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Exercise Sheet 2 on Jan 18, 2018
Calculus of Variations

Exercise 4 [Some vector calculus]

Let $v, w \in \mathbb{R}^N$ given by

$$v = (v_1, \ldots, v_N), \quad w = (w_1, \ldots, w_N).$$

Recall that the scalar product of two vectors is given as

$$v \cdot w = \sum_{i=1}^{N} v_i w_i.$$

Show that if $v \in \mathbb{R}^N$ so that

$$v \cdot w = 0 \quad \forall w \in \mathbb{R}^N$$

Then $v = 0$, i.e. $v_1 = 0, v_2 = 0, \ldots v_N = 0$.

Exercise 5 [Fundamental Lemma, k-th iteration] Fix some interval $I = (a, b)$ some $f \in L^1_{loc}(I)$ and some $k \in \{1, 2, \ldots\}$. Assume that

$$\int_I f(x)\, \eta^{(k)}(x)\, dx = 0 \quad \forall \eta \in C_c^\infty(I),$$

where $\eta^{(k)}$ denotes the k-th derivative of η. Prove that f is a polynomial of degree $(k-1)$.

Exercise 6 [Diffeomorphism] For an interval $I = (a, b)$ let $\lambda \in C_c^\infty(I)$, and set

$$\xi(t, \varepsilon) := t + \varepsilon\, \lambda(t).$$

Show that there exists $\varepsilon_0 > 0$ so that ξ is an admissible parameter variation on I, that is

(i) $\xi(t, \varepsilon) \in C^2(\overline{I} \times (-\varepsilon_0, \varepsilon_0))$

(ii) $\xi(\cdot, \varepsilon) : I \to I$ for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$),

(iii) $\xi(a, \varepsilon) = a$, $\xi(b, \varepsilon) = b$ and $\xi(t, 0) = t$.

(iv) Show that there is (for ε_0 small enough) some $\tau(\cdot, \varepsilon) = \xi(\cdot, \varepsilon)^{-1}$ i.e. so that

$$\xi\Big(\tau(t, \varepsilon), \varepsilon\Big) = t \quad \forall t \in I.$$

(v) Show that $\tau \in C^2(I \times (-\varepsilon_0, \varepsilon_0))$.
