INTRODUCTION TO ANALYSIS (MATH 420)
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In Analysis
there are no theorems
only proofs
These lecture notes are substantially based on the book [Leb], also several exercises are taken from there. Some exercises are also substantially inspired from [BS92].

For more exercises see also the standard reference [Rud76], which often is lovingly referred to as “Baby Rudin”.

Pictures are taken from wikipedia or otherwise available sources. Self-made pictures are often made with geogebra.

If you find typos (most likely there are many) please email me: armin@pitt.edu.

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1. Review

1.1. Numbers.

- \( \mathbb{N} \) denotes the natural numbers \( \{1, 2, \ldots \} \)
- \( \mathbb{Z} \) denotes the integers numbers \( \{\ldots, -2, -1, 0, 1, 2, \ldots \} \)
- \( \mathbb{Q} \) denotes the rational numbers \( \{p/q : p, q \in \mathbb{Z}, q \neq 0\} \)
- We are going to discuss our main number field, the real numbers \( \mathbb{R} \), below.
- We are not really going to work with complex numbers \( \mathbb{C} \).

Recall the notion of an upper bound and lower bound:

**Definition 1.1.** Let \( X \) be a totally ordered set (i.e. there exist the operation \(<\) with the usual reasonable properties and for any two \( x, y \in X \) we have either \( x = y \) or \( x < y \) or \( x > y \))

- A set \( A \subset X \) has an upper bound \( c \in X \) if for any \( a \in A \) we have \( a \leq c \) (i.e. either \( a < c \) or \( a = c \)).
- A set \( A \subset X \) has a lower bound \( c \in X \) if for any \( a \in A \) we have \( a \geq c \) (i.e. either \( a < c \) or \( a = c \)).

A set \( A \subset X \) with an upper bound is called bounded from above. A set \( A \subset X \) with a lower bound is called bounded from below. A set \( A \) which is bounded from above and below is called bounded.

The supremum of a set is the smallest upper bound, the infimum is the largest lower bound – if that exists (because e.g. in \( \mathbb{Q} \) it often doesn’t).

**Definition 1.2** (Supremum and infimum). Let \( X \) be a totally ordered set and let \( A \subset X \).

- A number \( c \in X \) is called the supremum of \( A \),
  \[ \sup A = c \]
  if
  (1) \( c \) is an upper bound of \( A \) and
  (2) for any other upper bound \( b \) of \( A \) we have \( c \leq b \).
  We call \( c \) the maximum of \( A \), \( c = \max A \), if \( c = \sup A \) and additionally \( c \in A \).
- A number \( c \in X \) is called the infimum of \( A \),
  \[ \inf A = c \]
  if \( c \) is a lower bound of \( A \) and for any other lower bound \( b \) of \( A \) we have \( c \geq b \).
  We call \( c \) the minimum of \( A \), if \( c = \inf A \) and \( c \in A \).

---

1 We do not consider 0 to be a natural number (this is not always the case in the literature)

2 so \( \mathbb{N}, \mathbb{Q}, \mathbb{R} \) are clearly totally ordered sets – but e.g. for \( \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \) it is a bit unclear how to define \( \leq \) – or the set of powersets \( (2^X, \subseteq) \) is often not a totally ordered set.
If $X = \mathbb{R}$ (as will be the case most of the time), then for notational convenience we often write

$$\sup A = +\infty \quad \text{if } A \text{ has no upper bound},$$

$$\inf A = -\infty \quad \text{if } A \text{ has no lower bound}.$$  

In the pathological case $A = \emptyset$ we write

$$\sup A = -\infty \quad \text{if } A = \emptyset,$$

$$\inf A = +\infty \quad \text{if } A = \emptyset.$$  

Example 1.3. 

- In $\mathbb{Q}$, the set

$$\{ q \in \mathbb{Q}, -\infty < q < 2 \} \equiv \mathbb{Q} \cap (-\infty, 2)$$

is bounded from above, not bounded from below.

- In $\mathbb{Q}$, the set

$$\{ q \in \mathbb{Q}, -\infty < q < 2 \} \equiv \mathbb{Q} \cap (-\infty, 2)$$

has no infimum (i.e. $\inf = -\infty$), the supremum is 2. 2 is not a maximum, though.

- In $\mathbb{Q}$, the set

$$\{ q \in \mathbb{Q}, -\infty < q \leq 2 \} \equiv \mathbb{Q} \cap (-\infty, 2]$$

has no infimum, but the maximum is 2.

- In $\mathbb{Q}$, the set

$$\{ q \in \mathbb{Q}, -1 < q < \sqrt{2} \} \equiv \mathbb{Q} \cap (-\infty, \sqrt{2}) \equiv \{ q \in \mathbb{Q}, -1 < q < \infty \text{ and } q^2 \leq 2 \}$$

is bounded from above and below. The infimum is $-1$. There is no supremum (it would be $\sqrt{2}$, but $\sqrt{2}$ does not belong to $\mathbb{Q}$).

- If a set $A \subset X$ has a supremum, it is necessarily bounded from above (similar statement for infimum)

- Any bounded set $A \subset \mathbb{Z}$ has a supremum and an infimum in $\mathbb{Z}$

Bounded sets in $\mathbb{Q}$ have always “almost” a supremum and an infimum – the only problem is this number may not belong to $\mathbb{Q}$. In other words, $\mathbb{Q}$ has infinitesimal holes, it is not complete. This is why we defined $\mathbb{R}$, the real numbers, which are the completion of $\mathbb{Q}$.

- $\mathbb{R}$ denotes the real numbers. There are many different ways to define them:

  - Element of $\mathbb{R}$ correspond to the supremum of bounded sets $A \subset \mathbb{Q}$:

    Define

    $$\mathbb{R} := \{ A \subset \mathbb{Q} : \text{A bounded} \} / \sim$$

    where $\sim$ is an equivalence relation defined as

    $$A \sim B : \iff \text{every upper bound } a \in Q \text{ of } A \text{ is an upper bound of } B \text{ and vice versa.}$$

    Then $\mathbb{R}$ can be ordered just as $\mathbb{Q}$, and any element in $q \in \mathbb{Q}$ corresponds to the set

    $$\{ q \} \sim \{ r \in \mathbb{Q}, r \leq q \} \sim \{ r \in \mathbb{Q}, 1 - q \leq r \leq q \} \sim \{ r \in \mathbb{Q}, 1 - q \leq r < q \}.$$
Figure 1.1. While some artefacts suggest that Babylonians simply used \( \pi = 3 \), like this one, there are also indications that people at the same time (not only in Babylon) knew that there was a more precise approximation. Source: Yale Babylonian Collection, 7302

Figure 1.2. Georg Cantor, 1845-1918. German, one of the founders of modern set theory and the notion of cardinality.

This definition of the real numbers is related to the so-called Dedekind cuts (which had been considered already by Bertrand)

- From Analysis aspects, this is not such a great definition, since it requires an ordering \(<\). Many generalized spaces (vector spaces, metric spaces, manifolds, function spaces) have no reasonable order. So instead, we will define (metric) “complete” and “completion” as plugging holes of limits (see Cauchy sequences, Section 4). From this point of view \( \mathbb{R} \) consist of all finite limits of sequences in \( \mathbb{Q} \).

The history of “rational numbers are not everything” is very long – people around the world understood that e.g. \( \sqrt{2} \) or \( \pi \) were not rational numbers thousands of years ago.\(^3\)

The modern understanding of \( \mathbb{R} \) is due to Cantor who axiomatized set theory.

For now (until we get to Cauchy sequences, Section 4) we use the following property of \( \mathbb{R} \):  

**Proposition 1.4.** For any bounded set \( A \subset \mathbb{R} \) both \( \sup A \) and \( \inf A \) exist in \( \mathbb{R} \).

A useful classification of suprema and infima is the following

\(^3\)Legend has it that Pythagoras, who lead some sort of number cult, had Hippasus murdered for figuring out that there were numbers not being able to be written as a ratio of two integers, namely \( \sqrt{2} \). Early approximations of \( \sqrt{2} \) are known e.g. from Shulva Sutras (India) or the Babylonian clay tablet YBC 7289

\(^4\)indeed it is the defining property of \( \mathbb{R} \): \( \mathbb{R} \) is the “smallest” set containing \( \mathbb{Q} \) with these properties
Figure 1.3. Richard Dedekind, 1831–1916. German, best known for his contributions to the definition of \( \mathbb{R} \) via the notion of Dedekind cuts (which Bertrand actually defined before him).

Figure 1.4. Joseph Louis Francois Bertrand, 1822 – 1900. French, did Dedekind cuts before Dedekind.

Lemma 1.5. Let \( S \subset \mathbb{R} \) and \( x \in \mathbb{R} \).

- The following are equivalent
  1. \( x = \sup S \)
  2. \( x \) is an upper bound of \( S \) and for any \( \varepsilon > 0 \) there exists \( s \in S \) with \( s > x - \varepsilon \).

- The following are equivalent
  1. \( x = \inf S \)
  2. \( x \) is a lower bound of \( S \) and for any \( \varepsilon > 0 \) there exists \( s \in S \) with \( s < x + \varepsilon \).

Proof. We only prove the first statement: Let \( S \subset \mathbb{R} \) and \( x \in \mathbb{R} \). The following are equivalent

1. \( x = \sup S \)
2. \( x \) is an upper bound of \( S \) and for any \( \varepsilon > 0 \) there exists \( s \in S \) with \( s > x - \varepsilon \).

(1) \( \Rightarrow \) (2). Assume that \( x = \sup S \), but assume that (2) is false. By definition of sup, \( x \) is an upper bound of \( S \). If (2) is false, there thus must be \( \varepsilon > 0 \) such that for all \( s \in S \) we have \( s \leq x - \varepsilon \). This implies that \( x - \varepsilon \) is an upper bound for \( S \). Since \( x - \varepsilon < x \), \( x \) cannot be the least upper bound of \( S \). Contradiction. So (2) must have been true.
(2) ⇒ (1). $x$ is an upper bound of $S$, we only need to show that $x$ is the least upper bound. So let $y \in \mathbb{R}$ be another upper bound of $S$, i.e. assume that $s \leq y$ for all $s \in S$. We need to show that $y \geq x$. Assume to the contrary that $y < x$. For $\varepsilon := \frac{|x-y|}{2}$ we then have $y < x - \varepsilon$. Since we assume that (2) holds, there exists an $s \in S$ with $s > x - \varepsilon$. But then $s > x - \varepsilon > y$ which means that $y$ is not an upper bound of $S$. contradiction, so (1) must have been true.

\[\square\]

Exercise 1.6. [Leb, Exercise 1.2.10] Let $A$ and $B$ be two nonempty bounded sets of non-negative real numbers. Define the set

$$C := \{ab : a \in A, \ b \in B\}.$$  

Show that $C$ is a bounded set and that

$$\sup C = (\sup A) (\sup B)$$  

and

$$\inf C = (\inf A) (\inf B)$$

1.2. The Euclidean metric – absolute value. For $x \in \mathbb{R}$ we define the absolute value $|x|$ as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x \leq 0. \end{cases}$$

The absolute value is incredibly important for the Analysis in $\mathbb{R}$, because it gives $\mathbb{R}$ a metric: we can use it to measure the (a reasonable) distance between to points $x, y \in \mathbb{R}$. Indeed, $d(x, y) := |x - y|$ is the so-called Euclidean metric.

Definition 1.7 (metric). A map $d : X \times X \to \mathbb{R}$ is called a metric for a set $X$ if

- $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry)
- $d(x, y) \geq 0$ for all $x, y \in X$ (positivity)
- $d(x, y) = 0$ if and only if $x = y$ (non-degeneracy)
- $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$ (triangular inequality).

A set $X$ with a metric $d$ is called a metric space.

