PARTIAL DIFFERENTIAL EQUATIONS I
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ARMIN SCHIKORRA

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In Analysis
there are no theorems
only proofs
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Basic formulas and concepts we’ll use a lot

Integration by parts, Green’s Theorem, Stokes theorem. If $\Omega \subset \mathbb{R}^n$ is a (nice) open bounded set with outwards facing unit normal $\nu = (\nu^1, \ldots, \nu^n) : \partial \Omega \to S^{n-1}$ ($S^{n-1}$ are simply the vectors $v \in \mathbb{R}^n$ with $|v| = 1$, i.e. the unit sphere) and $f, g$ are (nice) functions then we have for any $\alpha \in \{1, \ldots, n\}$

\begin{equation}
\int_{\Omega} f \partial_\alpha g = \int_{\partial \Omega} fg \nu^\alpha - \int_{\Omega} \partial_\alpha fg \tag{0.1}
\end{equation}

Observe that if $n = 1$ and $\Omega = (a, b)$ then $\nu(a) = -1$ and $\nu(b) = +1$, and then we have the usual one-dimension integration by parts formula

\begin{equation}
\int_{(a, b)} f \partial_\alpha g = f(b)g(b) - f(a)g(a) - \int_{(a, b)} \partial_\alpha fg \tag{0.2}
\end{equation}

- works also for $\mathbb{R}^n$ or unbounded set $\Omega$ – as long as
  \[
  \lim_{|x| \to \infty} f(x) = \lim_{|x| \to \infty} g(x) = 0
  \]
- Green’s formula (divergence theorem) is normally written for vector fields $G = (G^1, G^2, \ldots, G^n) : \Omega \to \mathbb{R}^n$,

\[
\int_{\Omega} \text{div}(G) = \sum_{\alpha=1}^n \int_{\Omega} \partial_\alpha G = \sum_{\alpha=1}^n \int_{\partial \Omega} \nu^\alpha G^\alpha - \sum_{\alpha=1}^n \int_{\Omega} (\partial_\alpha 1) G^\alpha = \int_{\partial \Omega} G \cdot \nu
\]

Exercise 0.1. Use Green’s formula

\[
\int_{\Omega} \text{div}(G) = \int_{\partial \Omega} G \cdot \nu
\]

to prove the integration by parts formula (0.1).

Exercise 0.2. Use (0.2) to show (0.1)

(You can use pictures and a simple set $\Omega$ – I care about the idea, not the most general case)

Polar coordinates. Let $f : B(0, R) \to \mathbb{R}^n$ (nice) then

\begin{equation}
\int_{B(0,R)} f(x)dx = \int_{0}^{R} \int_{\partial B(0,\rho)} f(\theta) \, d\theta \, d\rho \tag{0.3}
\end{equation}

This is actually Fubini’s theorem (or Cavalieri’s principle), and really isn’t that related to polar coordinates (well, there is a sphere...) We call it polar coordinates anyways. By a substitution we can write this as

\begin{equation}
\int_{B(0,R)} f(x)dx = \int_{0}^{R} \rho^{n-1} \int_{\partial B(0,\rho)} f(\rho \theta) \, d\theta \, d\rho \tag{0.4}
\end{equation}

Exercise 0.3. Prove (0.4) using (0.3)
Figure 0.1. $M$ is a (sub-)manifold iff around any point $x$ there exist a chart $\Phi$.

A special case is the case when $f : B(0, R) \to \mathbb{R}$ is radial. $f$ is called radial if $f(x) = f(Qx)$ for all rotation matrices $Q \in O(n)$ (i.e. $Q \in \mathbb{R}^{n \times n}$, $Q^tQ = I$).

Exercise 0.4. Show that if $f$ is radial then there exists $g : [0, \infty) \to \mathbb{R}$ such that

$$f(x) = g(|x|).$$

Thus, one often writes “$f$ radial” as “$f(x) = f(|x|)$” (this is an idiotic notation, but we’ll still use it).

If $f$ is radial then

$$\int_{B(0,R)} f(x)dx = \int_0^R \rho^{n-1} f(\rho)d\rho |\partial B(0,1)|,$$

where $|\partial B(0,1)|$ denotes the area of $\partial B(0,1)$, i.e. $|\partial B(0,1)| = \mathcal{H}^{n-1}(\partial B(0,1))$.

Exercise 0.5. Show (0.5) using (0.4) or (0.3)

Regular Sets

We are often going to talk about open sets $\Omega$ with smooth boundary, $\partial \Omega \in C^k$ or $\partial \Omega \in C^\infty$ or similar. When we say $\Omega \subset \mathbb{R}^n$ with $\partial \Omega \in C^k$ we mean that $M := \partial \Omega$ is a $C^k$-manifold. That is, for each $x \in \partial \Omega$ there exists a small ball $B(x, r) \subset \mathbb{R}^n$ and an associated chart $\Phi : B(x, r) \to \mathbb{R}^n$, which must be a $C^k$-diffeomorphism ($\Phi$ is invertible and $\Phi, \Phi^{-1}$ are $C^k$-maps in their respective domain) and

$$\Phi(B(x, r) \cap \partial \Omega) \subset \mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$$

and

$$\Phi(B(x, r) \setminus \partial \Omega) \subset \mathbb{R}^n_- = \{(x', x_n) \in \mathbb{R}^n : x_n \leq 0\}$$

and $\Phi(x) = 0$. Cf. Figure 0.1.
Part 1. PDE 1

1. Introduction and Some Basic Notation

When studying Partial Differential Equations (PDEs) the first question that arises is: what are partial differential equations.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $u : \Omega \to \mathbb{R}$ be differentiable. The partial derivatives $\partial_1$ is the directional derivative

$$
\partial_1 u(x) \equiv \partial_{x_1} u(x) = \frac{d}{dx_1} u(x) = \left. \frac{d}{dt} \right|_{t=0} u(x + t e_1),
$$

where $e_1 = (1, 0, \ldots, 0)$ is the first unit vector. The partial derivatives $\partial_2, \ldots \partial_n$ are defined likewise.

Sometimes it is convenient to use multiindices: an n-multiindex $\gamma$ is a vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ where $\gamma_1, \ldots, \gamma_n \in \{0, 1, 2, \ldots\}$. The order of a multiindex is $|\gamma|$ defined as

$$
|\gamma| = \sum_{i=1}^n \gamma_i.
$$

For a suitable often differentiable function $u : \Omega \to \mathbb{R}$ and a multiindex $\gamma$ we denote with $\partial^\gamma u$ the partial derivatives

$$
\partial^\gamma u(x) = \partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2} \cdots \partial_{x_n}^{\gamma_n} u(x).
$$

For example, for $\gamma = (1, 0, 0, \ldots, 0)$ we have

$$
\partial^\gamma u(x) = \partial_{x_1} u,
$$

i.e. a partial derivative of first order; and for $\gamma = (1, 2, 0, \ldots, 0)$ we have

$$
\partial^\gamma u = \partial_{122} u = \partial_1 \partial_2 \partial_2 u,
$$

i.e. a partial derivative of 3rd order.

The collection of all partial derivatives of $k$-th order of $u$ is usually denoted by $D^k u(x) \in \mathbb{R}^{n^k}$ or (the "gradient") $\nabla^k u$. Usually these are written in matrix form, namely

$$
Du(x) = (\partial_1 u(x), \partial_2 u(x), \partial_3 u(x), \ldots, \partial_n u(x))
$$

and

$$
D^2 u(x) = (\partial_{ij} u)_{i,j=1,\ldots,n} \equiv \begin{pmatrix}
\partial_{11} u(x) & \partial_{12} u(x) & \partial_{13} u(x) & \cdots & \partial_{1n} u(x) \\
\partial_{21} u(x) & \partial_{22} u(x) & \partial_{23} u(x) & \cdots & \partial_{2n} u(x) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\partial_{n1} u(x) & \partial_{n2} u(x) & \partial_{n3} u(x) & \cdots & \partial_{nn} u(x)
\end{pmatrix}
$$

Definition 1.1. Let $\Omega \subset \mathbb{R}^n$ an open set and $k \in \mathbb{N} \cup \{0\}$. A partial differential equation (PDE) of $k$-th order is an expression of the form

$$
F(D^k u(x), D^{k-1} u(x), D^{k-2} u(x), \ldots, Du(x), u(x), x) = 0 \quad x \in \Omega,
$$
where \( u : \Omega \rightarrow \mathbb{R} \) is the unknown (also the “solution” to the PDE) and \( F \) is a given structure (i.e. map)

\[
F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \ldots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}
\]

- (1.1) is called \textit{linear} if \( F \) is linear in \( u \): meaning if we can find for every \( n \)-multiindex \( \gamma \) with \( |\gamma| \leq k \) a function \( a_\gamma : \Omega \rightarrow \mathbb{R} \) (independent of \( u \)) such that

\[
F(D^k u(x), D^{k-1} u(x), D^{k-2} u(x), \ldots, Du(x), u(x), x) = \sum_{|\gamma| \leq k} a_\gamma(x) \partial^\gamma u(x)
\]

- (1.1) is called \textit{semilinear} if \( F \) is linear with respect to the highest order \( k \), namely if

\[
F(D^k u(x), D^{k-1} u(x), D^{k-2} u(x), \ldots, Du(x), u(x), x) = \sum_{|\gamma|=k} a_\gamma(x) \partial^\gamma u(x) + G(D^{k-1} u(x), D^{k-2} u(x), \ldots, Du(x), u(x), x)
\]

- (1.1) is called \textit{quasilinear} if \( F \) is linear with respect to the highest order \( k \) but the coefficient for the highest order may depend on the lower order derivatives of \( u \).

