Problem 14  Using the weak maximum principle, Theorem 2.2, show Corollary 2.4:

Namely, let $L$ be as above a non-divergence form linear elliptic operator, $\Omega \subset \subset \mathbb{R}^n$ with smooth boundary, $c \leq 0$. Fix $f \in C^0(\Omega)$, $g \in C^0(\partial \Omega)$. Show there exists at most one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of the Dirichlet boundary problem

$$
\begin{cases}
Lu = f & \text{in } \Omega \\
u = g & \text{on } \partial \Omega
\end{cases}
$$

Problem 15  Using the weak maximum principle, Theorem 2.2, show Corollary 2.5:

Let $L$ be a linear elliptic differential operator (non-divergence form), and assume that $c \leq 0$ in $\Omega \subset \subset \mathbb{R}^n$. Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy $-Lu \leq -Lv$ in $\Omega$. Show that then $u \leq v$ on $\partial \Omega$ implies $u \leq v$ in $\Omega$.

Problem 16  Using the weak maximum principle, Theorem 2.2, show Corollary 2.6:

Let $L$ be a linear elliptic differential operator (non-divergence form), and assume that $c \leq 0$ in $\Omega \subset \subset \mathbb{R}^n$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$
\begin{cases}
-Lu = f & \text{in } \Omega \\
u = g & \text{in } \Omega
\end{cases}
$$

where $f \in C^0(\overline{\Omega})$ and $g \in C^0(\partial \Omega)$.

Then for some constant $C = C(\Omega, b, \Lambda)$ we have

$$\sup_{\Omega} |u| \leq \sup_{\partial \Omega} |g| + C \sup_{\Omega} |f|.$$  

**Hint:**

(i) W.l.o.g. we can assume that $\Omega \subset \{x \in \mathbb{R}^n : 0 \leq x_1 \leq A\}$

(ii) By choosing $B$ (possibly depending on $L$ and $A$) appropriately construct $v$ such that

$$-Lv \geq 1 \text{ in } \Omega \subset \{x \in \mathbb{R}^n : 0 \leq x_1 \leq A\}$$

Choose $v$ of the form $v(x) = e^{BA} - e^{Bx_1}$.

(iii) Consider

$$w := u - \sup_{\partial \Omega} |g| - v \sup_{\Omega} |f|.$$  

and estimate $Lw$.

(iv) use the maximum principle on $w$.  