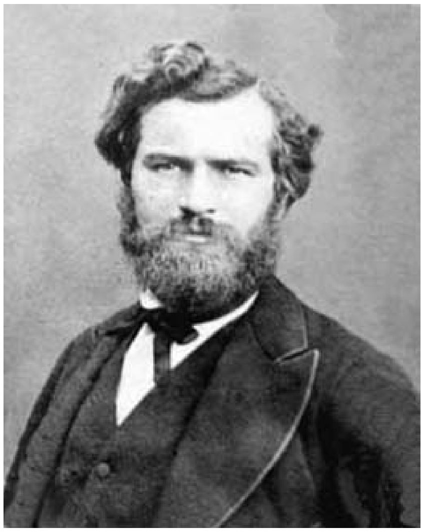
Dirac is best known for the Dirac equation, which describes the behavior of relativistic electrons and predicted the existence of antimatter. This equation combined quantum mechanics and special relativity and became a cornerstone of quantum field theory.

In mathematics and physics, Dirac introduced what is now called the Dirac delta function and the Dirac measure. The Dirac delta is not a function in the classical sense but rather a generalized function or distribution, defined to be zero everywhere except at one point where it is ”infinite” in such a way that its integral is one. It is used to model point charges, impulses, and localized sources in physical systems. The Dirac measure is the corresponding measure-theoretic object: it assigns all the measure to a single point and zero elsewhere, playing a crucial role in probability theory, functional analysis, and the theory of distributions.

Dirac believed deeply in the aesthetic principle that "it is more important to have beauty in one's equations than to have them fit experiment." His pursuit of mathematical beauty often led to physical insights, including the prediction of the positron and foundational ideas in gauge theory and quantum statistics.

Despite his revolutionary impact, Dirac was known for his terse and reclusive nature. An oft-repeated anecdote is that Niels Bohr once said, ”Dirac is the strangest man who ever visited my institute," and colleagues joked that a unit of silence could be measured in ”one Dirac."

Dirac received many honors, including the Nobel Prize in Physics in 1933 (shared with Erwin Schrödinger), and he held positions at Cambridge and later at Florida State University.



Camille Jordan (1838–1922) was a French mathematician whose work helped shape modern analysis, topology, and algebra. Born in Lyon, he studied at the École Polytechnique and later became a professor in Paris. Jordan is especially associated with the concept of Jordan content (or Jordan measure), an early attempt to assign a ”size” to subsets of Euclidean space by approximating them from inside and outside with finite unions of rectangles. Though later superseded in generality by Lebesgue measure, Jordan's approach was influential in the rigorous development of integration theory and is still introduced in elementary analysis as a bridge to more advanced measure theory.
Jordan also made major contributions to group theory and linear algebra. His Traité des substitutions et des équations algébriques was foundational in the development of permutation group theory. In linear algebra, the Jordan normal form (Jordan canonical form) bears his name and provides a structural classification of linear operators up to similarity, central in both theory and applications. In topology and complex analysis, the Jordan curve theorem (that a simple closed curve in the plane divides the plane into an inside and an outside) is another landmark associated with his name, though its complete proof required later refinement.

1.3. **Example: Hausdorff measure.** Let (X, d) be a metric space.

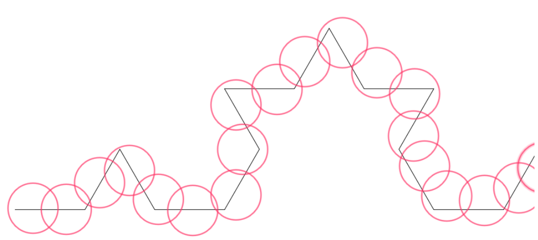
Definition 1.14. The s-dimensional *Hausdorff measure*, $s > 0$ is defined as follows.

Let $\delta \in (0, \infty]$, then for any $A \subset X$ we define

$$\mathscr{H}_\delta^s(A) := \alpha(s) \inf \left\{ \sum_{k=1}^{\infty} r_k^s : \quad A \subset \bigcup_{k=1}^{\infty} B(x_k, r_k), \quad r_k \in (0, \delta) \right\}.$$

Here $B(x_k, r_k)$ are open balls with radius r centered at x_k, i.e.

$$B(x_k, r_k) := \{y \in X : d(x_k, y) < r_k\}.$$



Moreover[4]

$$\alpha(s) := \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}.$$

where Γ is the Γ-function.

Now observe that $\delta \mapsto \mathcal{H}^s_\delta(A)$ is monotonce decreasing. So we can write

$$\mathscr{H}^s(A) := \lim_{\delta\to 0^+} \mathscr{H}^s_\delta(A) \equiv \sup_{\delta>0} \mathscr{H}^s_\delta(A) \in [0,\infty].$$

Often one writes $\mathscr{H}^0(A) := \#A$, the *counting measure*.

$\mathscr{H}^s_\infty$ is called the *Hausdorff content*

Remark 1.15.
- Observe that while $\mathscr{H}^s_\delta(A) < \infty$ whenever $s > 0$, $\delta > 0$ and A is any bounded set, as $\delta \to 0$ $\mathscr{H}^s(A)$ will be infinite whenever s is smaller than the "dimension of A" (a notion we will define more carefully below).

Exercise 1.16. *Show that*

(1) For $\delta > 0$ the map $\mathscr{H}^s_\delta$ defines a measure
(2) The map $\mathscr{H}^s_\infty$ defines a measure

Lemma 1.17. *For any $s \in [0,\infty)$, $\mathscr{H}^s$ is a measure in $\mathbb{R}^n$.*

Proof. $\mathscr{H}^s_\delta(\cdot)$ is a measure for each $\delta > 0$, Exercise 1.16.

We clearly have $\mathscr{H}^s(\emptyset) = 0$. Moreover, since $\mathscr{H}^s_\delta$ is a measure for any $\delta > 0$, we have for any $A \subset \bigcup_{k=1}^\infty A_k$,

$$\mathcal{H}^s_\delta(A) \leq \sum_{k=1}^\infty \mathscr{H}^s_\delta(A_k) \leq \sum_{k=1}^\infty \mathscr{H}^s(A_k).$$

Taking the supremum over δ in this inequality we have σ-additivity for $\mathscr{H}^s$.

$$\mathscr{H}^s(A) \leq \sum_{k=1}^\infty \mathscr{H}^s(A_k).$$

□

Exercise 1.18. *Show that*

$$\mathscr{H}^0_\delta(\mathbb{Q}) \xrightarrow{\delta\to 0} \infty.$$

If $k \in \mathbb{N}$ it is conceivable that $\mathscr{H}^k$ measures something of "dimension k". For example assume that $C = [0,1]^2 \times \{0\} \subset \mathbb{R}^3$ is a 2D-square of sidelength 1.