Namely if we have a representation of the form

\[
F(D^k u(x), D^{k-1} u(x), D^{k-2} u(x), \ldots, Du(x), u(x), x) = \sum_{|\gamma|=k} a_\gamma(D^{k-1} u(x), D^{k-2} u(x), \ldots, Du(x), u(x), x) \partial^\gamma u(x)
\]

- If all the above do not apply then we call \( F \) \textit{fully nonlinear}.

We have a system of partial differential equations of order \( k \), if \( u : \Omega \rightarrow \mathbb{R}^m \) is a vector and/or the structure function \( F \) is also a vector

\[
F : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \ldots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times \Omega \rightarrow \mathbb{R}^\ell
\]

for \( m, \ell \geq 1 \).

The goal in PDE is usually (besides modeling what PDE describes what situation) to solve PDEs, possibly subject to side-condition (such as prescribed boundary data on \( \partial \Omega \)).

This is rarely possible explicitly (even in the linear case) – which is a huge contrast to ODE. E.g.

\[
u''(x) = 2u(x) \quad x \in \mathbb{R},
\]

then we know that \( u(x) = e^{\sqrt{2}x} A \), and we can compute \( A \) by prescribing some initial value at \( x = 0 \) or similar.

Now if we try that in two dimensions, and consider

\[
\Delta u(x) \equiv \partial_{11} u(x) + \partial_{22} u(x) = 2u(x) \quad x \in B(0,1) \subset \mathbb{R}^2,
\]

it is really difficult to see what \( u \) is (observe that also the amount of initial data – e.g. values at \( \partial B(0,1) \) is not one, but infinitely many!

So in general the best one can hope for is address the following main questions for PDEs are
• Is there a solution to a problem (and if so: in what sense? – we will learn the
distributional/weak sense and strong sense)
• Are solutions unique (under reasonable assumptions like initial data, boundary
data?)?
• What are properties of the solutions (e.g. does the solution depend continuously
on the data of the problem)?

It is important to accept that there are PDEs without (classical) solutions and there is no
general theory of PDEs. There is theory for several types of PDEs.

Example 1.2 (Some basic linear equations).

• Laplace equation
  \[ \Delta u := \sum_{i=1}^{n} u_{x_i x_i} = 0. \]

• Eigenvalue equation (aka Helmholtz equation)
  \[ \Delta u = \lambda u. \]

• Transport equation
  \[ \partial_t u - \sum_{i=1}^{n} b^i u_{x_i} = 0 \]

• Heat equation
  \[ \partial_t u - \Delta u = 0 \]

• Schrödinger equation
  \[ i\partial_t u + \Delta u = 0 \]

• Wave equation
  \[ u_{tt} - \Delta u = 0 \]

Second order linear equations are classified into elliptic, parabolic, hyperbolic PDE. Roughly
this is understood as follows. Assume that \( u \) depends on \( x \) and \( t \), then

• **elliptic** means the equation is of the form
  \[ u_{xx} + u_{tt} = G(x, y, u, u_t, u_x) \]

• **parabolic** means
  \[ u_{xx} = G(x, y, u, u_t, u_x) \]

• **Hyperbolic**
  \[ u_{xx} - u_{tt} = G(x, y, u, u_t, u_x) \]
  or
  \[ u_{x,t} = G(x, y, u, u_t, u_x) \]
Let us have a generic second order linear equation

\[ Au_{xx} + Bu_{xy} + Cu_{yy} + Du_{x} + Eu_{y} + Fu = g \]

(for now let us assume that \( A, B, \ldots \) are constant.) We can write the second-order part as

\[ Au_{xx} + Bu_{xy} + Cu_{yy} = \left( \begin{array}{cc} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{array} \right) : \left( \begin{array}{cc} \partial_{xx}u & \partial_{yx}u \\ \partial_{xy}u & \partial_{yy}u \end{array} \right), \]

where : denotes the matrix scalar product (sometimes: Hilbert-Schmidt product). If \( AC - \frac{1}{4}B^2 > 0 \) the determinant of the coefficient matrix is positive, i.e. either the matrix has two positive eigenvalues \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \) or two negative eigenvalues \( \lambda_1 < 0 \) and \( \lambda_2 < 0 \), and there exists orthogonal matrices \( P \in SO(2) \) such that

\[ P^T \left( \begin{array}{cc} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{array} \right) P = \text{diag}(\lambda_1, \lambda_2). \]

Then we have

\[ \left( \begin{array}{cc} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{array} \right) : \left( \begin{array}{cc} \partial_{xx}u & \partial_{yx}u \\ \partial_{xy}u & \partial_{yy}u \end{array} \right) = \text{diag}(\lambda_1, \lambda_2) : P^T D^2 u P. \]

Now consider \( \tilde{u}(x, y) = u(P(x, y)^t) \), then by the chain rule,

\[ D^2 \tilde{u}(x, y) = P^t (D^2 u)(P(x, y)) P, \]

so that if we set \( (\tilde{x}, \tilde{y})^t := P(x, y)^t \) we have

\[ \lambda_1 u_{\tilde{x} \tilde{x}} + \lambda_2 u_{\tilde{y} \tilde{y}} = G(u, u_x, u_y), \]

that is if \( AC - \frac{1}{4}B^2 > 0 \) we can transform our equation into an elliptic equation.

Similarly, if \( AC - \frac{1}{4}B^2 = 0 \), at least one eigenvalue of the matrix in question is negative, one is positive, so we can transform the equation into

\[ \lambda_1 u_{\tilde{x} \tilde{x}} - \lambda_2 u_{\tilde{y} \tilde{y}} = G(u, u_x, u_y), \]

i.e. a hyperbolic equation.

And if \( AC - \frac{1}{4}B^2 < 0 \), one of the eigenvalues is zero, so that we have the structure

\[ \lambda_1 u_{\tilde{x} \tilde{x}} = G(u, u_x, u_y), \]

i.e. we are parabolic.

Whether one is elliptic, parabolic, hyperbolic is not purely an algebraic question – it often determines the ways we can understand properties of the equation in question. Often we think of elliptic equation as equilibrium or stationary equations, parabolic equations as a flow of an energy, and hyperbolic of a wave-like equation – but this is not really always the case, since the Schrödinger equation is parabolic in the previous sense, but it is wave-like. It generally holds: every type of equation warrants its own methods.
One can extend this theory, of course, to higher dimensions. If
\[ \sum_{i,j=1}^{n} A_{ij} \partial_{x_i,x_j} u + \sum_{i=1}^{n} B_i \partial_{x_i} u + Cu = D, \]
then we may assume that \( A \) is symmetric (any antisymmetric part vanishes because \( (\partial_{x_i,x_j} u)_{ij} \) is symmetric) – and thus we can discuss its eigenvalues.

- The equation is **elliptic** if all eigenvalues are nonzero and have the same sign.
- The equation is **parabolic** if exactly one eigenvalue is zero, all others are nonzero and have the same sign.
- The equation is **hyperbolic** if no eigenvalue is zero, and \( n - 1 \) eigenvalues have the same sign, and the other one has the opposite sign.

Of course there are more cases (and they may be very challenging to treat). In principle: elliptic means the second order derivatives “move in the same direction”, parabolic means “all but one direction move in the same direction and the remaining direction is of first order only”, and hyperbolic “the second derivatives compete with each other”.

Of course, since in general \( A \) and \( B \) are nonconstant, the type of equation may change and depend on the point \( x \) (e.g. \( tu_{xx} + u_t = 0 \)).

