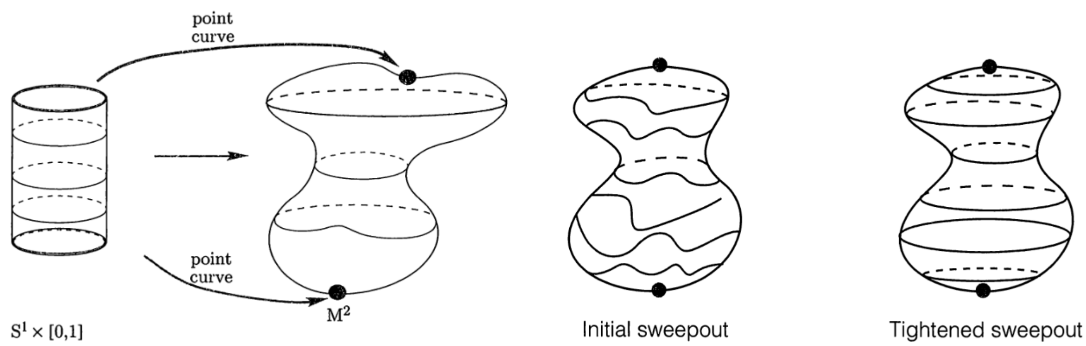
University of Pittsburgh
Armin Schikorra

Exercise Sheet 9 on Mar 29, 2018
Calculus of Variations



Sweepouts of a cylindrical surface, (C) Minicozzi, Holck-Colding: “A course in minimal Surfaces”

Exercise 21

Prove the *one-dimensional* Version of Theorem 6.1 under continuity assumptions: Let $E \in C^0(\mathbb{R})$, i.e. E a continuous function on $\mathbb{R}$. If $x_1 \neq x_2 \in \mathbb{R}$ are two *strict* local minima, i.e. there are two open sets $U_1 \ni x_1$, $U_2 \ni x_2$ such that

$$E(x_1) < E(x) \quad \forall x \in U\backslash\{x_2\}, \quad E(x_2) < E(x) \quad \forall x \in U_2\backslash\{x_2\}.$$

(i) There exits $x_3 \in \mathbb{R}$ so that

$$E(x_3) = \inf_{p \in \mathcal{P}} \max_{x \in p} E(x),$$

where $\mathcal{P}$ is the set of all compact intervals that contain x_1 and x_2.

$$\mathcal{P} = \{[a,b], \quad a, b \in \mathbb{R}, a \leq x_1, x_2 \leq b\}$$

(ii) This x_3 ist a local maximum of E.

(iii) Show that $x_3 \in (x_1, x_2)$.

(iv) Moreover show: x_3 ist das largest local maximum in (x_1, x_2). That is, if there is a further point $x_4 \in (x_1, x_2)$ such that $E(x_4) \geq E(x)$ for all $x \in U_4$ for an open neighborhood $U_4 \ni x_4$, then $E(x_4) \leq E(x_3)$.

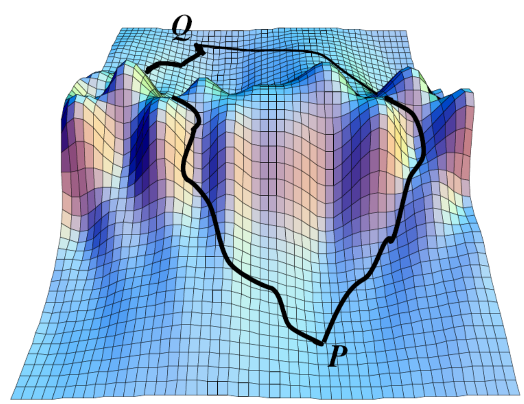


Figure 1: Mountain Pass, (C) A. Chapiro, ETHZ

Exercise 22

A set $p \subset \mathbb{R}^n$ is called *connected* if for all closed sets $A, B \subset \mathbb{R}^n$ such that $A \cup B \supset p$, $A \cap p \neq \emptyset$, $B \cap p \neq \emptyset$ we have $A \cap B \cap p \neq \emptyset$. Show the following:

(i) The closure $\overline{p}$ of a connected set p is again connected.

(ii) for all closed sets A and any connected set p we have $p \subset A$, or $p \cap \partial A \neq \emptyset$, or $p \cap A = \emptyset$ *Hint:* $p \subset A \cup (\overline{p} \setminus \operatorname{int} A)$.

(iii) A set $p \subset \mathbb{R}^n$ is connected if and only if for all disjoint open sets U, V mit $p \subset U \cup V$ it holds that $p \cap U = \emptyset$ or $p \cap V = \emptyset$.

(iv) If $(p_k)_{k \in \mathbb{N}}$ are each connected and it holds that $\bigcap_{k \in \mathbb{N}} p_k \neq \emptyset$, then $\bigcup_{k \in \mathbb{N}} p_k$ is connected.

(v) Assume $(p_k)_{k \in \mathbb{N}}$ each connected and compact. Moreover assume $\bigcap_{k \in \mathbb{N}} p_k \neq \emptyset$ as well as $p_{k+1} \subset p_k$. Then $\bigcap_{k \in \mathbb{N}} p_k$ is necessarily connected.

Hint: W.l.o.g. $B \cap \operatorname{int} A = \emptyset$. Observe now that $p_k \cap \partial A$. Conclude that $\bigcap_k p_k \cap \partial A \neq \emptyset$. The same holds for B. If $A \cap B \cap \bigcap p_k = \emptyset$, then there are open sets $U \supset A \cap \bigcap p_k$ and $V \supset B \cap \bigcap p_k$, which are disjoint.

(vi) Disprove the following: $(p_k)_{k \in \mathbb{N}}$ each connected and it holds that $\bigcap_{k \in \mathbb{N}} p_k \neq \emptyset$. then $\bigcap_{k \in \mathbb{N}} p_k$ is always connected.

(vii) Show that if $x_1 \neq x_2 \in p$ for some connected $p \subset \mathbb{R}^n$. Then there are $(y_k)_{k=1}^{\infty} \subset p \setminus \{x_1\}$ so that

$$\lim_{k \to \infty} |x_1 - y_k| = 0.$$

Hint: Show the claim first for $\overline{p}$. For this consider balls $A := \overline{B_\lambda(x_1)}$, $B := \overline{p} \cap \mathbb{R} \setminus B_\lambda(x_1)$

Exercise 23

Show the following extension of Theorem 6.1:

If E, x_1, x_2 are as in Theorem 6.1. Set

$$\beta := \inf_{p \in P} \max_{x \in p} E(x),$$

where

$$P := \{p \subset \mathbb{R}^n : \quad p \text{ compact and connected, } x_1, x_2 \in p\}.$$

Then for any $\overline{p} \in P$ mit

$$\max_{x \in \overline{p}} E(x) = \beta,$$

there exists $x_3 \in$ satisfying $E(x_3) = \beta$, $E'(x_3) = 0$.