We need $\approx \frac{1}{\delta^2}$ many balls of radius δ to cover C. Then

$$\mathscr{H}^s_\delta(C) \leq \alpha(s)\frac{1}{\delta^2}\delta^s.$$

[4]Warning: Some authors set $\alpha(s) := 1$. The main reason to not do that is so that $\mathscr{H}^n = \mathscr{L}^n$ in $\mathbb{R}^n$

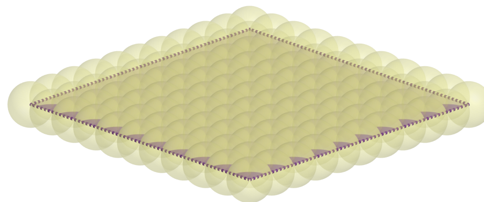


FIGURE 1.3. To cover a square of sidelength one with balls of radius δ, we need roughly $\frac{1}{\delta^2}$ many balls

So if $s > 2$ we see that $\mathscr{H}^s(C) \leq \lim_{\delta \to 0} \delta^{s-2} = 0$. That is C has no s-volume for $s > 2$.

For $s = 2$ one can argue that covering uniformly by balls of radius δ is optimal and thus we have

$$0 < \mathscr{H}^2(C) < \infty.$$

In particular $\mathscr{H}^s(C) = \infty$ for any $s < 2$. We want to investigate this more, but first let us gather some more properties

Remark 1.19. One can, and we will in Corollary 1.97, show that the n-dimensional Hausdorff measure in $\mathbb{R}^n$ coincides with the Lebesgue measure $\mathscr{L}^n$, i.e.

$$\mathscr{L}^n(A) = \mathscr{H}^n(A).$$

Exercise 1.20. *Let $U \subset \mathbb{R}^n$ be any non-empty open set. Then $\mathscr{H}^s(U) = \infty$ for all $s < n$.*

Exercise 1.21 (translation and rotation invariant)**.** *Let $A \subset \mathbb{R}^n$ and $s \in (0, \infty)$. Show the following*

(1) If $p \in \mathbb{R}^n$ then $\mathscr{H}^s(p + A) = \mathscr{H}^s(A)$.
(2) If $O \in O(n)$ (i.e. $O \in \mathbb{R}^{n \times n}$ and $O^t O = I$) then $\mathscr{H}^s(OA) = \mathscr{H}^s(A)$.
(3) If $A \subset \mathbb{R}^\ell \times \{0\}$ for $0 < \ell < n$ and $\pi : (x_1, \ldots, x_n) := (x_1, \ldots, x_\ell)$ is the projection from $\mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}^{n-\ell}$ to $\mathbb{R}^\ell$, then $\mathscr{H}^s_{\mathbb{R}^n}(A) = \mathscr{H}^s_{\mathbb{R}^\ell}(\pi(A))$.

Exercise 1.22. *Let for $1 < k \leq n$*

$$K := \{0\}^{n-k} \times \mathbb{R}^k \subset \mathbb{R}^n$$

Show that

$$\mathscr{H}^s(K) = \begin{cases} \infty & \text{if } s \leq k \\ 0 & \text{if } s > k \end{cases}$$

Exercise 1.23. *Let $\mathcal{M} \subset \mathbb{R}^n$ be a compact k-dimensional submanifold. Show that*

$$\mathscr{H}^s(\mathcal{M}) = \begin{cases} \infty & \text{if } s < k \\ < \infty & \text{if } s = k \\ 0 & \text{if } s \geq k \end{cases}$$

We want to investigate more this "dimension" of the Hausdorff measure. For this we observe that for any set A there is exactly one threshhold s where $s \mapsto H^s(A)$ changes from ∞ to 0. More precisely we have

Lemma 1.24. *Let $0 \leq s < t < \infty$.*

(1) If $\mathscr{H}^s(A) < \infty$ then $\mathscr{H}^t(A) = 0$
(2) If $\mathscr{H}^t(A) > 0$ then $\mathscr{H}^s(A) = \infty$.

Proof. Indeed, whenever $r_k \leq \delta$ and $(B(x_k, r_k))_{k\in\mathbb{N}}$ cover A we have

$$\mathscr{H}^t_\delta(A) \leq \alpha(t) \sum_{k=1}^\infty r_k^t \leq \alpha(t)\delta^{t-s} \sum_{k=1}^\infty r_k^s.$$

Taking the infimum over any such covering $B(x_k, r_k)$ of A we find

$$\mathscr{H}^t_\delta(A) \leq \frac{\alpha(t)}{\alpha(s)} \delta^{t-s} \mathscr{H}^s_\delta(A).$$

Taking $\lim_{\delta\to 0}$ on both sides we obtain

$$\mathscr{H}^t(A) \leq \frac{\alpha(t)}{\alpha(s)} 0 \cdot \mathscr{H}^s(A).$$

This implies that if $\mathscr{H}^t(A) > 0$ then necessarily $\mathscr{H}^s(A) = \infty$, and if $\mathscr{H}^s(A) < \infty$ then $\mathscr{H}^t(A) = 0$. □

Indeed, with the Hausdorff measure we can define a dimension

Definition 1.25. The *Hausdorff dimension* is defined as

$$\dim_{\mathscr{H}} A := \inf\left\{s \geq 0 : \quad \mathscr{H}^s(A) = 0\right\}.$$

If $\mathscr{H}^s(A) > 0$ for all $s > 0$ then $\dim_{\mathscr{H}}(E) := \infty$.

Lemma 1.26. *Let C be a set in a metric space (X, d) and let $s \geq 0$*

(1) If $\mathscr{H}^s(C) = 0$ then $\dim_{\mathscr{H}}(E) \leq s$.
(2) If $\mathscr{H}^s(C) > 0$ then $\dim_{\mathscr{H}}(E) \geq s$.
(3) If $0 < \mathscr{H}^s(C) < \infty$ then $\dim_{\mathscr{H}}(E) = s$.
(4) If $\mathscr{H}^s_\infty(C) > 0$ and $\mathscr{H}^s(C) < \infty$ then $\dim_{\mathscr{H}}(E) = s$.

