Exercise (6.1.7). Suppose that there exists a sequence of functions \( \{g_n\} \) uniformly converging to 0 on \( A \). Now suppose that we have a sequence of functions \( \{f_n\} \) and a function \( f \) on \( A \) such that

\[
|f_n(x) - f(x)| \leq g_n(x)
\]

for all \( x \in A \). Show that \( \{f_n\} \) converges uniformly to \( f \) on \( A \).

Proof. Let \( \varepsilon > 0 \) be given. Since \( g_n \to 0 \) uniformly, there exists an \( N \in \mathbb{N} \) such that \( n \geq N \Rightarrow |g_n(x)| < \varepsilon \) for all \( x \in A \).

For this same \( N \), when \( n \geq N \),

\[
|f_n(x) - f(x)| \leq |g_n(x)| < \varepsilon
\]

for all \( x \in A \). Therefore, \( \{f_n\} \) converges uniformly to \( f \) on \( A \). \( \square \)

Exercise (6.2.2). Let \( f_n(x) := \frac{x^n}{n} \). Show that \( f_n \) converges uniformly to a differentiable function \( f \) on \([0,1]\). However, show that \( f'(1) \neq \lim_{n \to \infty} f_n'(1) \).

Proof. Note that

\[
||f_n - 0||_u = \sup_{x \in [0,1]} \left| \frac{x^n}{n} \right|
\]

\[
\leq \frac{1}{n}
\]

\[
\rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Therefore, \( \{f_n\} \) converges uniformly to the zero function.

However,

\[
\lim_{n \to \infty} f_n'(1) = \lim_{n \to \infty} 1^{n-1} = 1 \neq 0 = f'(1).
\]

\( \square \)

Exercise (6.2.3). Let \( f : [0,1] \to \mathbb{R} \) be a Riemann integrable (hence bounded) function. Find \( \lim_{n \to \infty} \int_0^1 \frac{f(x)}{n} \, dx \).

Exercise (6.1.7). Suppose that there exists a sequence of functions \( \{g_n\} \) uniformly converging to 0 on \( A \). Now suppose that we have a sequence of functions \( \{f_n\} \) and a function \( f \) on \( A \) such that

\[
|f_n(x) - f(x)| \leq g_n(x)
\]

for all \( x \in A \). Show that \( \{f_n\} \) converges uniformly to \( f \) on \( A \).

Proof. Let \( \varepsilon > 0 \) be given. Since \( g_n \to 0 \) uniformly, there exists an \( N \in \mathbb{N} \) such that \( n \geq N \Rightarrow |g_n(x)| < \varepsilon \) for all \( x \in A \).

For this same \( N \), when \( n \geq N \),

\[
|f_n(x) - f(x)| \leq |g_n(x)| < \varepsilon
\]

for all \( x \in A \). Therefore, \( \{f_n\} \) converges uniformly to \( f \) on \( A \). \( \square \)

Exercise (6.2.2). Let \( f_n(x) := \frac{x^n}{n} \). Show that \( f_n \) converges uniformly to a differentiable function \( f \) on \([0,1]\). However, show that \( f'(1) \neq \lim_{n \to \infty} f_n'(1) \).

Proof. Note that

\[
||f_n - 0||_u = \sup_{x \in [0,1]} \left| \frac{x^n}{n} \right|
\]

\[
\leq \frac{1}{n}
\]

\[
\rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Therefore, \( \{f_n\} \) converges uniformly to the zero function.

However,

\[
\lim_{n \to \infty} f_n'(1) = \lim_{n \to \infty} 1^{n-1} = 1 \neq 0 = f'(1).
\]

\( \square \)

Exercise (6.2.3). Let \( f : [0,1] \to \mathbb{R} \) be a Riemann integrable (hence bounded) function. Find \( \lim_{n \to \infty} \int_0^1 \frac{f(x)}{n} \, dx \).
Proof. Define \( f_n(x) = \frac{f(x)}{n} \). Let \( M \) be a bound for \( f \) on \([0,1]\). Since \( |f_n(x) - 0| \leq \frac{M}{n} \to 0 \) as \( n \to \infty \) independent of \( x \), \( f_n \to 0 \) uniformly on \([0,1]\). Therefore,

\[
\lim_{n \to \infty} \int_0^1 \frac{f(x)}{n} \, dx = \int_0^1 \lim_{n \to \infty} f_n(x) \, dx = \int_0^1 0 \, dx = 0
\]

\[\square\]

**Exercise (6.2.4).** Show \( \lim_{n \to \infty} \int_1^2 e^{-nx^2} \, dx = 0 \).

**Solution.** Let \( f_n(x) := e^{-nx^2} \) and note that \( f'_n(x) = -2nx e^{-nx^2} < 0 \) on \([1,2]\). Then we have that

\[
||f_n - 0||_u = \sup_{x \in [1,2]} |f_n(x)|
\]

\[
\leq e^{-n} \to 0 \text{ as } n \to \infty.
\]

Therefore, \( \{f_n\} \) converges uniformly to the zero function, and

\[
\lim_{n \to \infty} \int_1^2 e^{-nx^2} \, dx = \int_1^2 0 \, dx = 0.
\]

\[\square\]

**Exercise (6.5.5).** (a) Show that

\[
f(x) = \sum_{n=0}^{\infty} \frac{1}{1 + n^2x}
\]

defines a continuous function on \((0, \infty)\).

(b) Prove that \( \lim_{x \to 0^+} f(x) = \infty \) and \( \lim_{x \to \infty} f(x) = 1 \).

**Proof.** (a) Fix \( a > 0 \). Then, on \([a, \infty)\),

\[
\left| \frac{1}{1 + n^2x} \right| \leq \frac{1}{a} \cdot \frac{1}{n^2}.
\]

Since \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges, the series \( \sum_{n=0}^{\infty} \frac{1}{1 + n^2x} \) converges uniformly by the \( M \)-Test.
Since each $f_n(x) := \frac{1}{1 + n^2 x}$ is continuous and the series converges uniformly, $f(x)$ is continuous on $[a, \infty)$. Since $a > 0$ was arbitrary, $f(x)$ is continuous on $(0, \infty)$.

(b) Note that $f(1/n) \geq n$. Therefore, $\lim_{x \to 0^+} f(x) = \infty$.

Also, note that

$$|f(x) - 1| = \left| \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x} \right| = \lim_{N \to \infty} \left| \sum_{n=1}^{N} \frac{1}{1 + n^2 x} \right|$$

$$\leq \lim_{N \to \infty} \frac{1}{x} \sum_{n=1}^{N} \frac{1}{n^2}$$

$$= \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (\(p\)-test), this gives that $|f(x) - 1| \to 0$ as $x \to \infty$.

I.e. $\lim_{x \to \infty} f(x) = 1$. 

\[\square\]