Partial Differential Equations 1 – Spring 2019
Exercise Sheet 6 — Due Date: April 8

Problem 19  Recall the Hölder inequality:
For \( p, q, r \in [1, \infty] \) with \( \frac{1}{r} = \frac{1}{q} + \frac{1}{p} \). If \( f \in L^p(\Omega) \), \( g \in L^q(\Omega) \) then \( fg \in L^r(\Omega) \), and
\[
\|fg\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.
\]

Show the following

(i) For \( N \geq 2 \) and \( p_1, \ldots, p_N \in [1, \infty) \) assume that \( r \in [1, \infty) \) such that \( \frac{1}{r} = \frac{1}{p_1} + \cdots + \frac{1}{p_N} \). Show that if \( f_i \in L^{p_i}(\Omega) \) for \( 1 \leq i \leq N \) then \( f_1 \cdot f_2 \cdots f_N \in L^r(\Omega) \) and we have
\[
\|f_1 \cdot f_2 \cdots f_N\|_{L^r(\Omega)} \leq \|f_1\|_{L^{p_1}(\Omega)} \|f_2\|_{L^{p_2}(\Omega)} \cdots \|f_N\|_{L^{p_N}(\Omega)}
\]

(ii) If the measure of \( \Omega \) is finite, \( |\Omega| < \infty \), then any \( f \in L^q(\Omega) \) for some \( q \in (1, \infty] \) also belongs to \( L^p(\Omega) \) for all \( p \in [1, q] \), and we have
\[
\|f\|_{L^p(\Omega)} \leq |\Omega|^{-\frac{1}{q}} \|f\|_{L^q(\Omega)}.
\]

Hint: \( g := \chi_\Omega \)

Problem 20  Let \( f \in L^p(\mathbb{R}^n) \). For \( r > 0 \) set \( g(x) := f(rx) \).

(i) Show that
\[
\|g\|_{L^p(\mathbb{R}^n)} = r^{-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)}
\]

(ii) Show that
\[
\|\partial^\gamma g\|_{L^p(\mathbb{R}^n)} = r^{\frac{|\gamma|}{p}} \|\partial^\gamma f\|_{L^p(\mathbb{R}^n)}
\]

Problem 21  Let \( \Omega \subset \mathbb{R}^n \) be open and nonempty. Let \( 1 \leq p, q, r \leq \infty \) such that \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \). Show that if \( f \in W^{k,p}(\Omega) \) and \( g \in W^{k,q}(\Omega) \) then \( fg \in W^{k,r}(\Omega) \), and that there exists a constant \( C = C(k) \) such that
\[
\|fg\|_{W^{k,r}(\Omega)} \leq C \|f\|_{W^{k,p}(\Omega)} \|g\|_{W^{k,q}(\Omega)}
\]

Hint: Use smooth approximations to show that \( D(fg) = Df \cdot g + fDg \) in distributional sense.

Problem 22

(i) Use the fundamental theorem of calculus (and approximation) to show that if \( p \in [1, \infty) \) and \( f \in W^{1,p}((a, b)) \) then for almost any \( x, y \in (a, b) \),
\[
|f(x) - f(y)| \leq |x - y|^{1 - \frac{1}{p}} \left( \int_x^y |f'(z)|^p \, dz \right)^{\frac{1}{p}}.
\]

Hint: First show the case \( p = 1 \). For \( p > 1 \) use Hölder’s inequality.

(ii) Conclude that in one dimension any \( f \in W^{1,1}((a, b)) \) is continuous

Hint: Absolute continuity of the integral.

(iii) Conclude that in one dimension any \( f \in W^{1,p}((a, b)) \) is Hölder continuous \( f \in C^{1 - \frac{1}{p}}((a, b)) \).

(iv) Show that these statements are not true in higher dimension, e.g. find a non-continuous function \( f \in W^{1,2}(B(0, 1)) \) on the ball \( B(0, 1) \subset \mathbb{R}^2 \).