Proof. This follows from Lemma 1.24 and the definition of Hausdorff measure.

(1) follows from the definition of the Hausdorff measure as infimum. then $\dim_{\mathscr{H}}(E) \leq s$.
(2) If $\mathscr{H}^s(C) > 0$ then by Lemma 1.24 $\mathscr{H}^t(C) = \infty$ for all $t < s$. Again from the definition it is clear that $\dim_{\mathscr{H}}(E) \geq s$.
(3) This is a consequence of the two above statements.
(4) Follows from the statement before since $\mathscr{H}^s_\infty(C) \leq \mathscr{H}^s(C)$

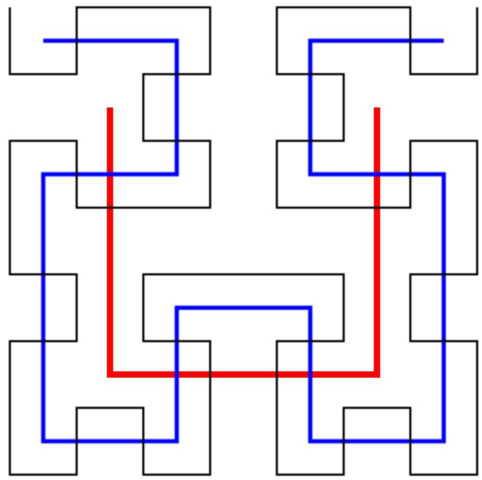


FIGURE 1.4. The Hilbert curve from is often described as a limit of a sequence of curves. It can, however, also be descriped by a fixed-point argument, Exercise 1.29

□

Exercise 1.27. *Let $C \subset X$ where X is a metric space (X, d).*

(1) Show that $\dim_{\mathscr{H}} C \leq \dim_{\mathscr{H}} X$
(2) Conclude that if $X = \mathbb{R}^n$ with euclidean metric, then $\dim_{\mathscr{H}} C \leq n$.

Exercise 1.28 (Hausdorff dimension under Lipschitz and Hölder maps)**.** *Let (X, d_x) and (Y, d_Y) be two metric spaces and let $f : X \to Y$. Assume that $A \subset X$ has Hausdorff-dimension $\dim_{\mathscr{H}}(A) = s$.*

(1) If f is uniformly Lipschitz continuous, i.e. for some $L > 0$,

$$d_Y(f(x), f(y)) \leq L\, d(x, y) \quad \forall x, y \in X$$

then $\dim_{\mathscr{H}}(f(A)) \leq s$.
(2) Give an example where $\dim_{\mathscr{H}}(A) < s$
(3) Assume f is uniformly Hölder continuous, i.e. for some $L > 0$ and $\alpha > 0$

$$d_Y(f(x), f(y)) \leq L\, d(x, y)^{\alpha} \quad \forall x, y \in X$$

Show that $\dim_{\mathscr{H}}(f(A)) \leq \frac{s}{\alpha}$.

Exercise 1.29 (Hilbert curve)**.** *The Hilbert curve $f : [0, 1] \to [0, 1]^2$, cf. Exercise 1.29, is a so-called spacefilling curve.*

It is an example of a $C^{1/2}$-map from $[0, 1]$ (Hausdorff dimension 1) onto $[0, 1]^2$ (Hausdorff dimension 2), showing that the estimates from Exercise 1.28 are really sharp. Prove this:

Consider the following four affine transformations f_i:

$$f_0(x,y) = \left(\frac{y}{2}, \frac{x}{2}\right)$$
$$f_1(x,y) = \left(\frac{x}{2}, \frac{y}{2} + \frac{1}{2}\right)$$
$$f_2(x,y) = \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{2} + \frac{1}{2}\right)$$
$$f_3(x,y) = \left(1 - \frac{y}{2}, \frac{1-x}{2}\right)$$

Each f_i maps $[0,1]^2$ to a subquadrant $p_i + [0, \frac{1}{2}]^2$.

Consider the operator $\mathcal{T} : C^0([0,1],[0,1]^2) \mapsto C^0([0,1],[0,1]^2)$ such that

$$(\mathcal{T}\gamma)(t) = \begin{cases} f_0(\gamma(4t)) & \text{if } t \in [0,1/4] \\ f_1(\gamma(4t-1)) & \text{if } t \in [1/4,2/4] \\ f_2(\gamma(4t-2)) & \text{if } t \in [2/4,3/4] \\ f_3(\gamma(4t-3)) & \text{if } t \in [3/4,1] \end{cases}$$

The Hilbert curve is defined as the unique continuous map $H \in C^0([0,1],[0,1]^2)$ such that $\mathcal{T}H = H$.

1. *Show that there is a unique continuous map H such that $\mathcal{T}(H) = H$.*
   For this let $\mathcal{C}$ be the space of all continuous functions from the interval $[0,1]$ to the unit square $[0,1]^2$, i.e. $\mathcal{C} = C([0,1],[0,1]^2)$, equipped with the L^∞-norm. We know this space is a complete metric space. Show that $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is a contraction.
2. *Show that H is Hölder continuous of order $\frac{1}{2}$. For this assume that $s,t \in [0,1]$ such that $4^{-n} \leq |s-t| < 4^{-n+1}$ for some $n \in \mathbb{N}$. Partition $[0,1]$ into 4^n-subintervals such that $[k/4^n, (k+1)/4^n]$. Then t,s can lie either in the same, or in two adjacent of these intervals. So H must map into two subsquares of sidelength 2^{-n} within the unit square $[0,1]^2$. Observe that by the Hilbert curve construction these two squares are adjacent. That is, the points $H(s)$ and $H(t)$ lie within the union of at most two adjacent squares of side length 2^{-n}, compute the size of this rectangle, and estimate $|H(s) - H(t)| \leq \sqrt{5}2^{-n}$.*
3. *Show that H is surjective. For this consider a point $p \in [0,1]^2$. Take a sequence of subquadrants Q_n of sidelength 2^{-n} such that $p \in Q_n$. Show that there is a sequence of points $t_n \in [0,1]$ with $H(t_n) \in Q_n$, and conclude the existence of $t \in [0,1]$ such that $H(t) = p$.*
4. *How does this show that the results in Exercise 1.28 are sharp?*
5. *Is H injective?*

Example 1.30. The Cantor set is defined as follows.

$$C_0 := [0,1]$$

FIGURE 1.5. The Cantor set

Let $C_0 := [0,1]$. In the k-th step we construct C_k by removing of each interval the open middle interval of size 3^{-n}. For exmaple

$$C_1 := [0, \frac{1}{3}] \cup [\frac{1}{3}, 1].$$

See Figure 1.5.

