Exercise (1.2.10). Let $A$ and $B$ be two nonempty bounded sets of nonnegative real numbers. Define the set $C := \{ab : a \in A, b \in B\}$. Show that $B$ is a bounded set and that

$$\sup C = (\sup A)(\sup B) \quad \text{and} \quad \inf C = (\inf A)(\inf B)$$

Proof. First, let $M$ be a bound for $A$ and $N$ a bound for $B$. Then for any $c = ab \in C$, $|c| = |ab| = |a||b| \leq MN$. Hence, $C$ is bounded.

Next, let $a \in A$ and $b \in B$ be arbitrary, so that $c = ab$ is an arbitrary member of $C$. Since $c = ab \leq (\sup A)(\sup B)$, $(\sup A)(\sup B)$ is an upper bound of $C$, and therefore $\sup C \leq (\sup A)(\sup B)$.

Also, since $c = ab \in C$, $ab \leq \sup C$. Since all numbers are nonnegative, this gives that $a \leq \frac{\sup C}{b}$. Since $a$ was arbitrary, $\frac{\sup C}{b}$ is an upper bound of $A$, and hence $\sup A \leq \frac{\sup C}{b}$.

Rearranging this equation gives $b \leq \frac{\sup C}{\sup A}$, and again, since $b$ was an arbitrary member of $B$, $\frac{\sup C}{\sup A}$ is an upper bound of $B$. Hence, $\sup B \leq \frac{\sup C}{\sup A}$ or $(\sup A)(\sup B) \leq \sup C$.

Since we have $\sup C \leq (\sup A)(\sup B)$ and $(\sup A)(\sup B) \leq \sup C$, it must be true that $\sup C = (\sup A)(\sup B)$.

The proof of the statement involving infimums is similar.

Exercise (1.3.1). Let $\epsilon > 0$. Show that $|x-y| < \epsilon$ if and only if $x-\epsilon < y < x+\epsilon$.

Proof. Let $\epsilon > 0$. Using Proposition 1.3.1 (ii) and (v),

$$|x-y| < \epsilon \iff |y-x| < \epsilon$$

$$\iff -\epsilon < y-x < \epsilon$$

$$\iff x-\epsilon < y < x+\epsilon$$
Exercise (2.1.12). Let \( S \subset \mathbb{R} \) be a nonempty bounded set. Then there exist monotone sequences \((x_n)\) and \((y_n)\) such that \( x_n, y_n \in S \) and

\[
\sup S = \lim_{n \to \infty} x_n \quad \text{and} \quad \inf S = \lim_{n \to \infty} y_n.
\]

Proof. Let \( a := \sup S \). By the definition of supremum, \( a - 1 \) is not an upper bound of \( S \), and so there exists \( y_1 \in S \) such that \( a - 1 < y_1 \leq a \). Let \( x_1 := y_1 \).

Next, since \( a - \frac{1}{2} \) is not an upper bound of \( S \), there exists \( y_2 \in S \) such that \( a - \frac{1}{2} < y_2 \leq a \). Since we want to construct a monotone sequence, we let \( x_2 := \max\{x_1, y_2\} \).

Continuing, since \( a - \frac{1}{3} \) is not an upper bound of \( S \), there exists \( y_3 \in S \) such that \( a - \frac{1}{3} < y_3 \leq a \). Let \( x_3 := \max\{x_2, y_3\} \).

Continuing in this manner, we construct a monotone increasing sequence \((x_n)\) such that for all \( n \in \mathbb{N} \), \( a - \frac{1}{n} < x_n \leq a \). By the Squeeze Lemma, \((x_n)\) converges to \( a \), as desired.

Note that we can construct a monotone decreasing sequence that converges to \( \inf S \) in a similar way.

Exercise (2.2.9). Suppose that \( \{x_n\} \) is a sequence and suppose that for some \( x \in \mathbb{R} \), the limit

\[
L := \lim_{n \to \infty} \frac{|x_{n+1} - x|}{|x_n - x|}
\]

exists and \( L < 1 \). Show that \( \{x_n\} \) converges to \( x \).

Proof. Let \( \{x_n\} \) be a sequence and \( x \in \mathbb{R} \). Define a sequence \( \{y_n\} \) by \( y_n = (x_n - x) \), for all \( n \in \mathbb{N} \). Then, by our assumption

\[
L := \lim_{n \to \infty} \frac{|y_{n+1}|}{|y_n|} = \lim_{n \to \infty} \frac{|x_{n+1} - x|}{|x_n - x|}
\]

exists and \( L < 1 \). Hence, by the Ratio Test for sequences, \( \{y_n\} \) converges to 0. Note that this implies that \( x_n = (x_n - x) + x \) converges to \( 0 + x = x \), as desired.