**Example 1.3** (Some basic nonlinear equations).

- **Eikonal equation**
  \[ |Du| = 1 \]

- **\( p \)-Laplace equation**
  \[ \text{div}(|Du|^{p-2}Du) \equiv \sum_{i=1}^{n} \partial_i(|Du|^{p-2} \partial_i u) = 0 \]

- **Minimal surface equation**
  \[ \text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0. \]

- **Monge-Ampere**
  \[ \det(D^2u) = 0. \]

- **Hamilton-Jacobi**
  \[ \partial_t u + H(Du, x) = 0 \]

The notion of what constitutes a solution is important, as a too weak notion allows for too many solutions, and a too strong notion of solution may allow for no solutions at all. We illustrate this for the Eikonal equation:

**Exercise 1.4.** We consider different notions of solutions for the Eikonal equation:
(1) Consider

\[
\begin{cases}
    |u'(x)| = 1 & x \in (-1, 1) \\
    u(-1) = u(1) = 0
\end{cases}
\]

Show that there is no \( u \in C^0([-1, 1]) \cap C^1((-1, 1)) \) such that (1.2) is satisfied.

(2) Consider instead

\[
\begin{cases}
    |u'(x)| = 1 & \text{all but finitely many } x \in (-1, 1) \\
    u(-1) = u(1) = 0
\end{cases}
\]

Show that there are infinitely many solutions \( u \in C^0([-1, 1]) \) that are differentiable in all but finitely many points in \((-1, 1)\) such that (1.3) is satisfied.

(3) Show that there is a sequence \( u_k \in C^0([-1, 1]) \) that are differentiable in all but finitely many points in \((-1, 1)\), such that

\[
\sup_{x \in [-1,1]} |u_k(x) - 0| \xrightarrow{k \to \infty} 0.
\]

(4) Consider instead

\[
\begin{cases}
    |u'(x)| = 1 & \text{in all but one } x \in (-1, 1) \\
    u(-1) = u(1) = 0
\end{cases}
\]

Show that there are still two solutions \( u \in C^0([-1, 1]) \) that are differentiable in all but at most one points in \((-1, 1)\) such that (1.4) is satisfied.

In this course we will focus on the linear theory (the nonlinear theory is almost always based on ideas on the linear theory). Almost each of the linear and nonlinear equations warrants its own course, so we will focus on the basics (namely: mainly elliptic equations).

2. LAPLACE EQUATION

2.1. Sort of a physical motivation. The following is often used to motivate Laplace’s equation

Assume \( \Omega \) is an open set in \( \mathbb{R}^n \) (usually \( \mathbb{R}^3 \)), and \( u \) describes the density of a fluid or heat that is at an equilibrium state, i.e. no fluid is moving in or out, or not heat is exchanged any more. This means that if we look at any subset \( \Omega' \subset \Omega \) nothing flows out or in that would change the density, that is

\[
\int_{\partial \Omega'} \nabla u \cdot \nu = 0.
\]

By Green’s divergence theorem this is equivalent to saying

\[
\int_{\Omega'} \text{div} \, (\nabla u) = 0.
\]
Figure 2.1. Solve $\Delta u = 0$ on the annulus (inner radius $r = 2$ and outer radius $R = 4$) with boundary condition $g(\theta) = 0$ if $|\theta| = 2$ and $g(\theta) = 4\sin(5\sigma)$ for $|\theta| = 4$ where $\sigma \in [0, 2\pi)$ is the angle such that $(\sin(\sigma), \cos(\sigma)) = \theta/|\theta|$. Source: Fourthirtytwo/Wikipedia CC-SA 3

Since this happens for all $\Omega'$ we obtain that

$$\text{div}(\nabla u) = 0$$

So we call $\text{div}(\nabla u) =: \Delta u$ and observe that $\Delta u = \sum_{i=1}^{n} \partial_{x_i x_i} u = \text{tr}(D^2 u)$.

Often one thinks of Laplace equation $\Delta u = 0$ in $\Omega$ as a heat distribution. Take $\Omega$ a solid, apply some heat at its boundary: at $\theta \in \partial \Omega$ we apply $g(\theta)$ heat. Wait until the heat had time to fully propagate. Then the solution $u : \Omega \to \mathbb{R}$ to the Dirichlet boundary problem

$$\begin{cases}
\Delta u = 0 & \text{in } \Omega \\
u = g & \text{on } \partial \Omega
\end{cases}$$

describes the temperature $u(x)$ at the point $x \in \Omega$. Cf. Figure 2.1.

2.2. Definitions. Let $\Omega \subset \mathbb{R}^n$ be an open set (this will always be the case from now on).

- We consider the homogeneous Laplace equation

$$\Delta u = 0 \quad \text{in } \Omega \tag{2.1}$$

where we recall that $\Delta u = \text{tr}(D^2 u) = \sum_{i=1}^{n} \partial_{x_i x_i} u$.

- The inhomogenous equation (sometimes: Poisson equation) is, for a given function $f : \Omega \to \mathbb{R}$,

$$\Delta u = f \quad \text{in } \Omega \tag{2.2}$$

Two types of boundary problems are common:
• Dirichlet-problem or Dirichlet-data \( g : \partial \Omega \to \mathbb{R} \)
\[
\begin{aligned}
\Delta u &= f \quad \text{in } \Omega \\
u &= g \quad \text{on } \partial \Omega
\end{aligned}
\]

• Neumann-problem or Neumann-data \( g : \partial \Omega \to \mathbb{R} \)
\[
\begin{aligned}
\Delta u &= f \quad \text{in } \Omega \\
\partial \nu u &= g \quad \text{on } \partial \Omega
\end{aligned}
\]

Here \( \nu : \partial \Omega \to \mathbb{R}^n \) is the \textit{outwards facing unit normal} of \( \partial \Omega \). (Often this is combined with a normalizing assumption like \( f_\Omega u = 0 \), because \( u + c \) is otherwise a solution if \( u \) is a solution – i.e. non-uniqueness occurs).

**Definition 2.1.** A function \( u \in C^2(\Omega) \) is called \textit{harmonic} if \( u \) pointwise solves
\[
\Delta u(x) = 0 \quad \text{in } \Omega
\]
We also say, \( u \) is a solution to the \textit{homogeneous Laplace equation}.
We say that \( u \) is a \textit{subsolution} or \textit{subharmonic} if
\[
\Delta u(x) \geq 0 \quad \text{in } \Omega.
\]
If
\[
\Delta u(x) \leq 0 \quad \text{in } \Omega
\]
we say that \( u \) is a \textit{supersolution} or \textit{superharmonic}.

This notion is very confusing, but it comes from the fact that \( -\Delta \) is a “positive operator” (i.e. has only positive eigenvalues).

2.3. **Fundamental Solution, Newton- and Riesz Potential.** There are many trivial solutions (polynomials of order 1) of Laplace equation. But these are not very interesting. There is a special type of solution which is called \textit{fundamental solution} (which, funny enough, is actually not a solution).

It appears when we want to compute the solution to an equation on the whole space
\[
\Delta u(x) = f(x).
\]
For this we make a brief (formal) introduction to Fourier transform:

The Fourier transform takes a map \( f : \mathbb{R}^n \to \mathbb{R} \) and transforms it into \( \mathcal{F}u \equiv \hat{f} : \mathbb{R}^n \to \mathbb{R} \) as follows
\[
\hat{f}(\xi) := \frac{1}{(2\pi)^\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\langle \xi, x \rangle} f(x) \, dx.
\]
The \textit{inverse Fouriertransform} \( f^\vee \) is defined as
\[
f^\vee(\xi) := \frac{1}{(2\pi)^\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle \xi, x \rangle} f(x) \, dx.
\]
It has the nice property that \((f^\wedge)^\lor = f\).

One of the important properties (which we will check in exercises) is that derivatives become polynomial factors after Fourier transform:

\[(\partial_x g)^\wedge (\xi) = -i\xi \hat{g}(\xi).\]

For the Laplace operator \(\Delta\) this implies

\[(\Delta u)^\wedge(\xi) = -|\xi|^2 \hat{u}(\xi).\]

(Side-remark: In this sense \(-\Delta\) is a positive operator).