Set $C := \bigcap_{k=1}^{\infty} C_k$. Observe that C is closed and bounded, so compact.

Lemma 1.31. $\dim_{\mathscr{H}}(C) = \frac{\log 2}{\log 3}$.

Proof. For each $k \in \mathbb{N}$ we have $C \subset C_k$. Observe that C_k consists of 2^k disjoint intervals each of diameter 3^{-k} (i.e. radius $\frac{1}{2}3^{-k}$). Thus for any $\delta > 0$ and for any $k \gg 1$ so that $\frac{1}{2}3^{-k} < \delta$ we have

$$\mathscr{H}^s_\delta(C) \leq \alpha(s) \sum_{\ell=1}^{2^k} (\frac{1}{2}3^{-k})^s = 2^{-s} \left(\frac{2}{3^s}\right)^k \xrightarrow{k\to\infty} \alpha(s) \begin{cases} 2^{-s} & s = \alpha(s)\frac{\log 2}{\log 3} \\ 0 & s > \frac{\log 2}{\log 3} \\ \infty & s < \frac{\log 2}{\log 3} \end{cases}$$

In particular we have

$$\mathscr{H}^s(C) = 0 \quad \forall s > \frac{\log 2}{\log 3}.$$

So from the definition of the Hausdorff dimension we get

$$\dim_{\mathscr{H}} C \leq \frac{\log 2}{\log 3}.$$

Now we need to show the <u>other direction</u>. From now on set $s := \frac{\log 2}{\log 3}$. Let $(B(x_i, r_i))_{i=1}^{\infty}$ be any covering of C. We claim that

$$\sum_{i=1}^{\infty} r_i^s \geq \frac{1}{2^s 4}. \tag{1.5}$$

Once we have (1.5) we are done, because (1.5) implies

$$\mathscr{H}^s_\infty(C) \geq \frac{1}{2^s 4}.$$

In particular (recall that $s = \frac{\log 2}{\log 3}$) we have $\infty > \mathscr{H}^s(C) \geq \mathscr{H}^s_\infty(C) > 0$.

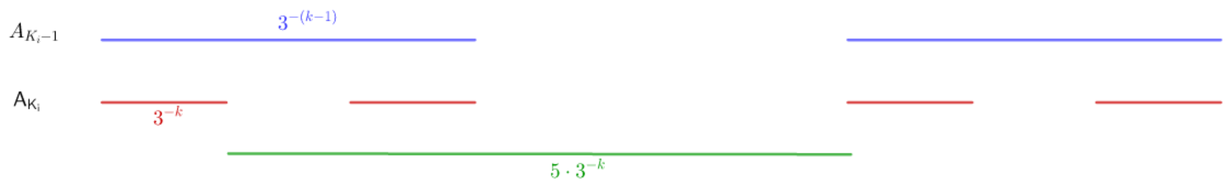


FIGURE 1.6. If a ball intersects three intervals of A_{K_i} its *diameter* is at least $5 \cdot 3^{-K_i}$

Let us make some notation. Denote by A_k the intervals of C_k, i.e. A_k consists of pairwise disjoint, closed intervals in $\mathbb{R}$ such that $C_k = \bigcup_{I \in A_k} I$. E.g.

$$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1], \quad A_1 = \{[0, \frac{1}{3}], [\frac{2}{3}, 1]\}.$$

Proof of (1.5) Since C is compact, we may assume that there finitely many, w.l.o.g. the first N balls $(B(x_i, r_i))_{i=1}^N$ already cover C. We may assume that each $r_i < \frac{1}{2}$, otherwise (1.5) is obvious.

Fix $i \in \{1, \ldots, N\}$.

Let $K_i \in \mathbb{N} \cup \{0\}$ so that

$$2r_i \in [3^{-K_i-1}, 3^{-K_i}).$$

Now we consider the construction step C_{K_i}. Each ball $B(x_i, r_i)$ has nonempty intersection with at most 2 intervals of C_{K_i}. Indeed, otherwise its *diameter* would be at least $5 \cdot 3^{-K_i}$, see Figure 1.6.

But then $B(x_i, r_i)$ has nonempty intersection with at most $2 \cdot 2^{j-K_i}$ intervals of C_j for any $j \geq K_i$. Since $s = \frac{\log 2}{\log 3}$ we have

$$2 \cdot 2^{j-K_i} = 2^{j+1} 2^{-K_i} = 2^{j+1} 3^{-K_i s} \leq 2^{j+1} 3^s (2r_i)^s = 2^{j+2} (2r_i)^s.$$

Set now $K := \max_{\{i=1,\ldots,N\}} K_i$.

Then for any $i \in \{1, \ldots, N\}$ each of the balls $B(x_i, r_i)$ has nonempty intersection with at most $2^{K+2}(2r_i)^s$ many intervals of A_K.

So if we set Γ_i to be the number of intervals in A_K that intersect $B_{r_i}(x_i)$ we have $\Gamma_i \leq 2^{K+2}(2r_i)^s$ and thus

$$\begin{aligned} \sum_{i=1}^N \Gamma_i (3^{-K})^s \leq& \sum_{i=1}^N \underbrace{(3^{-K})^s}_{=2^{-K}} 2^{K+2} (2r_i)^s \\ =& 4\ 2^s \sum_{i=1}^N (r_i)^s \end{aligned} \tag{1.6}$$

FIGURE 1.7. The fat cantor set for $a = \frac{1}{4}$, see Example 1.32

Now for each $x \in C$ there is exactly one interval I in A_K such that $x \in I$. Since $(B_{r_i}(x_i))_{i=1}^N$ covers all of C we have the following: for each interval I in A_K there exists some $i \in \{1, \ldots, N\}$ such that $B_{r_i}(x_i) \cap I \neq \emptyset$. That is,

$$\sum_{i=1}^N \Gamma_i \geq \text{number of intervals in } A_K = 2^K.$$

Thus,

$$\sum_{i=1}^N \Gamma_i 3^{-Ks} \geq 2^K 3^{-Ks} = 1. \tag{1.7}$$

Together, (1.6) and (1.7) imply (1.5). □

Example 1.32. The *Smith–Volterra–Cantor set*, aka *fat Cantor set* is defined as follows.

