Problem 11  Let $u \in C^1(\overline{\Omega})$. Denote by $\nu$ the outwards facing unit normal of $\Omega$. Assume that $u = 0$ on $\partial \Omega$. Show that then

$$Du = 0 \quad \text{on } \partial \Omega \quad \Leftrightarrow \quad \partial_\nu u = 0 \quad \text{on } \partial \Omega.$$  

Problem 12  Extend Theorem 2.2. and Theorem 2.24 to the following equation

$$\begin{aligned} \Delta^2 u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial \Omega \\ \partial_\nu u &= 0 \quad \text{on } \partial \Omega 
\end{aligned} \quad (1)$$

for $f \in C^0(\overline{\Omega}), \Omega \subset \subset \mathbb{R}^n$ with smooth boundary.

Here $\Delta^2$ is the bi-Laplace operator,

$$\Delta^2 u = \Delta(\Delta u) = \sum_{i=1}^n \sum_{j=1}^n \partial_{x_i x_i} \partial_{x_j x_j} u.$$ 

More precisely,

(i) Find an energy $E(u)$ and a set of permissible functions $X \subset C^4(\Omega) \cap C^1(\overline{\Omega})$ such that a minimizer of $E$ is a solution to the PDE $(1)$ and so that a solution to the PDE is a minimizer

(ii) Show uniqueness, i.e. if there are two solutions $u, v \in C^4(\Omega) \cap C^1(\overline{\Omega})$ of the PDE $(1)$ then $u \equiv v$.

Problem 13  Show that for the equation

$$\begin{aligned} \Delta^2 u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial \Omega 
\end{aligned} \quad (2)$$

no uniqueness result holds and explain why the proof of Theorem 2.24 fails.

More precisely show

(i) if $u$ solves $(2)$ then for $v$ a solution to

$$\begin{aligned} \Delta v &= 1 \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \partial \Omega 
\end{aligned}$$

the for any $\lambda \in \mathbb{R}$ function

$$w := u + \lambda v$$

still solves $(2)$.

(ii) Where does the proof of Theorem 2.24 fail?