This means that if we look at the equation (2.3) and apply Fourier transform on both sides we have

\[-|\xi|^2 \hat{u}(\xi) = \hat{f}(\xi),\]

that is

\[\hat{u}(\xi) = -|\xi|^{-2} \hat{f}(\xi),\]

Inverting the Fourier transform we get an explicit formula for \(u\) in terms of the data \(f\).

\[u(x) = -\left(|\xi|^{-2} \hat{f}(\xi)\right)^\lor(x).\]

This is not a very nice formula, so let us simplify it. Another nice property of Fourier transform (and its inverse) is that products become convolutions. Namely

\[(g(\xi)f(\xi))^\lor(x) = \int_{\mathbb{R}^n} g^\lor(x-z) f^\lor(z) \, dz.\]

In our case, for \(g(\xi) = -|\xi|^{-2}\) we get that

\[u(x) = \int_{\mathbb{R}^n} g^\lor(x-z) f(z) \, dz.\]

Now we need to compute \(g^\lor(x-z)\), and for this we restrict our attention to the situation where the dimension is \(n \geq 3\). In that case, just by the definition of the (inverse) Fourier transform we can compute that since \(g\) has homogeneity of order 2 (i.e. \(g(t\xi) = t^{-2}g(\xi)\)), then \(g^\lor\) is homogeneous of order \(2 - n\). In particular

\[g^\lor(x) = |x|^{2-n} g^\lor(x/|x|).\]

Now an argument that radial functions stay radial under Fourier transforms leads us to conclude that

\[g^\lor(x) = c_1 |x|^{2-n}.\]

That is, we have arrived that (by formal computations) a solution of (2.3) should satisfy (2.4)

\[u(x) = c_1 \int_{\mathbb{R}^n} |x-z|^{2-n} f(z) \, dz.\]

The constant \(c_1\) can be computed explicitly, and we will check below that this potential representation of \(u\) really is true. This potential is called the Newton potential (which is a special case of so-called Riesz potentials). The kernel of the Newton potential is called the fundamental solution of the Laplace equation (which, let us stress this again, is not a solution).
Definition 2.2. The fundamental solution \( \Phi(x) \) of the Laplace equation for \( x \neq 0 \) is given as
\[
\Phi(x) = \begin{cases} 
-\frac{1}{2\pi} \log |x| & \text{for } n = 2 \\
-\frac{1}{n(n-2)\omega_n} |x|^{2-n} & \text{for } n \geq 2
\end{cases}
\]
Here \( \omega_n \) is the Lebesgue measure of the unit ball \( \omega_n = |B(0,1)| \).

One can explicitly check that \( \Delta \Phi(x) = 0 \) for \( x \neq 0 \) (indeed, \( \Delta \Phi(x) = \delta_0 \) where \( \delta_0 \) is the Dirac measure at the point zero, cf. remark 2.7).

Exercise 2.3. Show that \( \Phi \in C^\infty(\mathbb{R}^n \setminus \{0\}) \) and compute that \( \Delta \Phi(x) = 0 \) for \( x \neq 0 \).

The following statement justifies (somewhat) the notion of fundamental solution: the fundamental solution \( \Phi(x) \) can be used to construct all solutions to the inhomogeneous Laplace equation:

Theorem 2.4. Let \( u \) be the Newton-potential of \( f \in C^2_c(\mathbb{R}^n) \), that is
\[
u(x) := \int_{\mathbb{R}^n} \Phi(x - y) f(y) \, dy.
\]
Here \( C^2_c(\mathbb{R}^n) \) are all those functions in \( C^2(\mathbb{R}^n) \) such that \( f \) is constantly zero outside of some compact set.

We have
\[
\begin{align*}
&\bullet \, u \in C^2(\mathbb{R}^n) \\
&\bullet \, -\Delta u = f \quad \text{in } \mathbb{R}^n.
\end{align*}
\]

Proof. First we show differentiability of \( u \). By a substitution we may write
\[
u(x) := \int_{\mathbb{R}^n} \Phi(x - y) f(y) \, dy = \int_{\mathbb{R}^n} \Phi(z) f(x - z) \, dz.
\]
Now if we denote the difference quotient
\[
\Delta_h^e u(x) := \frac{u(x + he_i) - u(x)}{h}
\]
where \( e_i \) is the \( i \)-th unit vector, then we obtain readily
\[
\Delta_h^e u(x) := \int_{\mathbb{R}^n} \Phi(x - y) f(y) \, dy = \int_{\mathbb{R}^n} \Phi(z) (\Delta_h^e f)(x - z) \, dz.
\]
One checks that \( \Phi \) is locally integrable (it is not globally integrable!), that is for every bounded set \( \Omega \subset \mathbb{R}^n \),
\[
\int_{\Omega} |\Phi| < \infty.
\]
Indeed, (we show this for \( n \geq 3 \), the case \( n = 2 \) is an exercise), if \( \Omega \subset \mathbb{R}^n \) is a bounded set, then it is contained in some large ball \( B(0, R) \).

\[
\int_{\Omega} |\Phi| \leq C \int_{B(0, R)} |x|^{2-n} dx
\]

Using Fubini’s theorem,

\[
\int_{B(0, R)} |x|^{2-n} dx
= \int_0^R \int_{\partial B(0,r)} |\theta|^{2-n} d\mathcal{H}^{n-1}(\theta) dr
= \int_0^R r^{2-n} \int_{\partial B(0,r)} d\mathcal{H}^{n-1}(\theta) dr
= c_n \int_0^R r^{2-n} r^{n-1} dr
= c_n \int_0^R r^1 dr
= c_n \frac{1}{2} R^2 < \infty.
\]

This establishes (2.5)

On the other hand \((\Delta^e_i f)\) has still compact support for every \( h \). In particular, by dominated convergence we can conclude that

\[
\lim_{h \to 0} \Delta^e_i u(x) = \int_{\mathbb{R}^n} \Phi(z) \lim_{h \to 0} (\Delta^e_i f)(x - z) dz.
\]

that is

\[
\partial_i u(x) = \int_{\mathbb{R}^n} \Phi(z) (\partial_i f)(x - z) dz.
\]

In the same way

\[
\partial_{ij} u(x) = \int_{\mathbb{R}^n} \Phi(z) (\partial_{ij} f)(x - z) dz.
\]

Now the right-hand side of this equation is continuous (again using the compact support of \( f \)). This means that \( u \in C^2(\mathbb{R}^n) \).

To obtain that \( \Delta u = f \) we first use the above argument to get

\[
\Delta u(x) = \int_{\mathbb{R}^n} \Phi(z) (\Delta f)(x - z) dz.
\]

Observe that

\[
(\Delta f)(x - z) = \Delta_x(f(x - z)) = \Delta_z(f(x - z)).
\]

Now we fix a small \( \varepsilon > 0 \) (that we later send to zero) and split the integral, we have

\[
\Delta u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy = \int_{B(0,\varepsilon)} \Phi(z) (\Delta f)(x-z) dz + \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(z) (\Delta f)(x-z) dz =: I_\varepsilon + II_\varepsilon.
\]
The term $I_\varepsilon$ contains the singularity of $\Phi$, but we observe that 

$$I_\varepsilon \xrightarrow{\varepsilon \to 0} 0.$$ 

Indeed, this follows from the absolute continuity of the integral and since $\Phi$ is integrable on $B(0,1)$:

$$|I_\varepsilon| \leq \sup_{\mathbb{R}^n} |\Delta f| \int_{B(x,\varepsilon)} |\Phi(z)| \xrightarrow{\varepsilon \to 0} 0.$$ 

The term $II_\varepsilon$ does not contain any singularity of $\Phi$ which is smooth on $\mathbb{R}^n \setminus B_\varepsilon(0)$, so we can perform an integration by parts

$$II_\varepsilon = \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(z) (\Delta f)(x-z) \, dz = \int_{\partial B(0,\varepsilon)} \Phi(z) \partial_\nu f(x-z) \, d\mathcal{H}^{n-1}(z) - \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \nabla \Phi(z) \cdot \nabla f(x-z) \, dz.$$ 

Here $\nu$ is the unit normal to the ball $\partial B(0,\varepsilon)$, i.e. $\nu = \frac{z}{\varepsilon}$.

By the definition of $\Phi$ one computes that (using (2.5))

$$\left| \int_{\partial B(0,\varepsilon)} \Phi(z) \partial_\nu f(x-z) \, d\mathcal{H}^{n-1}(z) \right| \leq \sup_{\mathbb{R}^n} |\nabla f| \int_{\partial B(0,\varepsilon)} |\Phi(z)| \xrightarrow{\varepsilon \to 0} 0.$$ 

So we perform another integration by parts and have

$$II_\varepsilon = o(1) - \int_{\partial B(0,\varepsilon)} \partial_\nu \Phi(z) f(x-z) \, dz + \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Delta \Phi(z) f(x-z) \, dz$$

$$= o(1) - \int_{\partial B(0,\varepsilon)} \partial_\nu \Phi(z) f(x-z) \, dz$$

Here in the last step we used that $\Delta \Phi = 0$ away from the origin, Exercise 2.3.