Almost everything\(^5\) we do with respect to convergence, continuity has a metric space generalization. The proofs are the same, the theorem changes from $\mathbb{R}$ to a general metric space $(X, d)$. Differentiability, however, becomes more tricky, then more structural assumptions on $d$ are helpful (e.g. a “norm” structure).

Exercise 1.8. Show the following

1. $d(x, y) = 2|x - y|$ is a metric in $\mathbb{R}$.

\(^5\)very importantly, not the Bolzano-Weierstrass theorem, Theorem 3.8, though
(2) \( d(x, y) = \sqrt{|x - y|} \) a metric in \( \mathbb{R} \)
(3) \( d(x, y) = |x - y|^2 \) is no metric in \( \mathbb{R} \)
(4) \( d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases} \) is a metric in \( \mathbb{R} \)

Exercise 1.9. [Leb, Exercise 1.3.1] Let \( \varepsilon > 0 \). Show that \( |x - y| < \varepsilon \) if and only if \( x - \varepsilon < y < x + \varepsilon \).

Exercise 1.10. [Leb, Exercise 1.3.2.]

1) Show that
\[
\max\{x, y\} = \frac{x + y + |x - y|}{2}
\]
2) Show that
\[
\min\{x, y\} = \frac{x + y - |x - y|}{2}
\]

1.3. functions: boundedness, infimum, supremum. We will mostly consider functions \( f : D \subset \mathbb{R} \rightarrow \mathbb{R} \). But of course one can also consider more general sets \( D \) (like \( D \subset \mathbb{R}^2 \) etc.)

Definition 1.11. A function \( f : D \rightarrow \mathbb{R} \) is

- **bounded from above** if there exists \( M \in \mathbb{R} \) with \( f(x) \leq M \) for all \( x \in D \).
- **bounded from below** if there exists \( M \in \mathbb{R} \) with \( f(x) \geq M \) for all \( x \in D \).
- **bounded** if it is bounded from above and below. In other terms: if there exists \( M \in \mathbb{R} \) with \( |f(x)| \leq M \) for all \( x \in D \).

For a function \( f : D \rightarrow \mathbb{R} \) we define (if existent)

- the **supremum** \( \sup_D f := \sup f(D) \). If there exists \( x \in D \) such that \( f(x) = \sup_D f \) then \( \max_D f := \sup_D f \) is called the **maximum (value)**.
- the **infimum** \( \inf_D f := \inf f(D) \). If there exists \( x \in D \) such that \( f(x) = \inf_D f \) then \( \min_D f := \inf_D f \) is called the **minimum (value)**.

For notational convenience we write
\[
\sup_D f = +\infty \quad \text{if } D \neq \emptyset \text{ and } f \text{ is not bounded from above}
\]
\[
\inf_D f = -\infty \quad \text{if } D \neq \emptyset \text{ and } f \text{ is not bounded from below}
\]
and the pathological cases
\[
\sup_D f = -\infty \quad \text{if } D = \emptyset
\]
\[
\inf_D f = +\infty \quad \text{if } D = \emptyset
\]
Exercise 1.12. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let
\[ g(x) := -f(x). \]
Show that for any $D \subset \mathbb{R}$ (including $D = \emptyset$)
\[ \sup_D g = -\inf_D f \]
and
\[ \inf_D g = -\sup_D f. \]

2. Sequences review

(it is a fun exercise to try to translate the statements here into notions on metric spaces, cf. Definition 1.7)

A sequence, usually denoted by $(x_n)_{n=1}^\infty \subset X$, is a map $x : \mathbb{N} \mapsto X$. But instead of writing $x(n)$ we prefer to write $x_n$. Every sequence induces a set $x(\mathbb{N}) := \{x_n, n \in \mathbb{N}\}$ (but not the other way around, since we do not know which element of the set to take first). Thus we can use set operations on sequences, e.g.,
\[ \sup(x_n)_{n \in \mathbb{N}} = \sup\{x_1, x_2, \ldots\}. \]

Definition 2.1. A sequence $(x_n)_{n=1}^\infty$
\begin{itemize}
  \item is \textit{bounded} if the set $\{x_1, \ldots, x_n, \ldots\} \subset \mathbb{R}$ is bounded.
  \item is \textit{unbounded} if the set $\{x_1, \ldots, x_n, \ldots\} \subset \mathbb{R}$ is not bounded.
  \item converges to a number $x \in \mathbb{R}$ if
  \[ \forall \varepsilon > 0 : \exists N \in \mathbb{N} \text{ s.t. } |x_n - x| < \varepsilon \quad \forall n \geq N. \]
  In words: \textit{all} sequence elements $x_n$ with sufficiently large index $n \geq N$ are very close to the limit point $x$.
  In this case we say that $x_n$ is convergent (to $x$).
  \[ \lim_{n \to \infty} x_n = x. \]
\end{itemize}

For a picture see Figure 2.1.

If the limit exists, then it is \textit{unique} that is

Exercise 2.2. Assume that $(x_n)_{n \in \mathbb{N}} \subset \mathbb{Q}$ is a sequence and for $x, y \in \mathbb{Q}$ we have
\[ \lim_{n \to \infty} x_n = x, \quad \lim_{n \to \infty} x_n = y \]
Show that $x = y$.

Exercise 2.3. Show the following
\begin{itemize}
  \item If $x_n = 1 + \frac{1}{n}$: $\lim_{n \to \infty} x_n = 1$.
\end{itemize}
Figure 2.1. A sequence \((a_n)_{n \in \mathbb{N}} \subset \mathbb{R}\) which seems to converge to \(a\), \(\lim_{n \to \infty} a_n = a\).

Figure 2.2. For a given \(\varepsilon\), several sequence elements (red) are not close to \(a\) at the scale \(\varepsilon\): \(|a_n - a| \geq \varepsilon\) for the red \(a_n\). But most of the sequence elements (blue) are close to \(a\) at the scale \(\varepsilon\): \(|a_n - a| < \varepsilon\). Indeed, we see that after some large enough number \(N\), all sequence elements are blue, i.e. close to \(a\), i.e. \(|a_n - a| < \varepsilon\) for all \(n > N\).

Figure 2.3. In order to show the convergence \(\lim_{n} a_n = a\) we have to show this for \(\varepsilon > 0\) we can find such an \(N\) from which on \(|a_n - a| < \varepsilon\). The \(N\) is allowed to change with \(\varepsilon\): for \(\varepsilon_0 > 0\) we find some \(N_0\), for \(\varepsilon_1\) we find another \(N_1\). In general, as \(\varepsilon > 0\) is smaller \(N\) needs to be chosen larger.

all pictures: Ceranilo, wikipedia.

- If \(x_n = (-1)^n\) does not converge.

Example 2.4. If \(x_n = \frac{n^2}{2n^2+n}\) then \(\lim_{n \to \infty} x_n = \frac{1}{2}\).
Indeed: Let $\varepsilon > 0$ be given. We need to find $N \in \mathbb{N}$ such that

$$|x_n - \frac{1}{2}| < \varepsilon \quad \forall n \geq N.$$ 

Now observe that

$$x_n - \frac{1}{2} = \frac{n^2}{2n^2 + n} - \frac{1}{2}$$

$$= \frac{2n^2 - (2n^2 + n)}{4n^2 + 2n}$$

$$= \frac{-n}{4n^2 + 2n}$$

$$= \frac{-1}{4n + 2}$$

Thus,

$$|x_n - \frac{1}{2}| = \frac{1}{4n + 2}$$

$$\leq \frac{1}{4n}.$$ 

So if we choose $N \in \mathbb{N}$ such that $N > \frac{1}{4\varepsilon}$ then for any $n \geq N$

$$|x_n - \frac{1}{2}| \leq \frac{1}{4n} \leq \frac{1}{4N} < \varepsilon.$$ 

Lemma 2.5. Every convergent sequence $(x_n)_{n \in \mathbb{N}}$ is bounded, i.e. there exists $M \in \mathbb{R}$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Proof. Since $x_n$ is convergent, there exists $x \in \mathbb{R}$ and and $N \in \mathbb{N}$ such that

$$|x_n - x| \leq 1 \quad \forall n > N.$$ 

* Set $\tilde{M} := \max\{|x_1|, \ldots, |x_N|\}$ – this maximum exists, because there are only finitely many points considered.
* Set $M := |x| + \tilde{M} + 1$. Then we have

$$|x_n| \leq \tilde{M} \leq M \quad \forall n \leq N$$

and

$$|x_n| \leq |x_n - x| + |x| \leq 1 + |x| \leq M \quad \forall n > N$$

That is $|x_n| \leq M$ for all $n \in \mathbb{N}$, and thus the sequence $x_n$ is bounded.

Corollary 2.6. Any unbounded sequence is not convergent.
Proof. This is just the logical equivalent of Lemma 2.5. Namely \( A \Rightarrow B \) is equivalent to \( \neg B \Rightarrow \neg A \), so

\[
((x_n)_{n \in \mathbb{N}} \text{ convergent}) \quad \Rightarrow \quad ((x_n)_{n \in \mathbb{N}} \text{ bounded})
\]

\[
\Leftrightarrow \quad \neg ((x_n)_{n \in \mathbb{N}} \text{ convergent}) \quad \Leftarrow \quad \neg ((x_n)_{n \in \mathbb{N}} \text{ bounded})
\]

\[
((x_n)_{n \in \mathbb{N}} \text{ not convergent}) \quad \Leftrightarrow \quad ((x_n)_{n \in \mathbb{N}} \text{ not bounded})
\]

\[
((x_n)_{n \in \mathbb{N}} \text{ not convergent}) \quad \Rightarrow \quad ((x_n)_{n \in \mathbb{N}} \text{ not bounded})
\]

Remark. In a very common abuse of notation we shall write

- “\( x_n \) converges to \(+\infty\)”, in formulas
  
  \[
  \lim_{n \to \infty} x_n = +\infty,
  \]

  if
  
  \[
  \forall M > 0 \ \exists N \in \mathbb{N} \quad \text{such that} \quad x_n > M \quad \forall n > N,
  \]

  that is all sequence elements are eventually very large.

- “\( x_n \) converges to \(-\infty\)”, in formulas
  
  \[
  \lim_{n \to \infty} x_n = -\infty,
  \]

  if
  
  \[
  \forall M > 0 \ \exists N \in \mathbb{N} \quad \text{such that} \quad x_n < -M \quad \forall n > N.
  \]

  that is all sequence elements are eventually very negative.

Definition 2.7. A sequence \( (x_n)_{n \in \mathbb{N}} \) is\(^6\)

- **monotone increasing** if \( x_n \leq x_m \) holds for any \( n, m \in \mathbb{N} \) with \( n \leq m \)
- **strictly monotone increasing** if \( x_n < x_m \) holds for any \( n, m \in \mathbb{N} \) with \( n < m \)
- **monotone decreasing** if \( x_n \geq x_m \) holds for any \( n, m \in \mathbb{N} \) with \( n \leq m \)
- **strictly monotone decreasing** if \( x_n > x_m \) holds for any \( n, m \in \mathbb{N} \) with \( n < m \)
- **monotone** if it is either monotone increasing or monotone decreasing.

Theorem 2.8 (Bounded monotone sequences are convergent). Let \( (x_n)_{n \in \mathbb{N}} \subset \mathbb{R} \) be a bounded monotone sequence. Then \( x = \lim_{n \to \infty} x_n \) exists, and

- \( x = \sup_n x_n \) (if \( (x_n)_{n \in \mathbb{N}} \) is increasing), or
- \( x = \inf_n x_n \) (if \( (x_n)_{n \in \mathbb{N}} \) is decreasing).

Proof. Assume w.l.o.g. that \( x_n \) is monotone increasing (the other case goes exactly the same way).

