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Partial Differential Equations 1 – Spring 2019
Exercise Sheet 5 — Due Date: March 18

Work in groups, write in LaTeX!

Problem 14 Using the weak maximum principle, Theorem 2.2, show Corollary 2.4:

Namely, let L be as above a non-divergence form linear elliptic operator, $\Omega \subset\subset \mathbb{R}^n$ with smooth boundary, $c \leq 0$. Fix $f \in C^0(\Omega)$, $g \in C^0(\partial\Omega)$. Show there exists at most one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of the Dirichlet boundary problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Problem 15 Using the weak maximum principle, Theorem 2.2, show Corollary 2.5:

Let L be a linear elliptic differential operator (non-divergence form), and assume that $c \leq 0$ in $\Omega \subset\subset \mathbb{R}^n$. Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy $-Lu \leq -Lv$ in Ω. Show that then $u \leq v$ on $\partial\Omega$ implies $u \leq v$ in Ω.

Problem 16 Using the weak maximum principle, Theorem 2.2, show Corollary 2.6:

Let L be a linear elliptic differential operator (non-divergence form), and assume that $c \leq 0$ in $\Omega \subset\subset \mathbb{R}^n$.

Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\begin{cases} -Lu = f & \text{in } \Omega \\ u = g & \text{in } \Omega \end{cases}$$

where $f \in C^0(\overline{\Omega})$ and $g \in C^0(\partial\Omega)$.

Then for some constant $C = C(\Omega, b, \Lambda)$ we have

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |g| + C \sup_{\Omega} |f|.$$

Hint:

(i) W.l.o.g. we can assume that $\Omega \subset \{x \in \mathbb{R}^n : 0 \leq x_1 \leq A\}$

(ii) By choosing B (possibly depending on L and A) appropriately construct v such that

$$-Lv \geq 1 \quad \text{in } \Omega \subset \{x \in \mathbb{R}^n : 0 \leq x_1 \leq A\}$$

Choose v of the form $v(x) = e^{BA} - e^{Bx_1}$.

(iii) Consider

$$w := u - \sup_{\partial\Omega} |g| - v \sup_{\Omega} |f|.$$

and estimate Lw.

(iv) use the maximum principle on w.