Now we observe that the unit normal on $\partial B(0,\varepsilon)$ is $\nu(z) = -\frac{z}{\varepsilon}$ and

$$D\Phi(z) = \begin{cases} 
-\frac{1}{2\pi |z|} \frac{\varepsilon}{|z|} & n = 2, \\
-\frac{1}{n(n-2)\omega_n} (2-n)|z|^{1-n} & n \geq 3.
\end{cases}$$

Thus, for $|z| = \varepsilon$,

$$\partial_\nu \Phi(z) = \nu \cdot D\Phi(z) = \frac{1}{n\omega_n} \varepsilon^{1-n}$$

---

$$\int_{\Omega} f \partial_i g = \int_{\partial\Omega} f g \nu^i - \int_{\Omega} \partial_i f g,$$

where $\nu$ is the normal of $\partial\Omega$ pointing outwards (from the point of view of $\Omega$). $\nu^i$ is the $i$-th component of $\nu$. Fun exercise: Check this rule in 1D, to see the relation what we all learned in Calc 1.
Thus we arrive at
\[ II_\varepsilon = o(1) - \int_{\partial B(0,\varepsilon)} \frac{1}{n\omega_n\varepsilon^{n-1}} f(x-z) \, dH^{n-1}(z) \]
\[ = o(1) - \int_{\partial B(0,\varepsilon)} f(x-z) \, dH^{n-1}(z) \]
\[ = o(1) - f(x) + \int_{\partial B(0,\varepsilon)} (f(x) - f(x-z)) \, dH^{n-1}(z) \]

Here we use the mean value notation
\[ \int_{\partial B(0,\varepsilon)} = \frac{1}{\mathcal{H}^{n-1}(\partial B(0,\varepsilon))} \int_{\partial B(0,\varepsilon)} \cdot \]

Now one shows (exercise!) that for continuous \( f \)
\[ \lim_{\varepsilon \to 0} \int_{\partial B(0,\varepsilon)} (f(x) - f(x-z)) \, dH^{n-1}(z) = 0. \]
(Indeed this is essentially Lebesgue’s theorem). That is
\[ II_\varepsilon = o(1) - f(x) \quad \text{as} \quad \varepsilon \to 0 \]
and thus
\[ \Delta u(x) = -f(x) + o(\varepsilon), \]
and letting \( \varepsilon \to 0 \) we have
\[ \Delta u(x) = -f(x), \]
as claimed. \( \square \)

**Exercise 2.5.** Show that \( \log |x| \) is locally integrable, i.e. that for any bounded set \( \Omega \subset \mathbb{R}^n \) we have
\[ \int_{\Omega} \log |x| < \infty. \]

**Exercise 2.6.** Assume \( f \) is continuous. Show that
\[ \lim_{\varepsilon \to 0^+} \int_{\partial B(0,\varepsilon)} |f(x) - f(x-z)| \, dH^{n-1}(z) = 0. \]

**Remark 2.7.** One can argue (in a distributional sense, which we learn towards the end of the semester)
\[ -\Delta \Phi = \delta_0, \]
where \( \delta_0 \) denotes the Dirac measure at 0, namely the measure such that
\[ \int_{\mathbb{R}^n} f(x) \, d\delta_0 = f(0) \quad \text{for all} \quad f \in C^0(\mathbb{R}^n). \]
Observe that $\delta_0$ is not a function, only a measure. In this sense one can justify that

$$-\Delta u(x) = \Delta \int_{\mathbb{R}^n} \Phi(x - z) f(z) dz$$

$$= \int_{\mathbb{R}^n} \Delta \Phi(x - z) f(z) dz$$

$$= \int_{\mathbb{R}^n} f(z) d\delta_x(z)$$

$$= f(x)$$

2.4. Green Functions. Our next goal are Green’s functions. In some way Green functions are a restriction of the fundamental solution to domains $\Omega \subset \mathbb{R}^n$ factoring in also boundary data. Recall that for the fundamental solution $\Phi$ we showed in Theorem 2.4 that for the Newton potential

$$u(x) := \int_{\mathbb{R}^n} \Phi(x - y) f(y) dy$$

we have $\Delta u = f$. It is an interesting observation that (for reasonable $f$) we have

$$\lim_{|x| \to \infty} u(x) = 0.$$ 

That is the Newton potential approach solves an equation of

$$\begin{cases}
\Delta u = f & \text{in } \mathbb{R}^n \\
u = 0 & \text{on the boundary, i.e. for } |x| \to \infty.
\end{cases}$$

The Greens function is a way to restrict this construction to domains $\Omega$. Instead of the Fundamental solution $\Phi(x - y)$ we get the Green kernel $G(x, y)$. Instead of the Newton potential we consider

$$u(x) = \int_{\Omega} G(x, y) f(y) dy$$

and hope that this object solves

$$\begin{cases}
\Delta u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}$$

The Greens function $G$ (which depends on $\Omega$) can be computed explicitely only for very specific $\Omega$ (balls, half-spaces) – which is somewhat related to the fact that there is not necessarily a reasonable Fourier transform for generic sets $\Omega$.

But one can abstractly show that the Green functions exists for reasonable sets $\Omega$. The idea is as follows: We know that the Newton potential as in (2.7) solves the right equation $\Delta u = f$, but it does not satisfy $u = 0$ on $\partial \Omega$. So let us try to correct the Newton potential and choose the Ansatz

$$u(x) := \int_{\Omega} \Phi(x - y) f(y) dy - \int_{\Omega} H(x, y) f(y) dy$$
By our computations for Theorem 2.4 we have that then for \( x \in \Omega \)
\[
\Delta u(x) := f(x) - \int_\Omega \Delta_x H(x, y) f(y) \, dy,
\]
so it would be nice if
\[
\Delta_x H(x, y) = 0 \quad \forall \, x, y \in \Omega.
\]
Moreover we would like that \( u(x) = 0 \) on \( \partial \Omega \), which would be satisfied if
\[
\Phi(x - y) = H(x, y) \quad \forall x \in \partial \Omega, y \in \Omega.
\]
That is, for each fixed \( y \in \Omega \) we should try to find a function \( H(\cdot, y) \) that solves
\[
\begin{cases}
\Delta_x H(\cdot, y) = 0 & \text{in } \Omega, \\
H(\cdot, y) = \Phi(\cdot - y) & \text{on } \partial \Omega.
\end{cases}
\]
Observe that for fixed \( y \in \Omega \) the boundary condition \( \Phi(\cdot - y) \in C^\infty(\partial \Omega) \) is a smooth function, since for \( y \in \Omega \) we clearly have
\[
\inf_{x \in \partial \Omega} |x - y| > 0.
\]
That is, there is a good chance to solve this equation \((2.8)\) (and from Theorem 2.22 we know that there is at most one solution).

**Definition 2.8 (Green function).** For given \( \Omega \), if there exists \( H \) as in \((2.8)\) then we call
\[
G(x, y) := \Phi(x - y) - H(x, y)
\]
the Green function on \( \Omega \).

One can show that \( G \) is symmetric, i.e. that
\[
G(x, y) = G(y, x) \quad \forall \, x \neq y \in \Omega
\]
While the Green function are usually not explicit, some properties and estimates can be shown, and there is an extensive research literature on the subject, e.g. see [Littman et al., 1963]. The Green function is also specially important from the point of view of stochastic processes, see e.g. [Chen, 1999].

We will only investigate the most basic property:

**Theorem 2.9.** Let \( \Omega \subset \subset \mathbb{R}^n \), \( \partial \Omega \in C^1 \) \( f \in C^0(\Omega) \) and \( g \in C^0(\partial \Omega) \). Assume that \( u \in C^2(\Omega) \cap C^0(\bar{\Omega}) \) is a solution to
\[
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
u = g & \text{on } \partial \Omega
\end{cases}
\]
Then if \( G \) is the Green function for \( \Omega \) from Definition 2.8 we have for any \( x \in \Omega \),
\[
u(x) = \int_\Omega G(x, y) f(y) \, dy - \int_{\partial \Omega} g(\theta) \partial_{\nu(\theta)} G(x, \theta) \, d\mathcal{H}^{n-1}(\theta).
\]
Proof. Recall the Gauss-Green formula\(^2\) on (smooth enough) domains \(A\),
\begin{equation}
(2.11) \quad \int_A u(y) \Delta v(y) - \Delta u(y) v(y) \, dy = \int_{\partial A} u(\theta) \partial_{\nu} v(\theta) - \partial_{\nu} u(\theta) v(\theta) \, d\mathcal{H}^{n-1}(\theta).
\end{equation}
We apply this to formula to \(A = \Omega \setminus B(x, \varepsilon)\) and \(v(y) := G(x, y)\). Observe that by symmetry of \(G\), (2.9),
\[\Delta_y G(x, y) = \Delta_x G(x, y) = 0 \quad x \neq y,\]
so, also in view of (2.10), (2.11) becomes
\begin{equation}
(2.12) \quad - \int_A G(x, y) f(y) \, dy = \int_{\partial A} u(\theta) \partial_{\nu} G(x, \theta) - \partial_{\nu} u(\theta) G(x, \theta) \, d\mathcal{H}^{n-1}(\theta).
\end{equation}
Now we argue as in the proof of Theorem 2.4. Observe that \(H\) is a smooth function. We have
\begin{align*}
\int_{\partial A} u(\theta) \partial_{\nu} G(x, \theta) d\mathcal{H}^{n-1}(\theta) \\
= \int_{\partial \Omega} g(\theta) \partial_{\nu} G(x, \theta) - \int_{\partial B(x, \varepsilon)} u(\theta) \partial_{\nu} \Phi(x - \theta) \, d\mathcal{H}^{n-1}(\theta) + \int_{\partial B(x, \varepsilon)} u(\theta) \partial_{\nu} H(x - \theta) \, d\mathcal{H}^{n-1}(\theta)
\end{align*}
\[\varepsilon \to 0 \quad \int_{\partial \Omega} g(\theta) \partial_{\nu} G(x, \theta) - u(x) + 0.
\]
and
\begin{align*}
\int_{\partial A} \partial_{\nu} u(\theta) G(x, \theta) d\mathcal{H}^{n-1}(\theta) \\
= \int_{\partial \Omega} \partial_{\nu} u(\theta) G(x, \theta) - \int_{\partial B(x, \varepsilon)} \partial_{\nu} u(\theta) G(x, \theta) \, d\mathcal{H}^{n-1}(\theta) \\
= 0 - \int_{\partial B(x, \varepsilon)} \partial_{\nu} u(\theta) G(x, \theta) \, d\mathcal{H}^{n-1}(\theta)
\end{align*}
\[\varepsilon \to 0 \quad 0.
\]
This proves the claim. \(\square\)