\(^6\) of course, this doesn’t make any sense in general metric spaces
Since \( \{x_n, n \in \mathbb{N}\} \subset \mathbb{R} \) is bounded by assumption, and \( \mathbb{R} \) is a complete space, Proposition 1.4, the supremum exists. We denote it by
\[
x := \sup_{n \in \mathbb{N}} x_n.
\]
We need to show that \( \lim_{n \to \infty} x_n = x \). For this let \( \varepsilon > 0 \) be arbitrary. We need to find \( N = N(\varepsilon) \in \mathbb{N} \) such that
\[
|x_n - x| < \varepsilon \quad \forall n > N.
\]
Equivalently we need to show that
\[
(2.1) \quad x_n - x < \varepsilon \quad \forall n > N,
\]
and
\[
(2.2) \quad x - x_n < \varepsilon \quad \forall n > N.
\]
Observe that \( (2.1) \) is true for any \( n \in \mathbb{N} \) because \( x \) is the supremum of the \( x_n \), and as such \( x \geq x_n \) for all \( n \in \mathbb{N} \).
So we only need to show \( (2.2) \). Assume to the contrary that for any \( N \) there exists an \( M > N \) such that
\[
x - x_M \geq \varepsilon \iff x_M \leq x - \varepsilon
\]
But by monotonicity this implies
\[
x_m \leq x_M \leq x - \varepsilon \quad \forall m \leq M.
\]
That is we would have
\[
\forall N \in \mathbb{N} \exists M > N : \quad x_m \leq x_M \leq x - \varepsilon \quad \forall m \leq M.
\]
In particular we have
\[
\forall N \in \mathbb{N} : \quad x_N \leq x - \varepsilon
\]
Just relabelling this, we have
\[
x_m \leq x - \varepsilon \quad \forall m \in \mathbb{N}
\]
But this contradicts that \( x \) is the sup \( x_n \), indeed \( x - \varepsilon \) is a smaller upper bound. Contradiction, so \( (2.2) \) must be true for some \( N \in \mathbb{N} \).

**Exercise 2.9.** Show that the statement of Theorem 2.8 is false if \( \mathbb{R} \) is replaced by \( \mathbb{Q} \).

For this give an example of a bounded monotone sequence in \( \mathbb{Q} \), \( (x_n)_{n \in \mathbb{N}} \subset \mathbb{Q} \), which does not converge in \( \mathbb{Q} \). That is, show that there is no \( x \in \mathbb{Q} \) with \( \lim_{n \to \infty} x_n = x \).

We can also reformulate the supremum and infimum definition of Definition 2.1:

**Exercise 2.10.** [Leb, Exercise 2.1.12] Show the following:

Let \( S \subset \mathbb{R} \) be a nonempty bounded set. Then there exist monotone sequences \( (x_n)_{n \in \mathbb{N}}, \) \( (y_n)_{n \in \mathbb{N}} \) such that \( x_n, y_n \in S \) for all \( n \) and
\[
\sup S = \lim_{n \to \infty} x_n.
\]
and

$$\inf S = \lim_{n \to \infty} y_n$$

**Hint:** Use the definition of supremum from Lemma 1.5 to find the sequence and Theorem 2.8 to ensure it converges.

The following lemma is also known as the sandwich theorem, cf. Figure 2.4.

**Lemma 2.11** (Squeeze theorem). Assume that we have three real sequences

$$(a_n)_{n \in \mathbb{N}}, \ (x_n)_{n \in \mathbb{N}}, \ (b_n)_{n \in \mathbb{N}}$$

such that

$$a_n \leq x_n \leq b_n \ \forall n \in \mathbb{N}. \tag{2.3}$$

If there exists $x \in \mathbb{R}$ with

$$x = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n$$

then

$$\lim_{n \to \infty} x_n = x.$$

**Proof.** Since by (2.3)

$$a_n - x \geq x_n - x \leq b_n - x,$$

we have

$$|x_n - x| \leq \max\{|a_n - x|, |b_n - x|\}. \tag{2.4}$$

Let now $\varepsilon > 0$. Since $a_n \to x$ and $b_n \to x$ there must be an $N(\varepsilon)\footnote{we take the maximum of the $N(\varepsilon) = \max\{N_1(\varepsilon), N_2(\varepsilon)\}$ where $N_1(\varepsilon)$ is such that the sequence $(a_n)$ satisfies $|a_n - x| < \varepsilon$ for all $n \geq N_1(\varepsilon)$ and $N_2(\varepsilon)$ is such that the sequence $(b_n)$ satisfies $|b_n - x| < \varepsilon$ for all $n \geq N_2(\varepsilon)$}$ such that

$$\max\{|a_n - x|, |b_n - x|\} < \varepsilon \ \forall n \geq N.$$

Thus, by (2.4),

$$|x_n - x| \leq \max\{|a_n - x|, |b_n - x|\} < \varepsilon.$$
Proposition 2.12. If \((x_n)_{n \in \mathbb{N}}\) is convergent, so is \((|x_n|)_{n \in \mathbb{N}}\), and we have
\[
\lim_{n \to \infty} |x_n| = |\lim_{n \to \infty} x_n|.
\]

Proof. This is what we will later call the continuity of the absolute value \(f(\cdot) := |\cdot|\).

Set
\[
x := \lim_{n \to \infty} x_n.
\]
The claim follows from the definition of a limit and the inverse triangle inequality which implies
\[
||x_n| - |x|| \leq |x_n - x|.
\]
Since \(x_n \to x\), for any \(\varepsilon > 0\) there must be \(N \in \mathbb{N}\) such that
\[
|x_n - x| < \varepsilon \quad \forall n \geq N
\]
From (2.5) we conclude that then
\[
||x_n| - |x|| < \varepsilon \quad \forall n \geq N
\]
which implies by definition that \(\lim_{n \to \infty} |x_n| = |x|\). \(\square\)

Definition 2.13. We say that a property \((A)\) holds for all but finitely many elements of a set \(S \subset X\) if there exists a finite number \(K\) and elements \(s_1, \ldots, s_K \in S\) such that property \((A)\) holds for any \(s \in S \setminus \{s_1, \ldots, s_K\}\).

It is an easy exercise to show that property \((A)\) holds for all but finitely many elements of a sequence \((x_n)_{n \in \mathbb{N}}\) if and only if there exists a large number \(N \in \mathbb{N}\) such that property \((A)\) holds for all \(x_n, n \geq N\). When talking about limits of sequences, we usually only care about all but finitely many elements of said sequence. For example:

Lemma 2.14. Let \((x_n)_{n \in \mathbb{N}}\) and \((y_n)_{n \in \mathbb{N}}\) be two sequences and assume that
\[
x_n \leq y_n \quad \text{for all but finitely many } n \in \mathbb{N}
\]
If \(\lim_{n \to \infty} x_n\) and \(\lim_{n \to \infty} y_n\) exist, then
\[
\lim_{n \to \infty} x_n \leq \lim_{n \to \infty} y_n.
\]

Exercise 2.15. Prove Lemma 2.14

We will also discuss a strengthened version of Lemma 2.14 in Exercise 3.3.

Exercise 2.16. [Leb, Ex. 2.1.3] Is the sequence \(\left(\frac{(-1)^n}{2^n}\right)_{n \in \mathbb{N}}\) convergent? If so, what is the limit?
2.1. Subsequences.

**Definition 2.17.** Suppose \( (x_n)_{n \in \mathbb{N}} \) is a sequence. Let \( (n_i)_{i \in \mathbb{N}} \) be a strictly increasing sequence of natural numbers (i.e., \( n_i < n_{i+1} \) for all \( i \)). The sequence
\[
(x_{n_i})_{i \in \mathbb{N}}
\]
is then called a subsequence of \( (x_n)_{n \in \mathbb{N}} \).

As sequence \( (x_n)_{n \in \mathbb{N}} \) has a convergent subsequence if there exists a subsequence \( (x_{n_i})_{i \in \mathbb{N}} \) which is convergent.

**Example 2.18.**
- Let \( (x_n)_{n \in \mathbb{N}} = (1, 5, 7, 8, 9, 10, 20, 33, \ldots) \) then
  \( (y_n)_n = (1, 7, 33, \ldots) \)
is a subsequence, whereas
  \( (z_n)_n = (1, 7, 5, 33, \ldots) \)
is (most likely) not a subsequence.
- Let
  \[
x_n := (-1)^{n+1}
  \]
Then
\[
x_{2n} = -1
\]
and
\[
x_{2n+1} = 1.
\]

Both subsequences are clearly convergent, but \( (x_n)_{n \in \mathbb{N}} \) is clearly not convergent.

**Exercise.** Let \( x_1 = 8 \) and \( x_{n+1} := \frac{1}{2} x_n + 2 \) for \( n \in \mathbb{N} \). Show that \( (x_n)_{n \in \mathbb{N}} \) is convergent and compute the limit.

*Hint: Use Theorem 2.8.*

**Lemma 2.19.** If \( (x_n)_{n \in \mathbb{N}} \) is a convergent sequence, then every subsequence of \( (x_n)_{n \in \mathbb{N}} \) is also convergent. Moreover if
\[
x := \lim_{n \to \infty} x_n
\]
then for any subsequence \( (x_{n_i})_{i \in \mathbb{N}} \),
\[
x = \lim_{i \to \infty} x_{n_i}
\]

**Exercise 2.20.** Prove Lemma 2.19.

**Exercise 2.21.** Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence and assume one of the following property:

1. there is some \( x \) such that any subsequence \( (x_{n_i})_{i \in \mathbb{N}} \) contains another subsequence \( (x_{n_{i,j}})_{j \in \mathbb{N}} \) which is convergent to \( x \).
(2) any subsequence \((x_{n_i})_{i \in \mathbb{N}}\) contains another subsequence \((x_{n_{ij}})_{j \in \mathbb{N}}\) which is convergent (a priori not necessarily to the same \(x\))

Show that in one of the cases the sequence \(x_n\) is convergent. Give a counterexample for the other case.

**Exercise 2.22.** [Leb, Exercise 2.1.15] Let \((x_n)_{n \in \mathbb{N}}\) be a sequence defined by

\[
x_n := \begin{cases} n & \text{if } n \text{ is odd}, \\ 1/n & \text{if } n \text{ is even}. \end{cases}
\]

a) Is the sequence bounded? (prove or disprove)
b) Is there a convergent subsequence? If so, find it.

**Exercise 2.23.** [Leb, Exercise 2.2.7] True or false, prove or find a counterexample. If \((x_n)_{n \in \mathbb{N}}\) is a sequence such that \((x_{2n})_{n \in \mathbb{N}}\) converges, then \((x_n)\) converges as well.

**Exercise 2.24.** Let \((x_n)_{n \in \mathbb{N}}\) be a sequence and assume one of the following properties:

1. there is some \(x\) such that any subsequence \((x_{n_i})_i\) contains another subsequence \((x_{n_{ij}})_j\) which is convergent to \(x\).
2. any subsequence \((x_{n_i})_i\) contains another subsequence \((x_{n_{ij}})_j\) which is convergent (a priori not necessarily to the same \(x\)).

Show in which cases \((x_n)_n\) is convergent. Give a counterexample for the other case.

**Exercise 2.25.** Find the following limit. Show all work.

\[
\lim_{n \to \infty} \left( \frac{1}{\sqrt{n^2 + 1}} + \frac{1}{\sqrt{n^2 + 2}} + \ldots + \frac{1}{\sqrt{n^2 + 2n}} \right)
\]

2.2. Further exercises for limits. Sequences are very important, so here we collect some (option) \(\varepsilon\)-\(N\)-type exercises.

