University of Pittsburgh
Armin Schikorra (armin@pitt.edu)

Partial Differential Equations 1 – Spring 2019
Exercise Sheet 1 — Due Date: January 21

Work in groups, write in LaTeX!

Problem 1 For a function $f : \mathbb{R}^n \to \mathbb{R}$ let the Fourier transform $\hat{f} : \mathbb{R}^n \to \mathbb{R}$ be defined as

$$\hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle} f(x)\, dx.$$

The *inverse Fourier Transform* $f^\vee$ is defined as

$$f^\vee(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{+i\langle \xi, x\rangle} f(x)\, dx.$$

Use formal computations (in particular assume that the relevant integrals converge and commute) to compute that

(i) the inversion formula holds,

$$f(y) = (\hat{f})^\vee(y), \quad f(y) = \widehat{f^\vee}(y)$$

Show moreover

$$\hat{\hat{f}}(x) = f(-x), \quad (f^\vee)^\vee(x) = f(-x).$$

Hint: You can use that

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle \xi, z\rangle} g(z)\, d\xi\, dz = g(0).$$

(ii) Let $f = \partial_{x_i} g$. Show that (formally) for any $\xi = (\xi_1, \ldots, \xi_n)$ and all $i = 1, \ldots, n$,

$$\hat{f}(\xi) = -i\xi_i\, \hat{g}(\xi).$$

Also show the converese: If $g(x) := -ix_i f(x)$ then

$$\partial_{\xi_i} \hat{f}(\xi) = \hat{g}(\xi).$$

(iii) Conclude from the previous computations that for $f = \Delta g$ we have

$$\hat{f}(\xi) = -|\xi|^2\, \hat{g}(\xi).$$

(iv) Show that for $f, g : \mathbb{R}^n \to \mathbb{R}$ (formally)

$$\widehat{fg}(\xi) = \int_{\mathbb{R}^n} \hat{f}(\xi - \eta)\, \hat{g}(\eta)\, d\eta.$$

(v) Show that if f is s-homogeneous, meaning that $f(\lambda x) = \lambda^s f(x)$ then

$$\hat{f}(\lambda\xi) = \lambda^{-n-s} \hat{f}(\xi).$$

(vi) Show that if f is a radial function, that is $f(x) = g(|x|)$ for some $g : \mathbb{R} \to \mathbb{R}$ then so is $\hat{f}$.

Hint: Show first that f radial is equivalent to saying that $f(Ox) = f(x)$ for any rotation $O \in O(n)$ ($O(n)$ are all the matrices $O \in \mathbb{R}^{n\times n}$ such that $O^T O = I$. Then show this last property for the Fourier transform.

Problem 2 Show that

$$\lim_{r\to 0^+} \fint_{\partial B_r(x)} u(y)\, d\mathcal{H}^{n-1}(y) = u(x) \quad \text{for all } u \in C^0(\mathbb{R}^n),\ x \in \mathbb{R}^n.$$

Problem 3 For $n \geq 2$ let $\Phi(x)$ be the fundamental solution, namely

$$\Phi(x) = \begin{cases} c \log|x| & \text{if } n = 2 \\ c|x|^{2-n} & \text{if } n \geq 3 \end{cases}$$

Show in both cases, $n = 2$ and $n \geq 3$, that

$$\Delta\Phi(x) = 0 \quad \text{for any } x \neq 0$$

Problem 4 For $n = 2$ let $\Phi(x)$ be the fundamental solution, namely

$$\Phi(x) = c \log|x|.$$

Show that $\Phi \in L^p_{loc}(\mathbb{R}^2)$ for any $p \in [1, \infty)$, that is: for any bounded, open set $\Omega \subset \mathbb{R}^2$

$$\|\Phi\|_{L^p(\Omega)} \equiv \left(\int_\Omega |\Phi(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

Hint: It suffices to consider $\Omega = B(0, R)$ a ball with radius R.