In special situations one can actually construct explicit Green’s function. Let us firstly consider the Half-space
\[\mathbb{R}^n_+ = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0 \}.
\]
So we need to find a solution to the equation
\[\begin{cases}
\Delta_x H(\cdot, y) = 0 & \text{in } \mathbb{R}^n_+, \\
H(\cdot, y) = \Phi(\cdot - y) & \text{on } \mathbb{R}^{n-1} \times \{0\} \equiv \partial \mathbb{R}^n_+.
\end{cases}\]
Since \(H\) at the boundary has to coincide with \(\Phi\) it is likely that \(H\) should be somewhat of the form of \(\Phi\) – only the singularity has to be gotten rid of – the idea is a reflection:
\[H(x, y) := \Phi(x - y^*)\]
\(^2\)this is a special case of the integration by parts formula
where
\[ y^* = (y_1, \ldots, y_n)^* = (y_1, \ldots, y_{n-1}, -y_n). \]

It is a good exercise to check that

1. \( H \) is symmetric, \( H(x, y) = H(y, x) \)
2. \( H \) is smooth in \( \mathbb{R}_+^n \times \mathbb{R}_+^n \) (since \( x^* = y \) implies \( x_n = -y_n \), so \( x \) and \( y \) cannot both lie in the upper half-space if this happens)
3. The reflection does not change the PDE, namely \( \Delta x^* H = 0 \) for \( x, y \in \mathbb{R}_+^n \).
4. Indeed \( H(x, y) = \Phi(x - y) \) for \( x \in \mathbb{R}_+^{n-1} \times \{0\} \) and \( y \in \mathbb{R}_+^n \).

So we set
\[ G(x, y) := \Phi(x - y) - \Phi(x - y^*) = \Phi(x - y) - \Phi(x^* - y) \]

When we now use the representation formula, as in Theorem 2.9, then we need to compute \( \partial_{\nu(y)} G(x, y) \) for \( y \in \mathbb{R}_+^{n-1} \times \{0\} \). Observe that the outwards unit normal \( \nu(y) = (0, \ldots, 0, -1) \), so we compute
\[
\partial_{\nu(y)} G(x, y) = -\partial_{y_n} \Phi(x - y) + \partial_{y_n} \Phi(x^* - y) = c_n \frac{x_n - y_n}{|x - y|^n} - c_n \frac{x_n + y_n}{|x - y|^n} = \tilde{c}_n \frac{x_n}{|x - y|^n}.
\]

If we write the variables in \( \mathbb{R}_+^n \) as \( x = (x', x_n), x' \in \mathbb{R}_+^{n-1} \) and \( x_n > 0 \), then as in Theorem 2.9 we indeed obtain, e.g., if
\[
(2.13) \quad u(x) := c_n \int_{\mathbb{R}_+^{n-1}} \frac{x_n}{|x' - y'|^2 + |x_n|^2} \, g(y') \, dy'
\]
then \( u \) satisfies indeed (for “reasonable” \( g \))
\[
\begin{align*}
\Delta u &= 0 \quad \text{in } \mathbb{R}_+^n, \\
\lim_{x_n \to 0} u(x) &= g(x'), \\
\lim_{x_n \to \infty} u(x) &= 0.
\end{align*}
\]

The formula for \( u \) is called the Poisson formula on the Half-space \( \mathbb{R}_+^n \), also the harmonic extension of \( g \) from \( \mathbb{R}_+^{n-1} \) to \( \mathbb{R}_+^n \).

Exercise 2.10. \quad (1) Show that the constant \( c_n \) in (2.13) is
\[
c_n = \left( \frac{1}{\int_{\mathbb{R}_+^{n-1}} \left( |x' - y'|^2 + |x_n|^2 \right)^{n/2} \, dy'} \right)^{-1}.
\]

**Hint:** Use the maximum principle for \( u \) assuming that \( g \equiv 1 \).
(2) Show that for any \( x_n > 0 \)
\[
c_n = \left( \frac{x_n}{\int_{\mathbb{R}_+^{n-1}} \left( |x' - y'|^2 + |x_n|^2 \right)^{n/2} \, dy'} \right)^{-1}
\]
**Example 2.11** (Dirichlet-to-Neumann formula). Let \( g \in C^\infty_c(\mathbb{R}^{n-1}) \). Define \( u \) via (2.13).

We consider the Neumann-data of \( u \):

\[
\partial_n u \bigg|_{x_n = 0} = \lim_{x_n \to 0^+} \frac{u(x', x_n) - u(x', 0)}{x_n} = \lim_{x_n \to 0^+} \frac{u(x', x_n) - g(x')}{x_n}
\]

\[
\frac{E.210}{c_n} \lim_{x_n \to 0^+} \int_{\mathbb{R}^{n-1}} \frac{x_n}{(|x' - y'|^2 + |x_n|^2)^{\frac{n}{2}}} \frac{g(y') - g(x')}{x_n} dy'
\]

\[
= c_n \lim_{x_n \to 0^+} \int_{\mathbb{R}^{n-1}} \frac{1}{(|x' - y'|^2 + |0|^2)^{\frac{n}{2}}} (g(y') - g(x')) dy'
\]

\[
= c_n \lim_{x_n \to 0^+} \int_{\mathbb{R}^{n-1}} \frac{(g(y') - g(x'))}{(|x' - y'|^2)^{\frac{n}{2}}} dy'
\]

This looks nice, but it has the problem that the integral does not converge absolutely (only in a *principal value* sense).

We try this again: Observe by substituting \( h' := x' - y' \) we can write

\[
u(x) := c_n \int_{\mathbb{R}^{n-1}} \frac{x_n}{(|h'|^2 + |x_n|^2)^{\frac{n}{2}}} g(x' - h') dh'.
\]

By substituting \( h' \) with \(-h'\) we also have

\[
u(x) := c_n \int_{\mathbb{R}^{n-1}} \frac{x_n}{(|h'|^2 + |x_n|^2)^{\frac{n}{2}}} g(x' + h') dh'
\]

So we can write

\[
u(x', x_n) - g(x') = \frac{1}{2} \frac{2u(x', x_n) - 2g(x')}{x_n}
\]

\[
= \frac{1}{2} c_n \int_{\mathbb{R}^{n-1}} \frac{x_n}{(|h'|^2 + |x_n|^2)^{\frac{n}{2}}} g(x' + h') + g(x' - h) - 2g(x') dh'
\]

\[
= \frac{1}{2} c_n \int_{\mathbb{R}^{n-1}} \frac{g(x' + h') + g(x' - h) - 2g(x')}{(|h'|^2 + |x_n|^2)^{\frac{n}{2}}} dh'
\]

\[
\xrightarrow{x_n \to 0} \frac{1}{2} c_n \int_{\mathbb{R}^{n-1}} \frac{g(x' + h') + g(x' - h) - 2g(x')}{|h'|^n + 1} dh'.
\]

In the last step we used that this integral really converges, Exercise 2.12.