**Exercise.** Use the precise \(\varepsilon\), \(N\) definition of limit to prove the following statements.

1. \(\lim_{n \to \infty} \frac{3n^2 + 2}{2n^2 - 5} = \frac{3}{2}\).
2. \(\lim_{n \to \infty} \frac{n + 5}{\sqrt{n^2 + 28}} = +\infty\)
3. \(\lim_{n \to \infty} \frac{n}{n^2 + 1} = 0\)
4. \(\lim_{n \to \infty} \frac{2n^2}{n + 1} = 1\)
5. \(\lim_{n \to \infty} \frac{n^2 + 3}{n + 5} = +\infty\).
6. \(\lim_{n \to \infty} \frac{1}{\sqrt{n + 7}} = 0\)
7. \(\lim_{n \to \infty} \frac{(-1)^n}{n + 1} = 0\)
8. More abstractly show that whenever \(a, b \neq 0\) we have \(\lim_{n \to \infty} \frac{an^2 + 2n + 7}{bn^2 + 5n - 5} = \frac{a}{b}\).
**Exercise.** Assume that \((x_n)_{n \in \mathbb{N}}\) is a sequence and \(\lim_{n \to \infty} x_n = 5\). Show that there exists some \(N \in \mathbb{N}\) such that \(x_n \geq 4\) for all \(n \geq N\).

**Exercise.** Show that

\[
\lim \left( \frac{2^n}{n!} \right) = 0.
\]

**Hint:** You can use without proof that for \(n \geq 3\) we have \(\frac{2^n}{n!} \leq 2 \left( \frac{2}{3} \right)^{n-2}\)

**Exercise.** Give an example of an unbounded sequence that has a convergent subsequence.

**Exercise.** Prove that the following sequences are divergent:

1. \(x_n := 1 + (-1)^n + 1/n\)
2. \(y_n := \sin \left( \frac{n\pi}{4} \right)\)

**Hint:** subsequences, Lemma 2.19

**Exercise 2.26.** Assume that \((x_n)_{n \in \mathbb{N}}\) satisfies \(x_n \geq 0\) for all \(n \in \mathbb{N}\) and assume \(\lim_{n \to \infty} (-1)^n x_n\) exists. Show that \((x_n)_{n \in \mathbb{N}}\) is convergent.
3. Limit superior, limit inferior, Bolzano-Weierstrass

Sequences can be subdivided into subsequences as discussed above, Section 2.1. The limit superior, \( \lim \sup \) is the largest possible limit (or \( +\infty \)) of any subsequence, the limit inferior, \( \lim \inf \) is the smallest possible limit (or \( -\infty \)) of any subsequence. More precisely,

**Definition 3.1.** Let \((x_n)_{n \in \mathbb{N}} \subset \mathbb{R}\) be any sequence.

Then \(\limsup_{n \to \infty} x_n, \liminf_{n \to \infty} x_n \in \mathbb{R} \cup \{-\infty, +\infty\}\) are defined as follows (cf. Figure 3.3)

- If \((x_n)_{n \in \mathbb{N}}\) is bounded from above we set
  \[
  \limsup_{n \to \infty} x_n := \inf_{n \in \mathbb{N}} \sup_{k \geq n} x_k \in \mathbb{R} \cup \{-\infty\}
  \]
  Observe that since \(n \mapsto \sup_{k \geq n} x_k\) is monotone decreasing we have equivalently
  \[
  \limsup_{n \to \infty} x_n := \lim_{n \to \infty} \sup_{k \geq n} x_k \in \mathbb{R} \cup \{-\infty\}
  \]
  so the \(\limsup_{n \to \infty} x_n\) computes the largest sequence element "at infinity".

- If \((x_n)_{n \in \mathbb{N}}\) is not bounded from above we set \(\limsup_{n \to \infty} x_n := +\infty\)

- If \((x_n)_{n \in \mathbb{N}}\) is bounded from below we set
  \[
  \liminf_{n \to \infty} x_n := \sup_{n \in \mathbb{N}} \inf_{k \geq n} x_k \equiv \lim_{n \to \infty} \inf_{k \geq n} x_k.
  \]
  so the \(\liminf_{n \to \infty} x_n\) computes the smallest sequence element "at infinity".

- If \((x_n)_{n \in \mathbb{N}}\) is not bounded from below we set \(\liminf_{n \to \infty} x_n := -\infty\)

**Exercise 3.2.** Let

\[
x_n := \begin{cases} 
\frac{1}{n} & n \text{ even} \\
-n & n \text{ odd}
\end{cases}
\]

Show that

\[
\lim_{n \to \infty} \sup x_n = 0
\]

and

\[
\lim_{n \to \infty} \inf x_n = -\infty.
\]

**Exercise 3.3.** Show the following version of Lemma 2.14:

Let \((x_n)_{n \in \mathbb{N}}\) and \((y_n)_{n \in \mathbb{N}}\) be two sequences such that

\[x_n \leq y_n \text{ for all but finitely many } n \in \mathbb{N}\]

Then we have

\[
\liminf_{n \to \infty} x_n \leq \liminf_{n \to \infty} y_n
\]

and

\[
\limsup_{n \to \infty} x_n \leq \limsup_{n \to \infty} y_n
\]
Exercise 3.4. [Leb, Ex. 2.3.7] Let \((x_n)_{n \in \mathbb{N}}\) and \((y_n)_{n \in \mathbb{N}}\) be bounded sequences.

(1) Show that \((x_n + y_n)_{n \in \mathbb{N}}\) is bounded.

(2) Show that
\[
(\liminf_{n \to \infty} x_n) + (\liminf_{n \to \infty} y_n) \leq \liminf_{n \to \infty} (x_n + y_n).
\]

(3) Find explicit \((x_n)_{n \in \mathbb{N}}\) and \((y_n)_{n \in \mathbb{N}}\) such that
\[
(\liminf_{n \to \infty} x_n) + (\liminf_{n \to \infty} y_n) < \liminf_{n \to \infty} (x_n + y_n).
\]

To match \(\limsup\) and \(\liminf\) with our intuition as computing “smallest subsequence” and “largest subsequence”, we observe

Lemma 3.5. Let \((x_n)_{n \in \mathbb{N}}\) be a sequence.

(1) Set \(a_n := \sup_{k \geq n} x_k\), then
\[
\limsup_{n \to \infty} x = \lim_{n \to \infty} a_n
\]

\(^9\)this stays true if \((x_n)\) and \((y_n)\) are not assumed to be unbounded, as long as we avoid \(\infty - \infty\) on the left-hand side.
in the sense that either both sides are finite or both sides are ±∞.

(2) Set \( b_n := \inf_{k \geq n} x_k \), then
\[
\liminf_{n \to \infty} x_n = \lim_{n \to \infty} b_n
\]
in the sense that either both sides are finite or both sides are ±∞.

(3) Let \((x_n)_{i \in \mathbb{N}}\) be any convergent subsequence. Then
\[
\liminf_{n \to \infty} x_n \leq \lim_{i \to \infty} x_{n_i} \leq \limsup_{n \to \infty} x_n.
\]

(4) If \( \limsup_{n \to \infty} x_n \in (-\infty, \infty) \) then there exists a convergent subsequence \((x_{n_i})_{i \in \mathbb{N}}\) with
\[
\lim_{i \to \infty} x_{n_i} = \limsup_{n \to \infty} x_n.
\]

(5) If \( \liminf_{n \to \infty} x_n \in (-\infty, \infty) \) then there exists a convergent subsequence \((x_{n_i})_{i \in \mathbb{N}}\) with
\[
\lim_{i \to \infty} x_{n_i} = \liminf_{n \to \infty} x_n.
\]

(6) If \( \limsup_{n \to \infty} x_n = \infty \) then there exists a subsequence \((x_{n_i})_{i \in \mathbb{N}}\) with \( \lim_{i \to \infty} x_{n_i} = \infty \). If \( \limsup_{n \to \infty} x_n = -\infty \) then all subsequences \((x_{n_i})_{i \in \mathbb{N}}\) satisfy \( \lim_{i \to \infty} x_{n_i} = -\infty \).

(7) If \( \liminf_{n \to \infty} x_n = -\infty \) then there exists a subsequence \((x_{n_i})_{i \in \mathbb{N}}\) with \( \lim_{i \to \infty} x_{n_i} = -\infty \). If \( \liminf_{n \to \infty} x_n = +\infty \) then all subsequences \((x_{n_i})_{i \in \mathbb{N}}\) satisfy \( \lim_{i \to \infty} x_{n_i} = +\infty \).

Proof. (1) If \((a_n)_{n \in \mathbb{N}}\) is not bounded from above, \((x_n)_{n \in \mathbb{N}}\) is not bounded from above, and so \( \lim_{n \to \infty} a_n = \limsup_{n \to \infty} x_n = \infty \).

If \( a_n \) is bounded from above then it is a monotone decreasing, bounded, sequence. From Theorem 2.8 we find that \( a_n \) is convergent and
\[
\lim_{n \to \infty} a_n = \inf_{n \to \infty} \sup_{k \geq n} x_k = \limsup_{n \to \infty} x_n.
\]

(2) \text{exercise!} (almost the same argument as as above)

(3) We only show
\[
\lim_{i \to \infty} x_{n_i} \leq \limsup_{n \to \infty} x_n.
\]

The other inequality follows the same way.

If \( \limsup_{n \to \infty} x_n = \infty \) then (3.1) is trivially satisfied. So let us assume \( \limsup_{n \to \infty} x_n < \infty \). Then
\[
x_{n_i} \leq \sup_{k \geq n_i} x_k =: a_i \quad \forall i \in \mathbb{N}.
\]

We observe that \((a_i)_{i \in \mathbb{N}}\) is a monotone increasing sequence. Since \( \limsup_{n \to \infty} x_n < \infty \) we have that \( a_i \) is bounded from above. So by Theorem 2.8 \( a_i \) is convergent and
\[
\lim_{i \to \infty} a_i = \inf_{i \to \infty} \sup_{k \geq i} x_k \leq \inf_{i \to \infty} \sup_{k \geq i} x_k = \limsup_{n \to \infty} x_n.
\]
By monotonicity of the limit, Lemma 2.14,

$$\lim_{i \to \infty} x_{n_i} \leq \lim_{i \to \infty} a_i = \limsup_{n \to \infty} x_n.$$  

(4) Set

$$a_n := \sup_{k \geq n} x_k.$$  

Since $\limsup_{n \to \infty} x_n < \infty$, by the definition of supremum as lowest upper bound (cf. Lemma 1.5), for any $n \in \mathbb{N}$ there must be a number $K = K(n) \geq n$ such that

$$a_n - \frac{1}{n} \leq x_K \leq a_n.$$  

Now we build our subsequence as follows. $n_1 := K(1)$, $n_2 := K(n_1 + 1)$, $n_i := K(n_{i-1} + 1)$. This is an strictly increasing sequence, and we have

$$a_{n_i} - \frac{1}{n_i} \leq x_{n_{i+1}} \leq a_{n_i} \quad \forall i.$$  

Since in particular $n_i \geq i$ we find

$$a_{n_i} - \frac{1}{i} \leq x_{n_{i+1}} \leq a_{n_i} \quad \forall i.$$  

By the squeeze theorem, Lemma 2.11, we have that

$$\lim_{i \to \infty} x_{n_i} = \lim_{i \to \infty} a_{n_i} = \limsup_{n \to \infty} x_n.$$  

(5) same as above

(6) If $\limsup_{n \to \infty} x_n = \infty$ then $\inf_{n \in \mathbb{N}} a_n = \infty$ where $a_n = \sup_{k \geq n} x_k$. That means that for any $M \in \mathbb{N}$ and for any $n \in \mathbb{N}$ there exists $k = k(n) \geq n$ with $x_k > M$. From this we can build a subsequence. Take $x_{n_1} := x_{k(1)}$, $x_{n_2} := x_{k(k(1)+1)}$ etc. This subsequence goes to infinity.

