University of Pittsburgh
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Partial Differential Equations 1 – Spring 2019
Exercise Sheet 3 — Due Date: February 18

Work in groups, write in LaTeX!

Problem 9 Using convolutions similar to the proof of Theorem 2.15, show the fundamental theorem of calculus of variations:

Let $\Omega \subset \mathbb{R}^n$ be open and assume $f \in L^1_{loc}(\Omega)$, i.e. for any $\Omega' \subset\subset \Omega$ we have

$$\int_{\Omega'} |f| < \infty.$$

(i) if

$$\int_\Omega f(x)\,\varphi(x) \geq 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega) \text{ that are nonnegative, } \varphi \geq 0,$$

then

$$f \geq 0 \quad \text{almost everywhere in } \Omega.$$

(ii) if

$$\int_\Omega f(x)\,\varphi(x) = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega) \text{ that are nonnegative, } \varphi \geq 0,$$

then

$$f \equiv 0 \quad \text{almost everywhere in } \Omega.$$

Problem 10 In Corollary 2.17 of the lecture we showed one version of the Liouville theorem.

If $u \in C^2(\mathbb{R}^n)$ is harmonic, i.e. $\Delta u = 0$ in $\mathbb{R}^n$ and u is bounded, i.e. if

$$\sup_{\mathbb{R}^n} u < \infty \quad \text{and} \quad \inf_{\mathbb{R}^n} u > -\infty \tag{1}$$

then u is a constant function.

Using Harnack's inequality, Theorem 2.18, show a refined statement, the same statement holds if instead of (1) we only assume u is bounded from below, i.e. if

$$\inf_{\mathbb{R}^n} u > -\infty. \tag{2}$$