Let $C_0 := [0, 1]$. In the k-th step we construct C_k by removing of each interval the open middle interval of size a^n. That is

$$C_1 = [0, \frac{1-a}{2}] \cup [\frac{1+a}{2}, 1].$$

$$C_2 = [0, \frac{1-a}{4} - \frac{a^2}{2}] \cup [\frac{1-a}{4} + \frac{a^2}{2}, \frac{1-a}{2}] \cup [\frac{1+a}{2}, \frac{1+\frac{1+a}{2}}{2} - \frac{a^2}{2}] \cup [\frac{1+\frac{1+a}{2}}{2} + \frac{a^2}{2}, 1].$$

Cf. Figure 1.7.

Set $C := \bigcap_{k=1}^\infty C_k$. For $a = \frac{1}{3}$ this is the typical *Cantor set*. For $a = \frac{1}{4}$ this is the *Fat Cantor set*.

Exercise 1.33. *The fat Cantor above set has positive $\mathscr{H}^1$-measure.*

Exercise 1.34. *Let $\mathcal{M} \subset \mathbb{R}^n$ be a smooth k-dimensional submanifold. Show that the Hausdorff-dimension of $\mathcal{M}$ is k.*

Exercise 1.35. *We construct the classical Sierpiński triangle S by the following iterative process:*

(1) *Start with an equilateral triangle of side length 1. Denote this set by S_0.*
(2) *To obtain S_{n+1} from S_n, divide each equilateral triangle in S_n into four congruent equilateral triangles of one-quarter the area (scaled by $\frac{1}{2}$ in side length), and remove the open middle triangle.*

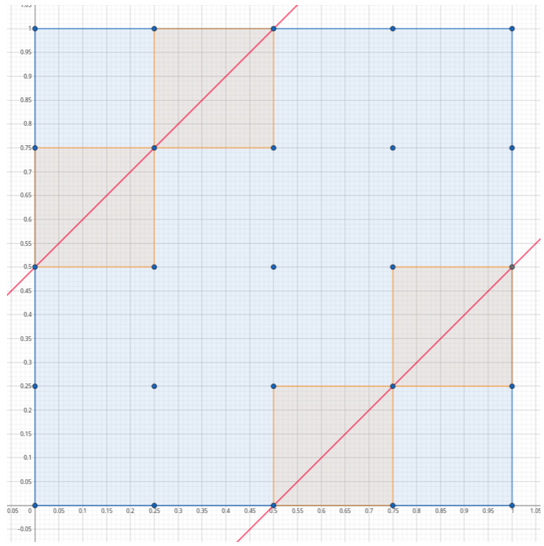


FIGURE 1.8. The construction step of the *Cantor Dust*, Exercise 1.36

(3) Define the Sierpiński triangle as the limit set:

$$S = \bigcap_{n=0}^{\infty} S_n.$$

Compute the Hausdorff dimension of the Sierpiński triangle S.

Exercise 1.36 (Cantor Dust)**.** *Let*

$$A_0 = [0,1]^2 \subset \mathbb{R}^2.$$

Subdivide A_0 by an axis-parallel grid of mesh size $1/4$ into 16 congruent subsquares. Let A_1 be the union of those 4 subsquares which meet the two horizontal line segments

$$\{(x,y) \in A_0 : y = x + \tfrac{1}{2}\} \quad \textit{and} \quad \{(x,y) \in A_0 : y = x - \tfrac{1}{2}\}.$$

Cf. Figure 1.8. Inductively, having defined A_k, form A_{k+1} by subdividing each subsquare of A_k by the same $1/4$-grid and again keeping just those subsquares meeting the two diagonal lines within that square. Finally set

$$A = \bigcap_{k=1}^{\infty} A_k.$$

Show that the Hausdorff dimension of A is 1

Exercise 1.37 (Hausdorff dimension of the Koch snowflake)**.** *The* Koch snowflake *is constructed as follows:*

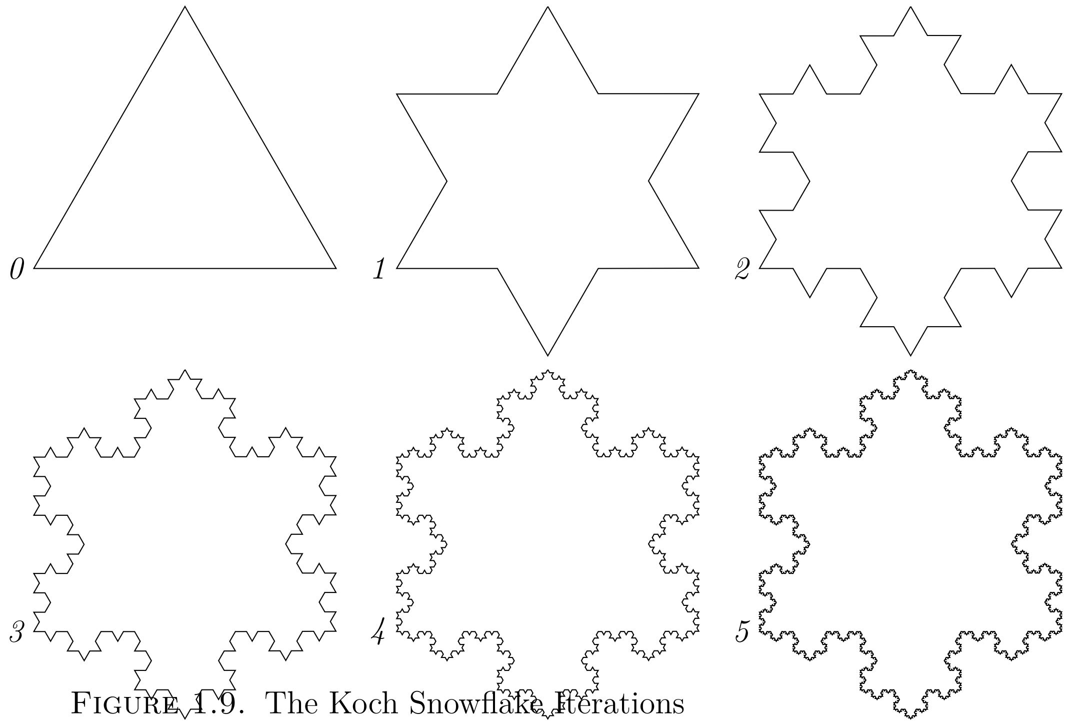


FIGURE 1.9. The Koch Snowflake Iterations

- *Start with an equilateral triangle* S_0 *of side length* 1.
- *Given the* n*-th iteration* S_n*, replace each line segment of length* 3^{-n} *by four segments of length* $3^{-(n+1)}$ *obtained by erecting on its middle third a new equilateral bump.*
- *Let* $S = \lim_{n\to\infty} S_n$ *be the limit set in the Hausdorff metric, i.e.*

$$S = \Big\{ x \in \mathbb{R}^2 : \exists\, x_n \in S_n,\ x_n \to x \text{ as } n \to \infty \Big\}.$$