Assume now that $\limsup_{n \to \infty} x_n = -\infty$ and take $(x_{n_i})_{i \in \mathbb{N}}$ any subsequence.

Then $\inf_{n \in \mathbb{N}} a_n = -\infty$ where $a_n = \sup_{k \geq n} x_k$. That is, for any $M > 0$ there must be some $N \in \mathbb{N}$ such that $a_N < -M$. But since $a_N = \sup_{k \geq N} x_k$, this implies $x_k \leq -M$ for all $k \geq N$. That is, for all $M > 0$ we have that $x_n < -M$ for all but finitely many $n \in \mathbb{N}$. In particular, for all $M > 0$ we have that $x_{n_i} < -M$ for all but finitely many $i \in \mathbb{N}$. This means that $\lim_{i \to \infty} x_{n_i} = -\infty$.

(7) analogue argument to above.

$$\square$$

**Lemma 3.6.** Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}$

(1) $\liminf_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n$

(2) For any subsequence $(x_{n_i})$,

$$\liminf_{n \to \infty} x_n \leq \liminf_{i \to \infty} x_{n_i} \leq \limsup_{i \to \infty} x_{n_i} \leq \limsup_{n \to \infty} x_n$$

(3) If $\lim_{n \to \infty} x_n = x$ then $\liminf_{n \to \infty} x_n = \limsup_{n \to \infty} x_n = x$. 


(4) If \( \liminf_{n \to \infty} x_n = \limsup_{n \to \infty} x_n \) and the value is finite then \((x_n)_{n \in \mathbb{N}}\) converges and we have \( \lim_{n \to \infty} x_n = \liminf_{n \to \infty} x_n = \limsup_{n \to \infty} x_n \).

We collect this in a smashy corollary for emphasis:

**Corollary 3.7.** Let \((x_n)_{n \in \mathbb{N}}\). Then

- \((x_n)_{n \in \mathbb{N}}\) is a convergent sequence if and only if
- \( \liminf_{n \to \infty} x_n = \limsup_{n \to \infty} x_n \) and this number is finite.

Also

- \( \lim_{n \to \infty} x_n = \pm \infty \) if and only if
- \( \liminf_{n \to \infty} x_n = \limsup_{n \to \infty} x_n = \pm \infty \)

**Proof of Lemma 3.6.**

1. Obvious from the definition
2. Obvious from the definition of lim sup, and monotonicity of the supremum/infinum.
3. From Lemma 3.5 we have that there exists a subsequence \((x_{n_i})_{i \in \mathbb{N}}\) such that
   \[
   \lim_{i \to \infty} x_{n_i} = \limsup_{n \to \infty} x_n.
   \]
   On the other hand, since \(x_n\) converges, so does any of its subsequences, so
   \[
   \lim_{i \to \infty} x_{n_i} = \lim_{n \to \infty} x_n.
   \]
   Together we find
   \[
   \limsup_{n \to \infty} x_n = \lim_{n \to \infty} x_n.
   \]
   The same argument works for the lim inf.
4. Let \(a_n := \inf_{k \geq n} x_k\) and \(b_n := \sup_{k \geq n} x_k\). Then
   \[
   a_n \leq x_n \leq b_n \quad \forall n \in \mathbb{N}.
   \]
   Since by assumption and Lemma 3.5,
   \[
   \liminf_{n \to \infty} x_n = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \limsup_{n \to \infty} x_n
   \]
   We conclude by the squeeze theorem, Lemma 2.11 that
   \[
   \lim_{n \to \infty} x_n = \liminf_{n \to \infty} x_n = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \limsup_{n \to \infty} x_n
   \]

A very useful theorem (indeed a consequence of Lemma 3.5) is that in \(\mathbb{R}\) every bounded sequence has a convergent subsequence\textsuperscript{10}

\textsuperscript{10}This remains true in finite dimensional metric spaces (whatever that means), but becomes false in infinite dimensional spaces. Since many important spaces are infinite dimensional (e.g. function spaces), for some function spaces a replacement is known: weak convergence, and the theorem by Banach-Alaoglu. This generalization is part of Functional Analysis and is one of the most crucial results in Analysis.
Figure 4.1. Augustin-Louis Cauchy, 1789 – 1857. French, mathematician, engineer, and physicist. Almost singlehandedly founded complex analysis (that’s why almost every theorem in complex analysis is Cauchy’s theorem).

**Theorem 3.8 (Bolzano-Weierstrass).** Suppose that \((x_n)_{n \in \mathbb{N}}\) is a bounded sequence in \(\mathbb{R}\). Then there exists a convergent subsequence.

**Proof.** Since \((x_n)_{n \in \mathbb{N}}\) is bounded, \(x := \limsup_{n \to \infty} x_n\) is a well-defined (finite!) number \(x \in \mathbb{R}\). From Lemma 3.5 we thus know that there must be a subsequence \((x_{n_i})_{i \in \mathbb{N}}\) with \(\lim_{i \to \infty} x_{n_i} = x\). \(\square\)

As a consequence of Theorem 3.8 and Exercise 2.21 one obtains the following statement.

**Corollary 3.9.** Assume that \((x_n)_{n \in \mathbb{N}}\) is a bounded sequence in \(\mathbb{R}\) and that there exists \(x \in \mathbb{R}\) such that any convergent subsequence \((x_{n_i})_{i \in \mathbb{N}}\) converges to \(x\). Then \(x_n\) converges to \(x\).

**Exercise 3.10.** Prove Corollary 3.9.

3.1. Further (optional) exercises.

**Exercise.** Assume \((x_n)_{n \in \mathbb{N}}\) is a sequence with \(x_n \neq 0\) for all but finitely many \(n \in \mathbb{N}\), and such that

\[
\limsup_{n \to \infty} \left| \frac{x_n}{x_{n+1}} \right| < 1.
\]

Show that \(\lim_{n \to \infty} x_n = 0\).

4. **Cauchy sequences**

A Cauchy sequence is a sequence where all sequence elements eventually lie arbitrarily close to each other. This is almost as good as converging – unless there is a whole in our underlying space.

Here is the formal definition.
Definition 4.1. A sequence \((x_n)_{n \in \mathbb{N}}\) is called a \textit{Cauchy sequence} if for any \(\varepsilon > 0\) there exists \(N = N(\varepsilon) \in \mathbb{N}\) such that
\[
|x_n - x_m| < \varepsilon \quad \forall n, m > N.
\]

Example 4.2. (1) The sequence \(x_n = \text{first } n \text{ digits of } \pi\) is a \textit{Cauchy sequence}. Indeed, fix \(\varepsilon > 0\) arbitrary. Let \(N \in \mathbb{N}\) such that \(10^{1-N} < \varepsilon\). Let \(n, m \geq N\) with w.l.g. \(n \leq m\). Then
\[
x_n - x_m = 0.0\ldots0 \quad \text{remaining } (m-n) \text{ digits of } \pi.
\]
That is
\[
|x_n - x_m| \leq 10^{1-n} \leq 10^{1-N} < \varepsilon.
\]
That is, \(x_n\) is a \textit{Cauchy} sequence.

Observe that the sequence \((x_n)_{n \in \mathbb{N}}\) is convergent in \(\mathbb{R}\) (\(\lim_{n \to \infty} x_n = \pi\)) but not in \(\mathbb{Q}\) (because \(\pi \notin \mathbb{Q}\)).

(2) \textbf{Warning:} The following is \textit{not} an equivalent definition for a \textit{Cauchy} sequence:
for any \(\varepsilon > 0\) there exists \(N = N(\varepsilon) \in \mathbb{N}\) such that
\[
|x_n - x_{n+1}| < \varepsilon \quad \forall n > N.
\]
Indeed, take
\[
x_n := \sum_{\ell=1}^{n} \frac{1}{\ell}.
\]
We have that
\[
|x_n - x_{n+1}| = \frac{1}{n+1} \xrightarrow{n \to \infty} 0.
\]

However, we know from Calculus 2 that
\[
\lim_{n \to \infty} \sum_{\ell=1}^{n} \frac{1}{\ell} = \sum_{\ell=1}^{\infty} \frac{1}{\ell} = \infty.
\]
So \(\lim_{n \to \infty} x_n\) does not exist, so by Theorem 4.4 below, \((x_n)_{n \in \mathbb{N}}\) is not a \textit{Cauchy} sequence.

This also can be seen explicetly: Relabelling this implies that for any \(n \in \mathbb{N}\)
\[
\sum_{\ell=n}^{\infty} \frac{1}{\ell} = \infty.
\]
This in turn (by a contradiction argument) implies that for any \(n \in \mathbb{N}\) there must be an \(m \in \mathbb{N}, m > n\) such that
\[
\sum_{\ell=n}^{m} \frac{1}{\ell} \geq 1.
\]
That is, for any \(N \in \mathbb{N}\) and any \(n \geq N\) there exists \(m \geq n \geq N\) such that
\[
|x_n - x_m| \not< 1.
\]
That is, \((x_n)_{n \in \mathbb{N}}\) is not a Cauchy sequence.

As we said before, Cauchy sequences are almost as good as converging sequences if the underlying space is complete (i.e. has no holes).

First we observe that any converging sequence is necessarily Cauchy.

**Lemma 4.3 (Converging sequences are Cauchy).** Let \((x_n)_{n \in \mathbb{N}}\) be a converging series. Then \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence \(^{11}\).

**Proof.** Set \(x := \lim_{n \to \infty} x_n\) (exists, because \((x_n)_{n \in \mathbb{N}}\) is converging). That is, for any \(\varepsilon > 0\) there exist \(N = N(\varepsilon)\) such that

\[
|x_n - x| < \frac{\varepsilon}{2} \quad \forall n \geq N.
\]

But then also

\[
|x_n - x_m| \leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall n, m \geq N.
\]

That is, \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence. \(\square\)

As many things of this course, the notion of a Cauchy sequences lives up to its full potential in metric spaces \((X, d)\). Metric spaces are complete if any Cauchy sequence has a limit (in the same space). If the metric space is not complete it has essentially an infinitesimal hole. Plugging these holes is called metric completion. For our purposes: \(\mathbb{Q}\) is not complete, and \(\mathbb{R}\) is the metric completion of \(\mathbb{Q}\).

