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Partial Differential Equations 1 – Spring 2019
Exercise Sheet 6 — Due Date: March 25

Work in groups, write in LaTeX!

Problem 17

For $s > 0$ let

$$f(x) := |x|^{-s}.$$

(i) Show $f \in L^p_{loc}(\mathbb{R}^n)$ for any $1 \leq p < \frac{n}{s}$.

(ii) compute for $x \neq 0$

$$\partial_i f(x) = -s\,|x|^{-s-2} x^i \tag{1}$$

(iii) show that $\partial_i f \in L^q_{loc}(\mathbb{R}^n)$ for any $1 \leq q < \frac{n}{s+1}$.

(iv) show that (1) holds also in the distributional sense, i.e. that if $n \geq 2$ and $0 < s < n-1$ then for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f(x)\,\partial_i \varphi(x)\,dx = \int_{\mathbb{R}^n} s\,|x|^{-s-2} x^i \varphi(x)\,dx.$$

Hint: Use a cutoff argument similar to the proof of Representation by Newton potential, Theorem 2.3: consider the integrals on $\mathbb{R}^n \backslash B(0,\varepsilon)$ where f is smooth, and show that the integrals on $B(0,\varepsilon)$, $\partial B(0,\varepsilon)$ vanish as $\varepsilon \to 0$.

(v) conclude that $f \in W^{1,q}_{loc}(\mathbb{R}^n)$ for any $1 \leq q < \frac{n}{s+1}$.

Problem 18

Let

$$f(x) := \log |x|.$$

(i) In any dimension n show that $f \in L^p_{loc}(\mathbb{R}^n)$ for any $1 \leq p < \infty$.

(ii) For dimension $n \geq 2$ show that and $f \in W^{1,n}_{loc}(\mathbb{R}^n)$.

Problem 19

Let

$$f(x) := \log \log \frac{2}{|x|} \quad \text{in } B(0,1)$$

(i) Show that for $n \geq 1$, $f \in W^{1,n}(B(0,1))$.

(ii) For $n = 2$, Show first that formally

$$\Delta f = |Df|^2$$

(iii) Show that this equation holds in distributional sense, i.e. for every $\varphi \in C_c^\infty(B(0,1))$,

$$-\int_{B(0,1)} Df \cdot D\varphi = -\int_{B(0,1)} |Df|^2 \varphi.$$
