**Exercise (3.2.10).** Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Suppose that for all rational numbers $r$, $f(r) = g(r)$. Show that $f(x) = g(x)$ for all $x$.

**Proof.** We want to show that $f(x) = g(x)$ for all $x \in \mathbb{R}$. We already have that $f(r) = g(r)$ if $r \in \mathbb{Q}$. Let $x$ be an arbitrary irrational number. If we can show that $f(x) = g(x)$ we are done.

Since $x \in \mathbb{R}$, there exists a sequence $\{r_n\} \subset \mathbb{Q}$ such that $r_n \to x$ as $n \to \infty$. Both $f$ and $g$ are continuous, and so

$$f(x) = \lim_{n \to \infty} f(r_n) = \lim_{n \to \infty} g(r_n) = g(x).$$

**Exercise (3.3.4).** Let

$$f(x) := \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that $f$ has the intermediate value property. That is, for any $a < b$, if there exists a $y$ such that $f(a) < y < f(b)$ or $f(b) < y < f(a)$, then there exists $c \in (a, b)$ such that $f(c) = y$.

**Proof.** Let $a < b$ and assume that there exists a $y$ such that $f(a) < y < f(b)$ or $f(b) < y < f(a)$. Note that $f(a), f(b) \in [-1, 1]$ and so $-1 < y < 1$.

**Case 1** $0 \in (a, b)$. There exists $t \in (a, b)$ such that $t > 0$ and $f(t) = f(a)$. Since $f|_{(t,b)}$ is continuous and $f(t) < y < f(b)$ or $f(b) < y < f(t)$, $\exists c \in (t, b) \subset (a, b)$ such that $f(c) = y$ by the Intermediate Value Theorem.

**Case 1** $0 \notin (a, b)$. Then $f|_{(a,b)}$ is continuous and $\exists c \in (a, b)$ such that $f(c) = y$ by the Intermediate Value Theorem.

**Exercise (3.3.7).** Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function. Prove that the direct image $f([a, b])$ is a closed and bounded interval or a single number.

**Proof.** If $f([a, b])$ is a single number, we are done. So, suppose otherwise. By the Min-Max Theorem, $f$ attains both an absolute maximum and an absolute minimum. Suppose that the minimum occurs at $c_1 \in [a, b]$ and the maximum
at \(c_2 \in [a, b]\). Then, \(f(c_1) \leq f(x) \leq f(c_2)\), for all \(x \in [a, b]\). I.e. \(f([a, b]) \subseteq [f(c_1), f(c_2)]\).

Without loss of generality, suppose that \(c_1 < c_2\). Note that the restriction of \(f\) to \([c_1, c_2]\) is continuous. Therefore, by the Intermediate Value Theorem, for any \(y\) such that \(f(c_1) < y < f(c_2)\), there is a \(c \in (c_1, c_2)\) such that \(f(c) = y\). Therefore, \(f([a, b]) = [f(c_1), f(c_2)]\), which is a closed and bounded interval.

**Exercise (3.4.3).** Show that \(f : (c, \infty) \rightarrow \mathbb{R}\) for some \(c > 0\) and defined by \(f(x) := 1/x\) is Lipschitz continuous.

**Proof.** Let \(K := 1/c^2\). Then, for any \(x, y \in (c, \infty)\),

\[
|f(x) - f(y)| = |1/x - 1/y| = \frac{|y - x|}{xy} \leq \frac{|x - y|}{c^2} = K |x - y|.
\]

Hence, \(f\) is Lipschitz continuous on \((c, \infty)\).

**Exercise (3.4.4).** Show that \(f : (0, \infty) \rightarrow \mathbb{R}\) defined by \(f(x) := 1/x\) is not Lipschitz continuous.

**Proof.** Assume for contradiction that \(\exists K \in \mathbb{R}\) such that \(|f(x) - f(y)| \leq K |x - y|\) for all \(x, y \in (0, \infty)\). Then, for all \(x, y \in (0, \infty)\),

\[
\frac{1}{x} - \frac{1}{y} \leq K |x - y| \Rightarrow \frac{y - x}{xy} \leq K |x - y| \Rightarrow \frac{|x - 1|}{|x||x - 1|} \leq K \Rightarrow \frac{1}{x} \leq K
\]

This is clearly a contradiction, and hence \(f\) is not Lipschitz continuous on \((0, \infty)\).

Alternatively, you can show that \(f(x) = 1/x\) is not uniformly continuous on \((0, \infty)\) and hence cannot be Lipschitz continuous.
Exercise. A function \( f : D \subset \mathbb{R} \to \mathbb{R} \) is called (sequentially) lower semicontinuous at a point \( x \in D \) if we have
\[
f(x) \leq \liminf_{D \ni y \to x} f(y),
\]
in the sense that for any sequence \((y_n)_{n \in \mathbb{N}} \subset D\) with \( \lim_{n \to \infty} y_n = x \) we have
\[
f(x) \leq \liminf_{n \to \infty} f(y_n).
\]
In a similar spirit, a function is called (sequentially) upper semicontinuous if
\[
f(x) \geq \limsup_{D \ni y \to x} f(y).
\]

(a) Give an example of a lower semicontinuous function which is not continuous.

(b) Give an example of an upper semicontinuous function which is not continuous.

(c) Show that \( f \) is continuous at \( x \in D \) if and only if \( f \) is lower and upper semicontinuous at \( x \).

(d) Show that if \( f : [a, b] \to \mathbb{R} \) is lower semicontinuous in every \( x \in [a, b] \), then \( f \) attains its minimum value in \( [a, b] \).

(e) Show that if \( f : [a, b] \to \mathbb{R} \) is upper semicontinuous in every \( x \in [a, b] \), then \( f \) attains its maximum value in \( [a, b] \).

Note: proofs of parts (d) and (e) will be given after the take-home exam.

Proof. (a)
\[
f(x) := \begin{cases} -1 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}
\]
is lower semicontinuous but not continuous (not continuous at 0).

(b)
\[
f(x) := \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}
\]
is upper semicontinuous but not continuous (not continuous at 0).

(c) Assume that \( f \) is continuous at \( x \in D \). For any sequence \((y_n) \subset D\) such that \( y_n \to x \) as \( n \to \infty \), \( f(y_n) \to f(x) \) and hence
\[
\liminf_{n \to \infty} f(y_n) = f(x) = \limsup_{n \to \infty} f(y_n).
\]
Therefore, \( f \) is lower and upper semicontinuous at \( x \).
Next assume that $f$ is lower and upper semicontinuous at $x$. Then for any sequence $(y_n) \subset D$ such that $y_n \to x$,

$$\limsup_{n \to \infty} f(y_n) \leq f(x) \leq \liminf_{n \to \infty} f(y_n).$$

However, since $\liminf_{n \to \infty} f(y_n) \leq \limsup_{n \to \infty} f(y_n)$, we must have

$$\limsup_{n \to \infty} f(y_n) = f(x) = \liminf_{n \to \infty} f(y_n)$$

and so $\lim_{n \to \infty} f(y_n) = f(x)$. Since $(y_n)$ was arbitrary, this shows that $f$ is continuous at $x$. 

\[ \square \]