**Theorem 4.4.** Any Cauchy sequence in \(\mathbb{R}\) is convergent, and any convergent sequence is a Cauchy sequence.

Before proving Theorem 4.4 we first show the following property (which holds in general metric spaces)

**Lemma 4.5.** Any Cauchy sequence is bounded \(^{12}\).

**Proof.** The argument is very similar to the proof of Lemma 2.5.

Let \((x_n)_{n \in \mathbb{N}}\) be a Cauchy sequence. Then there exists an \(N \in \mathbb{N}\) such that

\[
|x_{N+1} - x_n| < 1 \quad \forall n > N. \tag{4.1}
\]

Set

\[
M := \max\{|x_1|, \ldots, |x_{N+1}|\} + 1
\]

Then we have

\[
|x_n| \leq M \quad \forall n = 1, \ldots, N + 1.
\]

\(^{11}\)can be in \(\mathbb{Q}\) or \(\mathbb{R}\) or \(\mathbb{R} \setminus \{\sqrt{2}\}\), it does not matter

\(^{12}\)again, this can be in \(\mathbb{Q}\) or \(\mathbb{R}\) or \(\mathbb{R} \setminus \{\sqrt{2}\}\), it does not matter
On the other hand by (4.1) we have that
\[ |x_n| \leq |x_n - x_{N+1}| + |x_{N+1}| \leq 1 + |x_{N+1}| \leq M \quad \forall n > N. \]
That is, we have shown that \( |x_n| \leq M \) for all \( n \in \mathbb{N} \). □

Now we can give

**Proof of Theorem 4.4.** Any converging sequence is **Cauchy**: This is Lemma 4.3.

Any **Cauchy** sequence is convergent. Let \((x_n)_{n \in \mathbb{N}}\) be a **Cauchy** sequence, we need to show it converges in \( \mathbb{R} \). In view of Lemma 4.5 \((x_n)_{n \in \mathbb{N}}\) is bounded. By Bolzano-Weierstrass, Theorem 3.8, there exist a convergent subsequence \((x_{n_i})_{i \in \mathbb{N}}\) with
\[
\lim_{i \to \infty} x_{n_i} = x.
\]
Now we show that the **Cauchy** sequence property implies that \( \lim_{n \to \infty} x_n = x \). For this let \( \varepsilon > 0 \) be given. By the limit property for \((x_{n_i})_{i \in \mathbb{N}}\), (4.2), there must be \( N_1 \in \mathbb{N} \) such that
\[ |x_{n_i} - x| < \frac{\varepsilon}{2} \quad \forall i > N_1. \]
By the **Cauchy** property of \((x_n)_{n \in \mathbb{N}}\) there must be another \( N_2 \in \mathbb{N} \) such that
\[ |x_n - x_m| < \frac{\varepsilon}{2} \quad \forall n, m > N_2. \]
Now choose \( i > N_1 \) such that \( n_i > N_2 \). Then by the above estimates,
\[ |x_n - x| \leq |x_{n_i} - x| + |x_n - x_{n_i}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \forall n > N_2. \]
This proves that \( \lim_{n \to \infty} x_n = x \) and the proof of Theorem 4.4 is finished. □

**Remark 4.6.** Theorem 4.4 is sometimes called the **Cauchy criterion**: A sequence \((x_n)_{n \in \mathbb{N}}\) (in \( \mathbb{R} \)) is convergent if and only if \((x_n)_{n \in \mathbb{N}}\) is a **Cauchy** sequence.

**Exercise 4.7.** In Theorem 4.4 we have shown
\[
\text{any **Cauchy** sequence } (x_n)_{n \in \mathbb{N}} \subset \mathbb{R} \text{ has a limit in } \mathbb{R},
\]
\[ \text{i.e. there exists } x \in \mathbb{R} \text{ with } \lim_{n \to \infty} x_n = x. \]
The same statement is false in \( \mathbb{Q} \). Namely, the following is false:
\[
\text{any **Cauchy** sequence } (x_n)_{n \in \mathbb{N}} \subset \mathbb{Q} \text{ has a limit in } \mathbb{Q},
\]
\[ \text{i.e. there exists } x \in \mathbb{Q} \text{ with } \lim_{n \to \infty} x_n = x. \]

a) Give a counterexample to (2).
b) Which part of the proof of (1) (from Theorem 4.4) fails when we attempt to prove (2)?
**Exercise 4.8.** Only using the definition of Cauchy sequence, in particular without using Theorem 4.4, show the following

Assume the \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence, and there exists a subsequence \((x_{n_i})_{i \in \mathbb{N}}\) such that \((x_{n_i})_{i \in \mathbb{N}}\) is convergent. Set

\[
    z := \lim_{i \to \infty} x_{n_i}.
\]

Show that \((x_n)_{n \in \mathbb{N}}\) converges to \(z\), i.e. show that

\[
    z = \lim_{n \to \infty} x_n.
\]

**Hint:** You are not allowed to use the (statement) of Theorem 4.4, but you can look at the proof of Theorem 4.4, where we have essentially shown this.

The notion of Cauchy sequence is very useful to

- Check if a sequence converges in \(\mathbb{R}\) (more generally complete metric spaces): Sometimes it is easier to check if a sequence is Cauchy than to guess the limit
- to “complete spaces” (this is called: metric completion). Let us illustrate this for the metric completion of \(\mathbb{Q}\) to \(\mathbb{R}\) (but this works for any metric space).

We define

\[
    \mathbb{R} = \{(x_n)_{n \in \mathbb{N}} \subset \mathbb{Q} : \text{ (}x_n)_{n \in \mathbb{N}}\text{ is Cauchy sequence}\} / \sim
\]

where \(\sim\) is the equivalence relation

\[
    (x_n)_{n \in \mathbb{N}} \sim (y_n)_{n \in \mathbb{N}} \iff \lim_{n \to \infty} |x_n - y_n| = 0.
\]

That is, we consider two sequences to be the same if their distance converges to zero.

- \(\mathbb{Q} \subset \mathbb{R}\) in the following sense: We identify an element \(q \in \mathbb{Q}\) with

\[
    [q] \sim \left\{(x_n)_{n \in \mathbb{N}} \subset \mathbb{Q} : \lim_{n \to \infty} x_n = q \right\}.
\]

- \(\mathbb{R}\) corresponds to the usual definition of \(\mathbb{R}\) since any element \(r \in \mathbb{R}\) can be approximated by a sequence in \((q_n)_{n \in \mathbb{Q}}, \lim_{n \to \infty} q_n = r\). If we have two such sequences, \((q_n)_{n \in \mathbb{Q}}, (s_n)_{n \in \mathbb{Q}}, \lim_{n \to \infty} q_n = r = \lim_{n \to \infty} s_n\) then \(\lim_{n \to \infty} |q_n - s_n| = 0\), so \((q_n)_{n} \sim (s_n)_{n}\).

- The fact that our new definition of \(\mathbb{R}\) as Cauchy sequences of \(\mathbb{Q}\) is indeed complete follows with a diagonal argument, we will not treat it here.

- This method is general for metric spaces \((X, d)\) (any metric space \((X, d)\) can be made complete by considering its Cauchy sequences). One advantage of this method is that uniformly continuous functions on \((X, d)\) will extend uniquely to uniform continuous functions on the larger space, cf. Exercise 9.9.

**Exercise 4.9.** [Leb, Ex. 2.4.1] Prove that \(\left(\frac{n^2-1}{n^2}\right)\) is Cauchy using directly the definition of Cauchy sequences.
The following result is very useful for contraction arguments, such as the Banach Fixed point theorem (which we will not treat in this course, but see Theorem 6.17).

Exercise 4.10. [Leb, Ex. 2.4.2] Let \((x_n)_{n \in \mathbb{N}}\) be a sequence such that there exists a \(0 < \lambda < 1\) such that

\[ |x_{n+1} - x_n| \leq \lambda |x_n - x_{n-1}|. \]

(1) Prove that \((x_n)_{n \in \mathbb{N}}\) is Cauchy.
(2) Why doesn’t this contradict Example 4.2(2)?

Hint: You can freely use the formula (for \(\lambda \neq 1\))

\[ 1 + \lambda + \lambda^2 + \cdots + \lambda^n = \frac{1 - \lambda^{n+1}}{1 - \lambda}. \]

4.1. Optional exercises.

Exercise. Give an example of a bounded sequence that is not a Cauchy sequence.

Exercise. Show that

\[ \sum_{i=1}^{n} \frac{1}{n!} \]

is a Cauchy sequence (directly from the definition of Cauchy sequence).

5. Limits of functions

Remark 5.1 (Further reading). • Interactive picture example of \(\varepsilon\)-\(\delta\) limit operation

https://www.desmos.com/calculator/4efsywgvtg

We want to describe how a function \(f\) behaves near to a point \(c \in \mathbb{R}\), i.e. we would like to give a notion for

\[ \lim_{x \to c} f(x). \]

To do this, \(f\) does not need to be defined at \(c\), but it needs to be defined “close to \(c\”).

Definition 5.2. Let \(D \subset \mathbb{R}\) be a set.

• We define \(\overline{D}\) the closure of the set \(D\) as follows

\[ \overline{D} := \left\{ x \in \mathbb{R} : \exists (x_n)_{n \in \mathbb{N}} \subset D \lim_{n \to \infty} x_n = x \right\} \]

That is \(\overline{D}\) are all points (in \(\mathbb{R}\)) that can be approximated by sequences from within \(D\).

• A set \(D \subset \mathbb{R}\) is closed, if \(D = \overline{D}\).

• A set \(D \subset \mathbb{R}\) is open, if \(\mathbb{R} \setminus D\) is closed\(^{13}\)

\(^{13}\)Below we will see a equivalent but nicer/more intuitive definition of open: a set \(D\) is open if around any point \(x \in D\) a whole neighborhood of that point belongs to \(D\).
While not so relevant for our purposes, let us also define the boundary of a set $D$, usually denoted by $\partial D$,

$$\partial D = \overline{D} \cap \left( \mathbb{R}^n \setminus \overline{D} \right).$$

Equivalently $\partial D$ is the set of all points $x$ such that there exists a sequence $(y_n)_{n \in \mathbb{N}} \subset D$ and another sequence $(z_n)_{n \in \mathbb{N}} \subset \mathbb{R}^n \setminus D$ such that $x = \lim_{n \to \infty} y_n = \lim_{n \to \infty} z_n$. That is $\partial D$ are the points that can be approximated from both within $D$, and from within the complement of $D$, $\mathbb{R}^n \setminus D$.

- A point $c \in \overline{D}$ is a cluster point of $D$, if there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset D \setminus \{c\}$ with $\lim_{n \to \infty} x_n = c$.

  That is, a point $c$ is a cluster point of $D$ if it can approximated by points within $D$ different from $c$ itself.

**Exercise 5.3.** Show that

a) the set $\mathbb{Z} = \{ \ldots, -1, 0, 1, \ldots \}$ has no cluster points.

b) every point in $\mathbb{R}$ is a cluster point of $\mathbb{Q}$.

**Exercise 5.4.** The empty set $\emptyset$ is both open and closed. So is $\mathbb{R} = \mathbb{R} \setminus \emptyset$.

**Exercise 5.5.** $\overline{\mathbb{Q}} = \mathbb{R}$, $\mathbb{Q}$ is neither open nor closed

**Lemma 5.6.** We always have $D \subset \overline{D}$
Proof. For any \( x \in D \), the sequence \((x_n)_{n \in \mathbb{N}} := (x, x, x, \ldots)\) clearly converges to \( x \), \( \lim_{n \to \infty} x_n = x \), so \( x \in \overline{D} \).

**Exercise 5.7.** Let \( D \subset \mathbb{R} \) be a set. Show that \( \overline{D} \) is closed, i.e. that \( \overline{D} = \bar{D} \).

**Example 5.8.** \([1, 2] = [1, 2]\)

**Proof.** Indeed, we already know \((1, 2) \subset [1, 2]\). Now \(1 \in (1, 2)\) because \( x_n := 1 + \frac{1}{n} \in (1, 2) \) for any \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} x_n = 1 \). Similar argument for \(2\). If \( x \notin [1, 2] \) then there must be a \( \delta > 0 \) such that \( x < 1 - \delta \) or \( x > 2 + \delta \). Now for any sequence \((x_n)_{n \in \mathbb{N}}\) with \( \lim_{n \to \infty} x_n = x \) there exists \( N \in \mathbb{N} \) such that

\[
|x_n - x| < \frac{\delta}{2}, \quad \forall n \geq N.
\]

But then if \( x < 1 - \delta \) we have
\[
x_n < x_n - x + x < 1 - \delta + \frac{\delta}{2} = 1 - \frac{\delta}{2} < 1, \quad \forall n \geq N
\]

or if \( x > 2 + \delta \),
\[
x_n > x_n - x + x > 2 + \delta - \frac{\delta}{2} = 2 + \frac{\delta}{2} > 2, \quad \forall n \geq N.
\]

That is, in either case \( x_n \notin (1, 2) \) for all \( n \geq N \). That means there is no sequence \((x_n)_{n \in \mathbb{N}} \in (1, 2)\) such that \( \lim_{n \to \infty} x_n = x \) if \( x \notin [1, 2] \).