This defines an operator

\[
(-\Delta)^{\frac{1}{2}} g(x') \equiv \sqrt{-\Delta} g(x') := \frac{1}{2} c_n \int_{\mathbb{R}^{n-1}} \frac{g(x' + h') + g(x' - h) - 2g(x')}{|h'|^n + 1} dh'
\]
which is called the \textit{half-Laplacian}. Indeed using the Fourier transform on $\mathbb{R}^{n-1}$ one can check that

$$\mathcal{F}\left((-\Delta)^{\frac{1}{2}} g\right)(\xi') = c|\xi'| \mathcal{F}g(\xi') = c\sqrt{|\xi'|^2} \mathcal{F}g(\xi')$$

So this is really the square-root of the Laplacian.

We have proven the \textit{Dirichlet-to-Neumann principle}

If

$$\begin{cases}
\Delta u = 0 & \text{in } \mathbb{R}^n_+ \\
u(x') = g(x') & \text{on } \mathbb{R}^{n-1} \times \{0\}
\end{cases}$$

then

$$\left. \partial_n u \right|_{\mathbb{R}^{n-1} \times \{0\}} = c\left((-\Delta)^{\frac{1}{2}} g\right) \text{ on } \mathbb{R}^{n-1} \times \{0\}$$

In 2007, [Caffarelli and Silvestre, 2007], this formula was generalized for $\sigma \in (0, 2)$ to

$$\begin{cases}
\text{div} \left((x_n)^{1-\sigma} \nabla u\right) = 0 & \text{in } \mathbb{R}^n_+ \\
u(x') = g(x') & \text{on } \mathbb{R}^{n-1} \times \{0\}
\end{cases}$$

then

$$\lim_{x_n \to 0^+} (x_n)^{1-\sigma} \left. \partial_n u \right|_{\mathbb{R}^{n-1} \times \{0\}} = c\left((-\Delta)^{\frac{\sigma}{2}} g\right) \text{ on } \mathbb{R}^{n-1} \times \{0\}$$

This paper has more than 1500 citations and is often referred to as the Caffarelli-Silvestre extension formula.

\textbf{Exercise 2.12.} Let $g \in C^\infty_c(\mathbb{R}^d)$.

1. For $s \in (0, 1)$ show that for each fixed $x \in \mathbb{R}^d$

   $$y \mapsto \frac{g(y) - g(x)}{|x - y|^{d+s}} \in L^1(\mathbb{R}^d),$$

   i.e.

   $$\int_{\mathbb{R}^d} \frac{|g(y) - g(x)|}{|x - y|^{d+s}} \, dy < \infty.$$

2. For $s \in (0, 2)$ show that for each fixed $x \in \mathbb{R}^d$

   $$y \mapsto \frac{g(x + h) - g(x - h)}{|h|^{d+s}} \in L^1(\mathbb{R}^d),$$

   i.e. that

   $$\int_{\mathbb{R}^d} \frac{|g(x + h) + g(x - h) - 2g(x)|}{|x - y|^{d+s}} \, dy < \infty.$$
2.4.1. On a ball. The other situation where we can compute the Green’s function is the ball. For simplicity let us consider $\Omega = B(0,1)$, the unit ball centered at zero. Again the first goal is to find $H(x,y)$ that corrects the fundamental solution. In the case of the half-space $\mathbb{R}^n_+$ we set $H(x,y) = \Phi(x - \tilde{y})$, i.e. we reflected the $y$-variable in a way that did not interfere with the PDE but removed the singularity (and coincided with $\Phi(x - y)$ on the boundary.

So lets do the same for the ball. The canonical operation that reflects points from the ball $B(0,1)$ outside and vice versa is called the inversion at a sphere, $y^* := \frac{y}{|y|^2} : B(0,1) \to B(0,1)^c$. (Although it is not explicitly used here, it is good to know: the inversion at the sphere is a conformal transform, i.e. it preserves angles). So a first attempt would be to set

$$H(x,y) := \Phi \left( \frac{|x - y|}{|y|^2} \right),$$

which takes care of the singularity of $\Phi$ (for $y, x \in B(0,1)$ we have $|x - \frac{y}{|y|^2}| \neq 0$, and does not disturb the PDE for $G(x,y)$. However such a $G(x,y)$ is not equal to $\Phi(x - y)$ for $|x| = 1$. So we need to adapt $G$ to the boundary data. Observe that for $|x| = 1$,

$$|y|^2 \left| x - \frac{y}{|y|^2} \right|^2 = |y|^2 \left( |x|^2 + \frac{1}{|y|^2} - 2 \langle x, \frac{y}{|y|^2} \rangle \right) = \left( |y|^2 |x|^2 + 1 - 2 \langle x, y \rangle \right)$$

$$= \left( |y|^2 + |x|^2 - 2 \langle x, y \rangle \right) = |x - y|^2.$$

That is why we set

$$G_{B(0,1)}(x,y) := \Phi \left( \frac{|y|}{|y|^2} \left| x - \frac{y}{|y|^2} \right| \right),$$

which satisfies all the requested properties.

From this we obtain (without proof)

**Theorem 2.13** (Poisson’s formula for the ball). Assume $g \in C^0(\partial B(0,r))$. Define

$$u(x) := c_n \int_{\partial B(0,r)} \frac{1}{r^2 - |x|^2} g(\theta) d\mathcal{H}^{n-1}(\theta)$$

Then

1. $u \in C^\infty(B(0,r))$
2. $\Delta u = 0$ in $B(0,r)$
(3) \[
\lim_{B(0,r) \ni x \to x_0} u(x) = g(x_0) \quad \text{for any } x_0 \in \partial B(0,r)
\]

2.5. Mean Value Property for harmonic functions. An important property (but very special to the “base Operator $\Delta$”, i.e. not that easily generalizable to more general PDEs) is the mean value property

**Theorem 2.14** (Harmonic functions satisfy Mean Value Property). Let $u \in C^2(\Omega)$ such that $\Delta u = 0$, then

\[
(2.14) \quad u(x) = \int_{\partial B(x,r)} u(z) dH^{n-1}(z) = \int_{B(x,r)} u(y) dy
\]

holds for all balls $B(x,r) \subset \Omega$.

If $\Delta u \leq 0$ then we have “$\geq$” in (2.14). If $\Delta u \geq 0$ then we have “$\leq$” in (2.14).

**Proof.** Set

$$\varphi(r) := \int_{\partial B(x,r)} u(y) dH^{n-1}(y).$$

Observe that by substitution $z := \frac{y-x}{r}$ we have

$$\varphi(r) := \int_{\partial B(0,1)} u(x + rz) dH^{n-1}(z).$$

Taking the derivative in $r$ we have

$$\varphi'(r) = \int_{\partial B(0,1)} Du(x + rz) \cdot z dH^{n-1}(z).$$

Transforming back we get

$$\varphi'(r) = \int_{\partial B(x,r)} Du(y) \cdot \frac{y-x}{r} dH^{n-1}(y).$$

Observe that $\frac{y-x}{r}$ is the outer unit normal of $\partial B(x,r)$. That is

$$\varphi'(r) = |\partial B(x,r)|^{-1} \int_{\partial B(x,r)} \partial_y u(y) dH^{n-1}(y).$$

By Stokes or Green’s theorem (aka, integration by parts)

$$\varphi'(r) = |\partial B(x,r)|^{-1} \int_{B(x,r)} \Delta u(y) dy \overset{(2.14)}{=} 0.$$

That is,

$$\varphi'(r) = 0 \quad \forall r \text{ if } B(x,r) \subset \Omega.$$

which implies that $\varphi$ is constant, and in particular

$$\varphi(r) = \lim_{\rho \to 0} \varphi(\rho).$$
But (Exercise 2.6) for continuous functions $u$,
\[\lim_{\rho \to 0} \varphi(\rho) = \lim_{\rho \to 0} \int_{\partial B(x,\rho)} u(y) \, d\mathcal{H}^{n-1}(y) = u(x),\]
we have shown that
\[(2.15) \quad u(x) = \int_{\partial B(x,\rho)} u(y) \, d\mathcal{H}^{n-1}(y)\]
holds whenever $B(x, r) \subset \Omega$.