Cf. Figure 1.9

(1) *Show that there exists a homeomorphism* $f : \mathbb{S}^1 \to S$*, where* $\mathbb{S}^1$ *is the unit circle.*
(2) *Show that the Hausdorff dimension of* S *is* $\frac{\ln 4}{\ln 3}$
(3) *Compute the Hausdorff* d*-measure, is it true that* $0 < \mathscr{H}^d(S) < \infty$*?*

The following is a special case of the *Moran–Hutchinson formula*

Exercise 1.38. *Show that for a self-similar set as Sierpiński triangle, Cantor set, etc. formed by* N *copies each scaled by a factor* r*, the Hausdorff dimension is the unique solution* d *of the equation:*

$$Nr^d = 1.$$

Helge von Koch (1870–1924) was a Swedish mathematician best known for the Koch snowflake, one of the earliest described fractal curves. Born in Stockholm, he studied in Uppsala and Lund, and held academic positions at several Swedish institutions, eventually becoming a professor at Stockholm University.

Koch introduced the Koch curve in a 1904 paper titled "On a continuous curve without tangents, constructible from elementary geometry." The curve is built by recursively replacing each line segment with a triangular "bump," resulting in a figure that is continuous everywhere but nowhere differentiable. When applied to the sides of an equilateral triangle, it forms the famous Koch snowflake–a closed curve of infinite length enclosing a finite area.

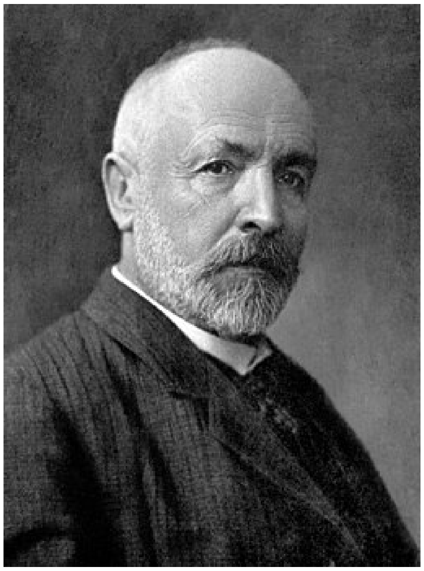
Koch's construction was groundbreaking because it challenged the prevailing intuition of geometry and calculus, showing that a curve could be smooth in appearance but defy classical notions of tangents and arc length. His work prefigured the modern theory of fractals, developed much later by Benoit Mandelbrot.

Although Koch made contributions to number theory and other areas, he is most remembered for this single striking geometric construction. The Koch curve became a classic example in real analysis, topology, and mathematical visualization, and remains a favorite in mathematical education and computer graphics.

Felix Hausdorff (8 November 1868–26 January 1942) was a German mathematician, philosopher, and music theorist, widely regarded as one of the founders of modern topology and set theory. Born in Breslau (now Wrocław, Poland), he studied mathematics and astronomy at the University of Leipzig and the University of Berlin, earning his doctorate in 1891 under Karl Weierstrass. After teaching at several Gymnasien, Hausdorff completed his habilitation in 1901 at the University of Bonn, where he remained a Privatdozent until 1909. He then accepted a professorship at the University of Greifswald and in 1910 moved to the University of Bonn, later joining the University of Leipzig. His landmark 1914 book *Grundzüge der Mengenlehre* ("Fundamentals of Set Theory") systematized set theory and introduced many concepts—among them, the notion of a "Hausdorff space" (a topological space where any two distinct points have disjoint neighborhoods) and what would become known as the "Hausdorff dimension" in fractal geometry. Hausdorff's work extended into measure theory, descriptive set theory, and philosophical foundations of mathematics. Despite growing anti-Semitic pressure in Nazi Germany (he was of Jewish descent), he continued teaching and writing until 1941. Faced with imminent deportation and certain murder, he tragically took his own life on 26 January 1942.

"At the basis of the distance concept lies, for example, the concept of a convergent point sequence and their defined limits, and one can, choosing these ideas as those fundamental to point set theory, eliminate the notions of distance ..." This reflection, from Grundzüge der Mengenlehre, captures Hausdorff's insight that topology can be built purely on notions of convergence and nearness, without presupposing any numerical metric.

Rudolf Lipschitz (1832–1903) was a German mathematician best known for the Lipschitz condition, a foundational concept in analysis and differential equations. Born in Königsberg, he studied at the universities of Königsberg and Berlin and later held academic positions at the University of Bonn and the University of Leipzig. He was a student of Dirichlet and was deeply influenced by the rigorous analytical tradition of 19th-century German mathematics.
Lipschitz is most famously associated with Lipschitz continuity, a strong form of uniform continuity. Lipschitz's work extended beyond analysis. He contributed significantly to number theory, mechanics, and the theory of quadratic forms. He also worked on generalizations of Dirichlet's principles in potential theory and on early formulations of what would become ideas in functional analysis.
Lipschitz also introduced a concept equivalent to the Clifford algebra, which would later become a major structure in modern geometry and physics–especially in the study of spinors and quantum mechanics–although this aspect of his work was not fully recognized until much later.
Although not as widely known as some of his contemporaries, Lipschitz's name is now permanently etched in mathematical vocabulary. The Lipschitz condition is a standard concept in every analysis course. Sometimes you got to be lucky...