**Exercise 5.9.** \([1, 2] \) is closed, all points are cluster points.

**Lemma 5.10.** A set \( D \subset \mathbb{R} \) is open, if and only if for any \( x_0 \in D \) there exists \( \varepsilon > 0 \) such that \((x_0 - \varepsilon, x_0 + \varepsilon) \subset D \) (in words: any point \( x_0 \in D \) has a small neighborhood also belonging to \( D \)).

**Proof.** Assume that \( D \) is open and \( x_0 \in D \), but for any \( \varepsilon > 0 \) there exists a point \( x_{\varepsilon} \in (x_0 - \varepsilon, x_0 + \varepsilon) \setminus D \). Choosing \( \varepsilon := \frac{1}{n} \) we then find a sequence \( x_n \in (x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \setminus D \). That is \(|x_n - x_0| \xrightarrow{n \to \infty} 0\). That is \( \lim_{n \to \infty} x_n = x_0 \). On the other hand, \( x_n \in \mathbb{R} \setminus D \) which is closed by assumption, so \( x_0 \in \mathbb{R} \setminus D \). Contradiction to \( x_0 \in D \).

For the other direction, assume that \( D \) is such that for any \( x_0 \in D \) there exists \( \varepsilon > 0 \) such that \((x_0 - \varepsilon, x_0 + \varepsilon) \subset D \). Let \((x_n)_{n \in \mathbb{N}} \subset \mathbb{R} \setminus D \) be a converging sequence and set \( x_0 := \lim_{n \to \infty} x_n \). We need to show \( x_0 \in \mathbb{R} \setminus D \). To the contrary assume that \( x_0 \in D \). By assumption there exists \( \varepsilon > 0 \) such that \((x_0 - \varepsilon, x_0 + \varepsilon) \subset D \), which means that \( x_n \notin (x_0 - \varepsilon, x_0 + \varepsilon) \). But this means that \(|x_n - x_0| \geq \varepsilon \) for all \( n \in \mathbb{N} \), i.e. \( x_n \) does not converge to \( x_0 \). Contradiction.

**Lemma 5.11.** Let \( D \) be an open set, then any point \( x_0 \in \overline{D} \) is a clusterpoint of \( D \) (and \( \overline{D} \)).
Proof. Indeed, let \( x_0 \in \overline{D} \). If \( x_0 \notin D \) there is no problem: by definition there must be a sequence \( (x_n)_{n \in \mathbb{N}} \subset D = D \setminus \{x_0\} \) with \( \lim_{n \to \infty} x_n = x_0 \).

If \( x_0 \in D \), then we construct a sequence \( (x_n)_{n \in \mathbb{N}} \subset D \setminus \{x_0\} \) as follows. Since \( D \) is open and \( x_0 \in D \), there must be some \( \varepsilon > 0 \) such that \( (x_0 - \varepsilon, x_0 + \varepsilon) \subset D \). Let \( N \) be such that \( \frac{1}{N} < \varepsilon \). Then take \( x_n \) any element from \( (x_0 - \frac{1}{N+n}, x_0 + \frac{1}{N+n}) \setminus \{x_0\} \subset D \setminus \{x_0\} \). It is easy to show that \( \lim_{n \to \infty} x_n = x_0 \). \( \square \)

Now we want to define

\[
\lim_{x \to c} f(x)
\]

There are issues to deal with: \( x \to c \) is not a sequence, second \( f(c) \) may not be defined.

**Definition 5.12.** Let \( f : D \subset \mathbb{R} \to \mathbb{R} \) be a function and \( c \in \overline{D} \) be a cluster point of \( D \). We say that the **limit as \( x \to D \) approaches \( c \) of \( f \) is a number \( L \in \mathbb{R} \),

\[
\lim_{x \to c} f(x) = L, \quad \text{or} \quad f(x) \xrightarrow{x \to c} L.
\]

if \(^{14}\)

\[
\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) > 0 : \quad \text{s.t.} \quad |f(x) - L| < \varepsilon \quad \forall x \in D \setminus \{c\} : \quad |x - c| < \delta.
\]

Cf. Figure 5.2.

\(^{14}\text{since } c \text{ is a cluster point of } D, \text{ the set } D \setminus \{c\} : \quad |x - c| < \delta \text{ is nonempty} \)
Example 5.13. Let 
\[ f(x) := \begin{cases} 
1 & x < 0 \\
0 & x > 0. 
\end{cases} \]
Then the domain of \( f \) is \(( -\infty, 0) \cup (0, \infty) \) and \( \lim_{x \to 0} f(x) \) does not exist.
On the other hand, if we consider \( f \) as a function \( f : (-\infty, 0) \to \mathbb{R} \) then \( \lim_{x \to 0} f(x) = 1. \)

First we show that if the limit exists, the limit is unique:

**Lemma 5.14.** Let \( f : D \subset \mathbb{R} \to \mathbb{R} \) be a function and \( c \in \overline{D} \) be a cluster point. If \( L_1, L_2 \in \mathbb{R} \) with \( \lim_{x \to c} f(x) = L_1 \) and \( \lim_{x \to c} f(x) = L_2 \) then \( L_1 = L_2. \)

**Proof.** By the definition of a limit, for any \( \varepsilon > 0 \) there must be \( \delta = \delta(\varepsilon) > 0 \) such that 
\[ \max \{|f(x) - L_1|, |f(x) - L_2|\} < \frac{\varepsilon}{2} \quad \forall x \in D \setminus \{c\} : |x - c| < \delta. \]
But then, if we pick any point \( x \in D, x \neq c, \) such that \( |x - c| < \delta \) (this point must exist, since \( c \) is a cluster point)
\[ |L_1 - L_2| \leq |f(x) - L_1| + |f(x) - L_2| < 2\frac{\varepsilon}{2} = \varepsilon \]
We can do this for any \( \varepsilon > 0, \) so \( |L_1 - L_2| < \varepsilon \) for any \( \varepsilon > 0. \) By the Archimedian principle this means that \( L_1 = L_2. \)

Recall again \( x \to c \) doesn't make too much sense, since \( x \) is not a sequence. The meaning of \( x \to c \) is: “take any possible sequence \((x_n)_{n \in \mathbb{N}}\) converging to \( c\)” (but no sequence element equal to \( c\)). More precisely, we have

**Lemma 5.15** (sequential limits). Let \( f : D \subset \mathbb{R} \to \mathbb{R} \) and \( c \in \overline{D} \) be a cluster point of \( D. \) Then the following are equivalent for any \( L \in \mathbb{R} \)

1. \( \lim_{x \to c} f(x) = L \)
2. for any sequence \((x_n)_{n \in \mathbb{N}} \subset D \setminus \{c\} \) with \( \lim_{n \to \infty} x_n = c \) we have that the sequence \((f(x_n))_{n \in \mathbb{N}}\) is convergent to \( L, \) i.e. \( \lim_{n \to \infty} f(x_n) = L. \)

**Proof.** (1) \( \Rightarrow (2)\): Assume that \( \lim_{x \to c} f(x) = L, \) and let \((x_n)_{n \in \mathbb{N}} \subset D \setminus \{c\} \) with \( \lim_{n \to \infty} x_n = c. \) We need to show that \( \lim_{n \to \infty} f(x_n) = L. \) For this let \( \varepsilon > 0 \) be arbitrary. We need to find \( N = N(\varepsilon) \) such that 
\[ |f(x_n) - L| < \varepsilon \quad \forall n \geq N. \]
Since by assumption \( \lim_{x \to c} f(x) = L \) there must be \( \delta > 0 \) such that 
\[ |f(x) - L| < \varepsilon \quad \forall x \in D \setminus \{c\} : |x - c| < \delta. \]
Moreover, since \( \lim_{n \to \infty} x_n = c, \) for this \( \delta \) there must be an \( N = N(\delta) \) such that 
\[ |x_n - c| < \delta \quad \forall n \geq N. \]
Figure 5.3. The graph of the function \( f(x) = \sin(1/x) \)

So in particular, for any \( n \geq N \) we have \( |f(x_n) - L| < \varepsilon \), which is what we needed to show.

\[(2) \Rightarrow (1):\]

Assume that \( \lim_{n \to \infty} f(x_n) = L \) holds for any sequence \( (x_n)_{n \in \mathbb{N}} \subset D\setminus \{c\} \) with \( \lim_{n \to \infty} x_n = c \). We need to show that \( \lim_{x \to c} f(x) = L \), that is

\[\forall \varepsilon > 0 \exists \delta > 0 : |f(x) - L| < \varepsilon \quad \forall x \in D\setminus \{c\}, |x - c| < \delta.\]

Assume this is not the case, then the logical negation is

\[\exists \varepsilon > 0 \forall \delta > 0 \exists x \in D\setminus \{c\} \text{ with } |x - c| < \delta \text{ but such that } |f(x) - L| > \varepsilon.\]

We can apply the above to \( \delta := \frac{1}{n} \) for each \( n \in \mathbb{N} \). Then for some \( \varepsilon > 0 \) fixed, we find for each \( n \in \mathbb{N} \) some \( x_n \in D\setminus \{c\} \) with \( |x_n - c| < \frac{1}{n} \) but \( |f(x_n) - L| > \varepsilon \).

The sequence \( (x_n)_{n \in \mathbb{N}} \subset D\setminus \{c\} \) then converges, \( \lim_{n \to \infty} x_n = c \). By assumption, this implies that \( \lim_{n \to \infty} f(x_n) = L \), which contradicts that \( |f(x_n) - L| > \varepsilon \) holds for any \( n \in \mathbb{N} \).

**Example 5.16.**

- \( \lim_{x \to 0} \sin(1/x) \) does not exist, cf. Figure 5.3. Indeed take the sequence \( x_n := \frac{1}{\pi n + \pi/2} \). Then \( \sin(x_n) = (-1)^n \), \( \lim_{n \to \infty} x_n = 0 \), but \( \lim_{n \to \infty} \sin(x_n) \) does not exist - Lemma 5.15 implies the limit of \( \sin(1/x) \) cannot exist.

- \( \lim_{x \to 0} x \sin(1/x) = 0 \), cf. Figure 5.4: Indeed,

\[|x \sin(1/x)| \leq |x|.
\]

So for any \( \varepsilon > 0 \) if we choose \( \delta := \varepsilon \) we have

\[|x \sin(1/x) - 0| \leq |x| < \varepsilon \quad \forall |x| < \delta.
\]

**Exercise 5.17.** [Leb, Ex. 3.1.9]: Let \( c_1 \) be a cluster point of \( A \subset \mathbb{R} \) and \( c_2 \) be a cluster point of \( B \subset \mathbb{R} \). Suppose that \( f : A \to B \) and \( g : B \to \mathbb{R} \) are functions such that \( f(x) \to c_2 \) as \( x \to c_1 \) and \( g(y) \to L \) as \( y \to c_2 \). Let \( h(x) := g(f(x)) \) and show \( h(x) \to L \) as \( x \to c_1 \).

Since we know from Lemma 5.15 that the limit of a function \( f \) can be described as the limit of sequence \( f(x_n) \) we can deduce the limit laws from the sequential limit laws.

