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Exercise Sheet 4 on Feb 15, 2018
Calculus of Variations

Exercise 10 Let $(\ell^2(\mathbb{N}), \|\cdot\|_{\ell^2(\mathbb{N})})$ be the space of all sequences

$$a = (a_i)_{i=1}^{\infty}$$

such that $\|a\|_{\ell^2(\mathbb{N})} < \infty$, where

$$\|a\|_{\ell^2(\mathbb{N})} := \left(\sum_{i=1}^{\infty} |a_i|^2\right)^{\frac{1}{2}}$$

A sequence of $\ell^2(\mathbb{N})$-sequences $(a^K)_{K=1}^{\infty} \subset \ell^2(\mathbb{N})$ converges (*strongly*) in $\ell^2(\mathbb{N})$ if there exists a limit sequence $a \in \ell^2(\mathbb{N})$ so that

$$\|a^K - a\|_{\ell^2(\mathbb{N})} \xrightarrow{K\to\infty} 0.$$

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$$\sum_{k=1}^{\infty} b_k\, (a_k^K - a_k) \xrightarrow{K\to\infty} 0$$

holds for any other sequence $b \in \ell^2(\mathbb{N})$.

(i) Show that if $a = (a_i)_{i=1}^{\infty} \in \ell^2(\mathbb{N})$ then $\lim_{i\to\infty} |a_i| = 0$.

(ii) Show that for any $K \in \mathbb{N}$ the sequence $e^K := (0, \ldots, 0, 1, 0, \ldots)$ belongs to $\ell^2(\mathbb{N})$ and compute its norm $\|e^K\|_{\ell^2(\mathbb{N})}$.

(iii) Show that the sequence $(e^K)_{K=1}^{\infty} \subset \ell^2(\mathbb{N})$ has no converging subsequence with respect to the $\ell^2(\mathbb{N})$-norm.

(iv) Show that *weakly* $e^K \xrightarrow{K\to\infty} 0$ in $\ell^2(\mathbb{N})$, where $0 = (0, \ldots, 0, \ldots)$.

Exercise 11 For some $N \in \mathbb{N}$ let $\mathcal{I} = \{1, \ldots, N\}$.
Let $(\ell^2(\mathcal{I}), \|\cdot\|_{\ell^2(\mathcal{I})})$ be the space of all sequences

$$a = (a_i)_{i=1}^{N}$$

such that $\|a\|_{\ell^2(\mathcal{I})} < N$, where

$$\|a\|_{\ell^2(\mathcal{I})} := \left(\sum_{i=1}^{N} |a_i|^2\right)^{\frac{1}{2}}$$

A sequence of $\ell^2(\mathcal{I})$-sequences $(a^K)_{K=1}^{\infty} \subset \ell^2(\mathcal{I})$ converges (*strongly*) in $\ell^2(\mathcal{I})$ if there exists a limit sequence $a \in \ell^2(\mathcal{I})$ so that

$$\|a^K - a\|_{\ell^2(\mathcal{I})} \xrightarrow{K\to\infty} 0.$$

A sequence of $\ell^2(\mathcal{I})$-sequences $(a^K)_{K=1}^{\infty} \subset \ell^2(\mathcal{I})$ converges (*weakly*) in $\ell^2(\mathcal{I})$ if there exists a limit sequence $a \in \ell^2(\mathcal{I})$ so that

$$\sum_{k=1}^{N} b_k\, (a_k^K - a_k) \xrightarrow{K\to\infty} 0$$

holds for any other sequence $b \in \ell^2(\mathcal{I})$.

(i) Show that any bounded sequence in $\ell^2(\mathfrak{J})$ has a convergent subsequence. Namely, if $(a^K)_{K=1}^\infty \subset \ell^2(\mathbb{N})$ is so that

$$\sup_{K\in\mathbb{N}} \|a^K\|_{\ell^2(\mathfrak{J})} < \infty$$

then there exists a subsequence $(K_i)_{i=1}^\infty$ so that $K_i \to \infty$, a limiting object $a = (a_1, \ldots, a_N) \in \ell^2(\mathbb{N})$ so that

$$\|a^K - a\|_{\ell^2(\mathbb{N})} \xrightarrow{K\to\infty} 0.$$

Hint: Bolzano–Weierstrass Theorem.

(ii) Conclude that every *weakly* converging sequence is actually *strongly* convergent in $\ell^2(\mathfrak{J})$.

Exercise 12 Let $I \subset \mathbb{R}^1$ be an open, bounded interval. A continuous function $F : I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is called quasi-convex (in p) if for any $\varphi \in C_c^\infty(I)$ and any $(x_0, z_0, p_0) \in I \times \mathbb{R}^N \times \mathbb{R}^N$ we have

$$\int_I F(x_0, z_0, p_0 + \varphi'(x))\, dx \geq |I|\, F(x_0, z_0, p_0).$$

Show that if F is convex in p, then F is also quasiconvex in p.

Hint: Jensen's inequality.