Georg Cantor (1845–1918) was a German mathematician best known as the founder of set theory and the concept of different sizes of infinity. He was born in St. Petersburg, Russia, but spent most of his life in Germany. Cantor introduced the idea that not all infinities are equal—some are bigger than others—a groundbreaking notion that deeply influenced modern mathematics, particularly analysis and logic.
He is also the creator of the famous Cantor set, a simple yet paradoxical construction formed by repeatedly removing the middle third of a line segment. Despite having zero length, the Cantor set contains uncountably many points. It became one of the first examples of a fractal-like set and is central to modern topology and measure theory.
His most famous achievement is the rigorous definition of cardinality and the proof that the set of real numbers is uncountably infinite, while the set of natural numbers is countably infinite. This led to the development of the continuum hypothesis, one of the major problems in mathematical logic.
Cantor's work was controversial in his time and met strong opposition from some of his contemporaries, notably Leopold Kronecker, who rejected the notion of actual infinity. Despite the resistance, Cantor persisted, though the struggle took a toll on his mental health, and he spent periods in psychiatric clinics later in life.
Cantor once wrote to a friend, "The essence of mathematics lies in its freedom," expressing his deep belief that mathematics should not be constrained by dogma—a statement that inspired generations of mathematicians.

Wacław Sierpiński (1882–1969) was a Polish mathematician renowned for his contributions to set theory, number theory, topology, and mathematical logic. Born in Warsaw, he studied at the University of Warsaw and later taught at Lwów and the University of Warsaw, where he played a leading role in the development of the Polish school of mathematics.
Sierpiński was one of the early and enthusiastic adopters of set theory and a close collaborator of Zermelo and others in formalizing foundational mathematics. He made important advances in the theory of cardinal and ordinal numbers, descriptive set theory, and the axiom of choice.
He is especially known for a number of self-similar and fractal constructions that now bear his name, including the Sierpiński triangle, carpet, and curve, and the Sierpiński numbers. These constructions are now fundamental in fractal geometry, computer graphics, and dynamical systems.
Sierpiński was a prolific author, writing over 700 papers and more than 50 books, including widely used monographs on set theory, number theory, and topology. He was also deeply committed to mathematical education and helped mentor a generation of Polish mathematicians.
Sierpiński continued to publish actively even during World War II, including clandestine teaching under Nazi occupation, when higher education was forbidden for Poles. His dedication to mathematics under such extreme conditions became a symbol of intellectual resistance.

1.4. **Measurable sets.** As we have discussed, our definition of measure does not include the "natural" condition that $\mu(B) = \mu(B \cap A) + \mu(B \backslash A)$ for all $A, B \subset X$ – because this "natural" condition leads to incompatibility such as the Banach-Tarski Paradoxon.

So we will denote the class of sets $A \subset 2^X$ where we have the above "natural" condition as the σ-algebra of measurable sets.

Definition 1.39 (Carathéodory)**.** Let μ be a measure on X.

$A \subset X$ is called *μ-measurable* if

$$\mu(B) = \mu(A \cap B) + \mu(B \backslash A) \quad \text{for any } B \subset X$$

Remark 1.40. By additivity of the measure, we always have

$$\mu(B) \leq \mu(A \cap B) + \mu(B \backslash A) \quad \text{for any } B \subset X$$

So a set A is μ-measurable if and only if

$$\mu(B) \geq \mu(A \cap B) + \mu(B \backslash A) \quad \text{for any } B \subset X$$

Exercise 1.41. *Let X be a set and $\mu : 2^X \to [0, \infty]$ a measure. Show that $\emptyset$ and X are measurable sets.*

Exercise 1.42. *Let $X \neq \emptyset$ be any set. Show the following.*

- *Assume $\mu(\emptyset) = 0$ and $\mu(A) = 1$ for any $A \neq \emptyset$. Then A is μ-measurable if and only if $A = \emptyset$ or $A = X$.*
- *Assume $\nu = \#$ is the counting measure. Then any set A is ν-measurable.*
- *Assume $\mu = \delta_a$ the Dirac measure for some $a \in X$. Then any set A is ν-measurable.*

Exercise 1.43. *Let X be a finite set and $\mu : 2^X \to [0, \infty]$ is a measure. Is any subset $A \subset X$ necessarily measurable?*

Exercise 1.44. *Assume $\mu : 2^{\mathbb{R}^n} \to [0, \infty]$ is a measure which is translation invariant, i.e.*

$$\mu(x + A) = \mu(A) \quad \forall A \subset \mathbb{R}^n, x \in \mathbb{R}^n$$

Let $A \subset \mathbb{R}^n$. Show that the following are equivalent

- *A is measurable*
- *$x + A$ is measurable for some $x \in \mathbb{R}^n$*
- *$x + A$ is measurable for all $x \in \mathbb{R}^n$*

Lemma 1.45. *Finite union of measurable sets are measurable, i.e.*

$$(A_i)_{i=1}^N \text{ are measurable} \quad \Rightarrow \quad \bigcup_{i=1}^N A_i \text{ is measurable}$$

Proof. We proof this by induction. Clearly this holds for $N = 1$. So to conclude, we only need to show:

If A_1, A_2 are μ-measurable, then so is $A_1 \cup A_2$.

So assume A_1 and A_2 are μ-measurable and $B \subset X$.

$$\begin{aligned}\mu(B) =&\mu(B \setminus A_1) + \mu(B \cap A_1) \\ =&\mu\left((B \setminus A_1) \cap A_2\right) + \mu\left((B \setminus A_1) \setminus A_2\right) \\ &+ \mu\left((B \cap A_1) \cap A_2\right) + \mu\left((B \cap A_1) \setminus A_2\right)\end{aligned}$$

Observe that

$$(B \setminus A_1) \setminus A_2 = B \setminus (A_1 \cup A_2).$$

Moreover

$$B \cap (A_1 \cup A_2) = ((B \setminus A_1) \cap A_2) \cup ((B \cap A_1) \cap A_2) \cup ((B \cap A_1) \setminus A_2).$$

Thus, by sublinearity

$$\mu\left((B \setminus A_1) \cap A_2\right) + \mu\left((B \cap A_1) \cap A_2\right) + \mu\left((B \cap A_1) \setminus A_2\right) \geq \mu\left(B \cap (A_1 \cup A_2)\right),$$

So we have

$$\begin{aligned}\mu(B) =&\mu\left((B \setminus A_1) \cap A_2\right) + \mu\left((B \setminus A_1) \setminus A_2\right) \\ &+ \mu\left((B \cap A_1) \cap A_2\right) + \mu\left((B \cap A_1) \setminus A_2\right) \\ \geq&\mu\left(B \setminus (A_1 \cup A_2)\right) + \mu(B \cap (A_1 \cup A_2))\end{aligned}$$

