Problem 1 For a function \( f : \mathbb{R}^n \to \mathbb{R} \) let the Fourier transform \( \hat{f} : \mathbb{R}^n \to \mathbb{R} \) be defined as

\[
\hat{f}(\xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle\xi, x\rangle} f(x) \, dx.
\]

The inverse Fourier Transform \( f' \) is defined as

\[
f'(\xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle\xi, x\rangle} f(x) \, dx.
\]

Use formal computations (in particular assume that the relevant integrals converge and commute) to compute that

(i) the inversion formula holds,

\[
f(y) = (\hat{f})'(y), \quad f'(y) = \hat{f}'(y).
\]

Show moreover

\[
\hat{f}(x) = f(-x), \quad (f')'(x) = f'(-x).
\]

Hint: You can use that \( \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle\xi, z\rangle} g(z) \, d\xi \, dz = g(0) \).

(ii) Let \( f = \partial_i g \). Show that (formally) for any \( \xi = (\xi_1, \ldots, \xi_n) \) and all \( i = 1, \ldots, n \),

\[
\hat{f}(\xi) = -i\xi_i \hat{g}(\xi).
\]

Also show the converse: If \( g(x) := -ix_i f(x) \) then

\[
\partial_{\xi_i} \hat{f}(\xi) = \hat{g}(\xi).
\]

(iii) Conclude from the previous computations that for \( f = \Delta g \) we have

\[
\hat{f}(\xi) = -|\xi|^2 \hat{g}(\xi).
\]

(iv) Show that for \( f, g : \mathbb{R}^n \to \mathbb{R} \) (formally)

\[
\hat{f}g(\xi) = \int_{\mathbb{R}^n} \hat{f}(\xi - \eta) \hat{g}(\eta) \, d\eta.
\]

(v) Show that if \( f \) is \( s \)-homogeneous, meaning that \( f(\lambda x) = \lambda^s f(x) \) then

\[
f(\lambda \xi) = \lambda^{-n-s} \hat{f}(\xi).
\]

(vi) Show that if \( f \) is a radial function, that is \( f(x) = g(|x|) \) for some \( g : \mathbb{R} \to \mathbb{R} \) then so is \( \hat{f} \).

Hint: Show first that \( f \) radial is equivalent to saying that \( f(Ox) = f(x) \) for any rotation \( O \in O(n) \) (\( O(n) \) are all the matrices \( O \in \mathbb{R}^{n\times n} \) such that \( O^T O = I \)). Then show this last property for the Fourier transform.
Problem 2  Show that
\[
\lim_{r \to 0} \int_{\partial B_r(x)} u(y) \, dH^{n-1}(y) = u(x) \quad \text{for all } u \in C_0(\mathbb{R}^n), \, x \in \mathbb{R}^n.
\]

Problem 3  For \( n \geq 2 \) let \( \Phi(x) \) be the fundamental solution, namely
\[
\Phi(x) = \begin{cases} 
  c \log |x| & \text{if } n = 2 \\
  c |x|^{2-n} & \text{if } n \geq 3
\end{cases}
\]
Show in both cases, \( n = 2 \) and \( n \geq 3 \), that
\[
\Delta \Phi(x) = 0 \quad \text{for any } x \neq 0
\]

Problem 4  For \( n = 2 \) let \( \Phi(x) \) be the fundamental solution, namely
\[
\Phi(x) = c \log |x|.
\]
Show that \( \Phi \in L^p_{loc}(\mathbb{R}^2) \) for any \( p \in [1, \infty) \), that is: for any bounded, open set \( \Omega \subset \mathbb{R}^2 \)
\[
\|\Phi\|_{L^p(\Omega)} \equiv \left( \int_{\Omega} |\Phi(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.
\]
*Hint:* It suffices to consider \( \Omega = B(0, R) \) a ball with radius \( R \).