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Partial Differential Equations 1 – Spring 2019
Exercise Sheet 6 — Due Date: April 8

Work in groups, write in LaTeX!

Problem 19

Recall the Hölder inequality:
For $p, q, r \in [1, \infty]$ with $\frac{1}{r} = \frac{1}{q} + \frac{1}{p}$. If $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ then $fg \in L^r(\Omega)$, and

$$\|fg\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)} \, \|g\|_{L^q(\Omega)}.$$

Show the following

(i) For $N \geq 2$ and $p_1, \ldots, p_N \in [1, \infty]$ assume that $r \in [1, \infty]$ such that $\frac{1}{r} = \frac{1}{p_1} + \ldots \frac{1}{p_N}$. Show that if $f_i \in L^{p_i}(\Omega)$ for $1 \leq i \leq N$ then $f_1 \cdot f_2 \cdot \ldots f_N \in L^r(\Omega)$ and we have

$$\|f_1 \cdot f_2 \cdot \ldots f_N\|_{L^r(\Omega)} \leq \|f_1\|_{L^{p_1}(\Omega)} \, \|f_2\|_{L^{p_2}(\Omega)} \, \ldots \|f_N\|_{L^{p_N}(\Omega)}$$

(ii) If the measure of Ω is finite, $|\Omega| < \infty$, then any $f \in L^q(\Omega)$ for some $q \in (1, \infty]$ also belongs to $L^p(\Omega)$ for all $p \in [1, q]$, and we have

$$\|f\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{p} - \frac{1}{q}} \, \|f\|_{L^q(\Omega)}.$$

Hint: $g := \chi_\Omega$

Problem 20

Let $f \in L^p(\mathbb{R}^n)$. For $r > 0$ set $g(x) := f(rx)$.

(i) Show that

$$\|g\|_{L^p(\mathbb{R}^n)} = r^{\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)}$$

(ii) Show that

$$\|\partial^\gamma g\|_{L^p(\mathbb{R}^n)} = r^{|\gamma| + \frac{n}{p}} \|\partial^\gamma f\|_{L^p(\mathbb{R}^n)}$$

Problem 21

Let $\Omega \subset \mathbb{R}^n$ be open and nonemtpy. Let $1 \leq p, q, r \leq \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Show that if $f \in W^{k,p}(\Omega)$ and $g \in W^{k,q}(\Omega)$ then $fg \in W^{k,r}(\Omega)$, and that there exists a constant $C = C(k)$ such that

$$\|fg\|_{W^{k,r}(\Omega)} \leq C \, \|f\|_{W^{k,p}(\Omega)} \, \|g\|_{W^{k,q}(\Omega)}$$

Hint: Use smooth approximations to show that $D(fg) = Dfg + fDg$ in distributional sense.

Problem 22

(i) Use the fundamental theorem of calculus (and approximation) to show that if $p \in [1, \infty)$ and $f \in W^{1,p}((a, b))$ then for almost any $x, y \in (a, b)$,

$$|f(x) - f(y)| \leq |x - y|^{1 - \frac{1}{p}} \left(\int_x^y |f'(z)|^p \, dz \right)^{\frac{1}{p}}.$$

Hint: First show the case $p = 1$. For $p > 1$ use Hölder's inequality.

(ii) Conclude that in one dimension any $f \in W^{1,1}((a, b))$ is continuous

Hint: Absolute continuity of the integral.

(iii) Conclude that in one dimension any $f \in W^{1,p}((a, b))$ is Hölder continuous $f \in C^{1 - \frac{1}{p}}((a, b))$.

(iv) Show that these statements are not true in higher dimension, e.g. find a non-continuous function $f \in W^{1,2}(B(0, 1))$ on the ball $B(0, 1) \subset \mathbb{R}^2$.