By Remark 1.40 we have that $(A_1 \cup A_2)$ is also measurable. $\square$

Indeed, the collection of measurable sets is closed under countable operations:

Proposition 1.46. *Let X be a set and μ be a measure on X.*

The collection $\mathcal{A} \subset 2^X$ of μ-measurable functions

$$\mathcal{A} := \{A \subset X : \quad A \text{ is } \mu\text{-measurable}\}$$

is a σ-algebra, that is

(1) $X \in \mathcal{A}$
(2) $A \in \mathcal{A}$ *implies that* $X \backslash A \in \mathcal{A}$
(3) *If* $(A_i)_{i=1}^\infty \subset \mathcal{A}$ *then* $\bigcup_{i=1}^\infty A_i \in \mathcal{A}$.[5]

In particular

- $\emptyset \in \mathcal{A}$
- *if* $(A_i)_{i=1}^\infty \subset \mathcal{A}$ *then* $\bigcap_{i=1}^\infty A_i \in \mathcal{A}$

Proof. (1) For any $B \subset X$: since $B \cap X = B$ and $B \backslash X = \emptyset$ we have

$$\mu(B) = \mu(B) + \mu(\emptyset) = \mu(B \cap X) + \mu(B \setminus X).$$

(2) Assume that $A \in \mathcal{A}$. Set $\tilde{A} := X \backslash A$. For any $B \subset X$ we have

$$\tilde{A} \cap B = (X \backslash A) \cap B = B \backslash A,$$

and

$$B \setminus \tilde{A} = B \setminus (X \backslash A) = B \cap A.$$

Since A is measurable we then have

$$\mu(B \cap \tilde{A}) + \mu(B \setminus \tilde{A}) = \mu(B \setminus A) + \mu(B \cap A) = \mu(B).$$

(3) Let $(A_i)_{i \in \mathbb{N}} \subset \mathcal{A}$. Set $A := \bigcup_{i=1}^\infty A_i$.

Without loss of generality we have that $A_i \cap A_j = \emptyset$ for $i \neq j$. Indeed, otherwise we set $\tilde{A}_1 := A_1$ and $\tilde{A}_k := A_k \backslash \bigcup_{i=1}^{k-1} A_i$. By the previously proven properties and Lemma 1.45 each $\tilde{A}_k$ belongs to $\mathcal{A}$ and we have $A = \bigcup_{k=1}^\infty \tilde{A}_k$ – so we could work with $\tilde{A}_k$ instead of A_k.

We have by measurability of each A_k and since A_N and $\bigcup_{k=1}^{N-1} A_k$ are disjoint,

$$\begin{aligned}\mu(B \cap \bigcup_{k=1}^N A_k) =& \mu(B \cap \left(\bigcup_{k=1}^N A_k\right) \cap A_N) + \mu(\left(B \cap \bigcup_{k=1}^N A_k\right) \setminus A_N) \\ =& \mu(B \cap A_N) + \mu(B \cap \bigcup_{k=1}^{N-1} A_k)\end{aligned}$$

[5]This is the σ in σ-algebra, σ means for countably many. If we only had for any $N \in \mathbb{N}$: $(A_i)_{i=1}^N \subset \mathcal{A}$ then $\bigcup_{i=1}^N A_i \in \mathcal{A}$, $\mathcal{A}$ would be merely an Algebra (no σ!)

Repeating this computation $N-1$ times we obtain

$$\mu(B \cap \bigcup_{k=1}^{N} A_k) = \sum_{k=1}^{N} \mu(B \cap A_k). \tag{1.8}$$

By Lemma 1.45 and the monotonicity of μ, Exercise 1.7, we then have

$$\mu(B) = \mu(B \cap \bigcup_{k=1}^{N} A_k) + \mu(B \setminus \bigcup_{k=1}^{N} A_k) \geq \sum_{k=1}^{N} \mu(B \cap A_k) + \mu(B \setminus \bigcup_{k=1}^{\infty} A_k)$$

This holds for any N, so we obtain

$$\mu(B) \geq \sum_{k=1}^{\infty} \mu(B \cap A_k) + \mu(B \setminus \bigcup_{k=1}^{\infty} A_k)$$

By the σ-subadditivity of μ we then have

$$\mu(B) \geq \mu(B \cap \bigcup_{k=1}^{\infty} A_k) + \mu(B \setminus \bigcup_{k=1}^{\infty} A_k)$$

In view of Remark 1.40 this implies measurability of $\bigcup_{k=1}^{\infty} A_k$.

□

Definition 1.47. Let $\mathcal{C} \subset 2^X$ any nonempty family of subsets of X, then

$$\sigma(\mathcal{C})$$

denotes the *σ-Algebra generated by $\mathcal{C}$*, namely the smallest σ-algebra containing $\mathcal{C}$.

Exercise 1.48.
- *$\{\emptyset, X\}$ is a σ-algebra of X*
- *2^X is a σ-algebra of X*
- *Let (X,d) be a metric space. Denote $\mathcal{O} \subset 2^X$ the family of all open sets.*

  Let $\mathcal{F}$ be the family of σ-Algebras that contain all open sets. That is, $\mathcal{A} \subset 2^X$ belongs to $\mathcal{F}$ if and only if $\mathcal{A}$ is a σ-Algebra, and any open set $O \in \mathcal{O}$ belongs to $\mathcal{A}$, i.e. $O \in \mathcal{A}$.

  Define

  $$\mathcal{B} := \bigcap \{\mathcal{A} : \quad \mathcal{A} \in \mathcal{F}\}.$$

  Show that (a) $\mathcal{F}$ is nonempty, (b) $\mathcal{B}$ is a σ-algebra and (c) $\mathcal{B}$ is the smallest σ-Algebra containing all open sets, i.e. show that $\mathcal{B} = \sigma(\mathcal{O})$.

  $\mathcal{B}$ is called the Borel σ-Algebra *and a set $B \in \mathcal{B}$ is called a* Borel set.

Definition 1.49. If $\mu : 2^X \to [0,\infty]$ is a measure on X, and Σ is the σ-algebra of μ-measurable sets, then one calls (X, Σ, μ) a *measure space*.

Remark 1.50 (Warning: Measure vs Outer Measure)**.** Some author choose to define measures only on their σ-algebra Σ of measurable sets. This makes sense in particular in probability (when you don't care what the value of the measure is on non-measurable sets). In those author's terminology our definition of a measure (defined on all subsets in X, not just on elements of Σ) is called an *outer measure*. For most purposes those two