**Corollary 5.18.** Let \( D \subset \mathbb{R} \) and \( c \in \overline{D} \) a cluster point of \( D \). Let \( f, g, h : D \to \mathbb{R} \) be functions.
Figure 5.4. The graph of the function $f(x) = x \sin(1/x)$

1) If $f(x) \leq g(x)$ for all $x \in D$, then if $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ both exist we have
$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

2) If for some $a, b \in \mathbb{R}$ we have $a \leq f(x) \leq b$ for all $x \in D$, then if $\lim_{x \to c} f(x)$ exists we have
$$a \leq \lim_{x \to c} f(x) \leq b.$$

3) If $f(x) \leq g(x) \leq h(x)$ for all $x \in D$, then if $\lim_{x \to c} f(x) = \lim_{x \to c} h(x)$ (i.e. they both exist and are equal) then $\lim_{x \to c} g(x)$ exists and we have
$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \lim_{x \to c} h(x).$$

Proof. (1) In view of (5.15) for any sequence $(x_n)_{n \in \mathbb{N}} \subset D \setminus \{c\}$ with $\lim_{n \to \infty} x_n = c$ we have
$$\lim_{n \to \infty} f(x_n) = \lim_{x \to c} f(x), \quad \lim_{n \to \infty} g(x_n) = \lim_{x \to c} g(x).$$

On the other hand $f(x) \leq g(x)$ for all $x \in D$ implies $f(x_n) \leq g(x_n)$ for all $n \in \mathbb{N}$, and by the limit laws for sequences (monotonicity of the limit):
$$\lim_{x \to c} f(x) = \lim_{n \to \infty} f(x_n) \leq \lim_{n \to \infty} g(x_n) \leq \lim_{x \to c} g(x).$$

(2) Follows from (1): E.g. taking $g(x) := b$, observing that $\lim_{x \to c} g(x) = b$ we conclude from $f(x) \leq b$ for all $x \in D$ that
$$\lim_{x \to c} f(x) \leq b.$$
(3) This is a consequence of the squeeze lemma, Lemma 2.11. Let \((x_n)_{n \in \mathbb{N}} \subset D \setminus \{c\}\) be an arbitrary sequence with \(\lim_{n \to \infty} x_n = c\).

If we set \(a_n := f(x_n), b_n := g(x_n), c_n := h(x_n)\) and

\[
\Gamma := \lim_{x \to c} f(x) = \lim_{x \to c} h(x),
\]

then we have by assumption (and Lemma 5.15)

\[
a_n \leq b_n \leq c_n \quad \text{for all } n \in \mathbb{N},
\]

and

\[
\Gamma = \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n.
\]

By the squeeze lemma, Lemma 2.11,

\[
\lim_{n \to \infty} b_n = \Gamma.
\]

Thus, we have shown for any sequence \((x_n)_{n \in \mathbb{N}} \subset D \setminus \{c\}\) with \(\lim_{n \to \infty} x_n = c\) that

\[
\lim_{n \to \infty} g(x_n) = \Gamma.
\]

By Lemma 5.15 we conclude that

\[
\lim_{x \to c} g(x) = \Gamma = \lim_{x \to c} f(x) = \lim_{x \to c} h(x).
\]

\[\square\]

From Lemma 5.15 we also obtain that the usual limit laws hold for \(\lim_{x \to c}\)-operation:

**Corollary 5.19.** Let \(D \subset \mathbb{R}\) and \(c \in \overline{D}\) a cluster point of \(D\). Let \(f, g : D \to \mathbb{R}\) be functions. Suppose that \(\lim_{x \to c} f(x)\) and \(\lim_{x \to c} g(x)\) both exist. Then

1. \(\lim_{x \to c} (f(x) + g(x)) = (\lim_{x \to c} f(x)) + (\lim_{x \to c} g(x))\).
2. \(\lim_{x \to c} (f(x) - g(x)) = (\lim_{x \to c} f(x)) - (\lim_{x \to c} g(x))\).
3. \(\lim_{x \to c} (f(x)g(x)) = (\lim_{x \to c} f(x)) (\lim_{x \to c} g(x))\).
4. If \(g(x) \neq 0\) for all \(x \in D\) and \(\lim_{x \to c} g(x) \neq 0\) then

\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}.
\]

**Exercise 5.20.** Prove Corollary 5.19.

To compute limits \(\lim_{n \to \infty} x_n\) we only care about all but finitely many sequence elements of \((x_n)_{n \in \mathbb{N}}\). A similar statement same is true for for \(\lim_{x \to c} f(x)\): we only care about points \(x\) “close” to \(c\), that is it suffices to consider \(f\) **restricted to a small (open) neighborhood of \(c\)**.
Definition 5.21. Let $f : D \to \mathbb{R}$ be a function and let $D_2 \subset D$. The function $f$ restricted to $D_2$, $f\big|_{D_2}$, is the function

$$f\big|_{D_2} : D_2 \to \mathbb{R}$$

$$f\big|_{D_2} : x \ni D_2 \mapsto f(x).$$

Lemma 5.22. Let $D_2 \subset D \subset \mathbb{R}$, $c \in \overline{D} \cap \overline{D_2}$ be a cluster point of $D$ and $D_2$. Let $f : D \to \mathbb{R}$ and let $f\big|_{D_2} : D_2 \to \mathbb{R}$ be its restriction to $D_2$

(1) If $\lim_{x \to c} f(x)$ exists then $\lim_{x \to c} f\big|_{D_2}$ exist, and

$$\lim_{x \to c} f\big|_{D_2} = \lim_{x \to c} f(x).$$

(2) If $\lim_{x \to c} f\big|_{D_2}$ exists, in general $\lim_{x \to c} f$ may not exist.

(3) Assume that $D_2$ contains a relative open neighborhood of $c$ in $D$. That is, assume there exists $\varepsilon > 0$ such that $(c - \varepsilon, c + \varepsilon) \cap D \subset D_2$. Then $\lim_{x \to c} f\big|_{D_2}$ exists if and only if also $\lim_{x \to c} f$ exists. Also if one of the limits exists, we have

$$\lim_{x \to c} f\big|_{D_2} = \lim_{x \to c} f(x).$$

Proof. (1) Let $(x_n)_{n \in \mathbb{N}} \subset D_2 \setminus \{c\}$ be any sequence with $\lim_{n \to \infty} x_n = c$. Since $D_2 \subset D$ we also have $(x_n)_{n \in \mathbb{N}} \subset D \setminus \{c\}$, and thus by assumption and Lemma 5.15,

$$\lim_{n \to \infty} f(x_n) = \lim_{x \to c} f(x).$$

Since for any $n \in \mathbb{N}$ we have $x_n \in D_2 \setminus \{c\}$,

$$f(x_n) = f\big|_{D_2} (x_n)$$

and consequently we have

$$\lim_{n \to \infty} f\big|_{D_2} (x_n) = \lim_{n \to \infty} f(x_n) = \lim_{x \to c} f(x).$$

This holds for any sequence $(x_n)_{n \in \mathbb{N}} \subset D_2 \setminus \{c\}$ with $\lim_{n \to \infty} x_n = c$, so again by Lemma 5.15

$$\lim_{x \to c} f\big|_{D_2} (x) \lim_{n \to \infty} f\big|_{D_2} (x_n) = \lim_{n \to \infty} f(x_n) = \lim_{x \to c} f(x).$$

(2) The typical example is the so-called Heaviside function,

$$f(x) := \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$
It is easy to show (exercise) that \( \lim_{x \to 0} f(x) \) does not exist. However if we consider \( f \big|_{(-\infty,0)} \) then
\[
f \big|_{(-\infty,0)} (x) = 0,
\]
so
\[
\lim_{x \to 0} f \big|_{(-\infty,0)} = 0.
\]

(3) In (1) we have shown that if \( \lim_{x \to c} f \) exists, then also \( \lim_{x \to c} f \big|_{D_2} \) exists and the two numbers are the same.

For the converse, assume that \( \lim_{x \to c} f \big|_{D_2} \) exists. Let \( (x_n)_{n \in \mathbb{N}} \subset D \setminus \{c\} \) be a sequence with \( \lim_{n \to \infty} x_n = c \). Since \( x_n \) converges to \( c \), there must be a large index \( N = N(\alpha) \in \mathbb{N} \) such that
\[
|x_n - c| < \alpha \quad \forall n > N.
\]
That is,
\[
(5.1) \quad x_n \in (c - \alpha, c + \alpha) \cap D \subset D_2 \quad \forall n > N.
\]
Set
\[
(z_n)_{n \in \mathbb{N}} := (x_{N+1}, \ldots, x_{N+1}, x_{N+2}, x_{N+3}, \ldots)
\]
Then from (5.1) we deduce that \( (z_n)_{n \in \mathbb{N}} \subset (D \cap D_2) \setminus \{c\} \) and we have
\[
\lim_{n} z_n = \lim_{n \to \infty} x_n = c.
\]
Thus
\[
f(z_n) = f \big|_{D_2} \left( z_n \right) \xrightarrow{n \to \infty} \lim_{x \to c} f \big|_{D_2} (z_n).
\]
In other words,
\[
\lim_{n \to \infty} f(z_n) = \lim_{x \to c} f \big|_{D_2} (z_n).
\]
Since \( z_n = x_n \) (and thus \( f(z_n) = f(x_n) \)) for all but finitely many \( n \in \mathbb{N} \) we have
\[
\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f(z_n) = \lim_{x \to c} f \big|_{D_2} (z_n).
\]
The above holds for any sequence \( (x_n)_{n \in \mathbb{N}} \subset D \setminus \{c\} \) with \( \lim_{n \to \infty} x_n = c \). By Lemma 5.15 we conclude that
\[
\lim_{x \to c} f(x) = \lim_{x \to c} f \big|_{D_2} (z_n).
\]

\[ \square \]

**Exercise 5.23.** Use the precise \( \varepsilon, \delta \)-definition of limit to prove
\[
\lim_{x \to 2} x^2 = 4.
\]
Further (optional) exercises. Computing limits of functions is also very important, so here some practice exams

**Exercise.** Use the precise $\varepsilon$-$\delta$-definition of limit to prove the following statements.

1. $\lim_{x \to 10}(2x + 4) = 24$
2. $\lim_{x \to -\frac{3}{2}}(1 - 4x) = 7$
3. $\lim_{x \to 1}(x^2 + 3) = 4$
4. $\lim_{x \to \frac{2}{x+3}} = \frac{1}{3}$
5. $\lim_{x \to -6}\frac{x+4}{2-x} = -\frac{1}{4}$
6. $\lim_{x \to 9}(\sqrt{x} + 2) = 5$
7. $\lim_{x \to -2}\frac{2+4x}{3} = 2$
8. $\lim_{x \to -2}(x^2 - 1) = 3$
9. $\lim_{x \to 2}x^3 = 8$

6. Continuous functions

A function $f : D \to \mathbb{R}$ is *continuous* is small changes in the domain $x \in D$ imply small changes in the target $f(x)$.

Here is the precise definition of continuous functions that we are going to use for the rest of our (mathematical) life.

**Definition 6.1** ($\varepsilon$-$\delta$-definition). Let $D \subset \mathbb{R}$ be a set and let $f : D \to \mathbb{R}$ be a function.

- $f$ is **continuous** at a point $x_0 \in D$, if
  \[ \forall \varepsilon > 0 \exists \delta = \delta(\varepsilon) > 0 : |f(x) - f(x_0)| < \varepsilon \text{ holds whenever } x \in D \text{ and } |x - x_0| < \delta. \]

- $f$ is **continuous in** $D$ if $f$ is continuous at any point $x_0 \in D$.

**Exercise 6.2.** [Leb, Ex. 3.2.1] Use the definition of continuity from Definition 6.1 to prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2$ is continuous.

**Exercise 6.3.** [Leb, Ex. 3.2.3] Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

\[
  f(x) = \begin{cases} 
    x & \text{if } x \text{ is rational} \\
    x^2 & \text{if } x \text{ is irrational}.
  \end{cases}
\]

Cf. Figure 6.2. Using the definition of continuity from Definition 6.1, prove that $f$ is continuous at 1 and discontinuous at 2.

Definition 6.1 is not the definition we have from Calculus 1 (which, we recall, was that $f$ is continuous at $x_0$ if $\lim_{x \to x_0} f(x) = f(x_0)$). But it is very related.

**Proposition 6.4** (Continuity via limits). Let $f : D \to \mathbb{R}$ be a function.