Moreover, by Fubini’s theorem
\[
\int_{B(x,r)} u(y) \, dy = \frac{1}{|B(x,r)|} \int_0^r \int_{\partial B(x,\rho)} u(\theta) \, d\mathcal{H}^{n-1}(\theta) \, d\rho
\]
\[
= \frac{1}{|B(x,r)|} \int_0^r |\partial B(x,\rho)| \int_{\partial B(x,\rho)} u(\theta) \, d\mathcal{H}^{n-1}(\theta) \, d\rho
\]
\[
= \frac{1}{|B(x,r)|} \int_0^r |\partial B(x,\rho)| u(x) \, d\rho
\]
\[
= u(x) \frac{1}{|B(x,r)|} \int_0^r \int_{\partial B(x,\rho)} 1 \, d\mathcal{H}^{n-1}(\theta) \, d\rho
\]
\[
= u(x) \frac{|B(x,r)|}{|B(x,r)|}
\]
\[
= u(x).
\]
Together with (2.15) we have shown the claim for $\Delta u = 0$. The inequality arguments are left as an exercise. \qed

The converse holds as well (and there is actually a whole literature on “how many balls” one has to assume the mean value property to get harmonicity, cf. [Llorente, 2015, Kuznetsov, 2019])

**Theorem 2.15** (Mean Value property implies harmonicity). Let $u \in C^2(\Omega)$. If for all balls $B(x, r) \subset \Omega$,
\[(2.16) \quad u(x) = \int_{\partial B(x,r)} u(\theta) \, d\mathcal{H}^{n-1}(\theta)\]
then
\[\Delta u = 0 \quad \text{in } \Omega\]

**Proof.** Assume the claim is false.

Then there exists some $x_0 \in \Omega$ such that $\Delta u(x_0) \neq 0$, so (by continuity of $\Delta u$) w.l.o.g. $\Delta u > 0$ in a small neighbourhood $B(x_0, R)$ of $x_0$. 

On the other hand, setting as above
\[ \varphi(r) := \int_{\partial B(x_0,r)} u(\theta) \quad (2.16) \]
we have \( \varphi'(r) = 0 \) for all \( r > 0 \) such that \( B(x_0,r) \subset \Omega \). But as computed before, for \( r < R \),
\[ \varphi'(r) = C(r) \int_{B(x_0,r)} \Delta u dy > 0. \]
This \( (0 = \varphi'(r) > 0) \) is a contradiction, so the claim is established.

2.6. **Maximum and Comparison Principles.** The mean value property as above is very rigid in the sense that it holds only for very special operators such as the Laplacian. A much more general property (which for the Laplacian \( \Delta \) is a direct consequence of the mean value property) are maximum principles, which should be seen as a “forced convexity/concavity property” for sub-/supersolutions of a large class of PDEs (2nd order elliptic).

In one-dimension a subsolution of Laplace’s equation satisfies
\[ u'' \geq 0 \]
that is, subsolutions are exactly the convex \( C^2 \)-functions.

On the hand, if \( u : \Omega \to \mathbb{R} \) is a smooth convex function, then \( D^2 u(x) \geq 0 \) (in the sense of matrices), so \( \Delta u = \text{tr} D^2 u = \sum (\text{eigenvalues of } D^2 u) \geq 0 \).

On the other hand, the converse does not hold: if we take \( u(x,y) = 2x^2 - y^2 \) then \( u \) is not convex, but \( \Delta u \geq 0 \).

Still, subsolutions have some properties of convex functions (and supersolutions have some properties of concave functions): comparison principles:

Convexity means that on any interval \( (a,b) \) the maximum of \( u \) is obtained at \( a \) or at \( b \) – and if the maximum is obtained in a point \( c \in (a,b) \) then \( u \) is constant. The curious fact is that these properties still hold in arbitrary dimension for solutions of the Laplace equation (and later a large class of elliptic 2nd order equations), they are the so-called weak maximum principle and strong maximum principle.

There is also a “physical” way to explain maximum principles: For example, assume that a solid \( \Omega \) is heated from the sides with a heat source \( g : \partial \Omega \to \mathbb{R} \) and assume there is some heat source from the middle, but it only subtracts heat, \( -\Delta u \leq 0 \), then what is the maximal heat at any point in the interior (letting the system become stationary)? well the maximum heat in the inside is the heat at the boundary (weak maximum principle). And if the heat at any point in the interior is exactly the maximum value of the heat, since the system is stationary, if it is colder at any other point then the heat would have distributed to that point – meaning any other point must have the same heat (strong maximum principle).
Corollary 2.16 (Strong Maximum-principle). Let \( u \in C^2(\Omega) \) be subharmonic, i.e. \( \Delta u \geq 0 \) in \( \Omega \). If there exists \( x_0 \in \Omega \) at which \( u \) attains a global maximum then \( u \) is constant in the connected component of \( \Omega \) containing \( x_0 \).

Proof. By taking a possibly smaller \( \Omega \) we can assume w.l.o.g. \( \Omega \) is connected and \( u \) still attains its global maximum in \( x_0 \in \Omega \).

Let 
\[
A := \{ y \in \Omega : u(y) = u(x_0) \}.
\]

We will show that \( A = \Omega \) (and thus \( u \) is constant) by showing that the following three properties hold

- \( A \) is nonempty
- \( A \) is relatively closed (in \( \Omega \)).
- \( A \) is open

Then \( A \) is an open and closed set in \( \Omega \), and since \( A \) is not the empty set it is all of \( \Omega \).

Clearly \( A \) is nonempty since \( x_0 \in A \).

Also \( A \) is relatively closed by continuity of \( u \): If \( \Omega \ni y_k \xrightarrow{k \to \infty} y_0 \in \Omega \) then
\[
u(y_0) = \lim_{k \to \infty} u(y_k) = u(x_0)
\]
and thus \( y_0 \in A \).

To show that \( A \) is open let \( y_0 \in A \). Since \( \Omega \) is open we can find a small ball \( B(y_0, \rho) \subset \Omega \).

Observe that \( x_0 \) is a global maximum of \( u \) in \( B(y_0, \rho) \).

The mean value property, Theorem 2.14, and then the fact that \( u(x_0) \geq u(y) \) for all \( y \) in \( B(y_0, \rho) \), imply
\[
u(x_0) = u(y_0) \leq \iint_{B(y_0, \rho)} u(y) \, dy \leq \iint_{B(y_0, \rho)} u(x_0) \, dy = u(x_0).
\]

Since left-hand side and right-hand side coincide the inequality is actually an equality.

That is, we have
\[
u(x_0) = \iint_{B(y_0, \rho)} u(y) \, dy,
\]
in other words
\[
\iint_{B(y_0, \rho)} u(y) - u(x_0) \, dy = 0.
\]

Since \( u(y) - u(x_0) \) by assumption \( \leq 0 \) the above integral becomes
\[
-\iint_{B(y_0, \rho)} |u(y) - u(x_0)| \, dy = 0.
\]
that is 
\[ u(y) \equiv u(x_0) \quad \text{in } B(y_0, \rho), \]
that is \( B(y_0, \rho) \subset A \). That is, \( A \) is open. □

**Remark 2.17.** The statement of Corollary 2.16 is false if one replaces global with local maximum (even though local maxima are locally global maxima). A counterexample is for example

\[
u(x) := \begin{cases} 
0 & x \leq 0 \\
 x^3 & x > 0 
\end{cases}
\]

Then \( u \in C^2(\mathbb{R}) \) and
\[
\Delta u = u'' \geq 0 \quad \text{in } (-1, 1)
\]
Clearly \( u \) attains several local maxima, namely in \((-1, 0)\) we have \( u \equiv 0 \), but also clearly \( u \) is not constant. The argument above in the proof of Corollary 2.16 fails, since the point 0 is not a local maximum, and thus the set
\[ A := \{ x \in (-1, 1) : u(x) = 0 \} \]
is not open.

For the next statement, and henceforth, we use the notation \( A \subset\subset B \) ("\( A \) is compactly contained in \( B \)) which means that \( A \) is bonded and its closure \( \overline{A} \subset B \). I.e. for two open sets \( A, B \) the condition \( A \subset\subset B \) means in particular that \( \partial A \) has positive distance from \( \partial B \).

**Corollary 2.18 (Weak maximum principle).** Let \( \Omega \subset\subset \mathbb{R}^n \) and \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) be subharmonic, i.e. \( -\Delta u \leq 0 \) in \( \Omega \). Then
\[
\sup_{\Omega} u = \sup_{\partial\Omega} u,
\]
i.e. "the maximal value is attained at the boundary"³.

**Remark 2.19.** This statement also holds on unbounded sets \( \Omega \), one just has to define the meaning of \( \sup_{\partial\Omega} \) in a suitable sense (i.e. \( \sup_{\partial\mathbb{R}^n} \) should be interpreted as \( \limsup_{|x| \to \infty} \)).

**Proof of Corollary 2.18.** Clearly by continuity
\[
\sup_{\Omega} u \geq \sup_{\partial\Omega} u.
\]
To prove the converse let us argue by contradiction and assume that
\[
(2.17) \quad \sup_{\Omega} u > \sup_{\partial\Omega} u.
\]
Since \( u \) is continuous and \( \Omega \) bounded this must mean that there exists a local maximum point \( x_0 \in \Omega \) such that
\[
(2.18) \quad u(x_0) = \sup_{\Omega} u > \sup_{\partial\Omega} u.
\]
³again: think of convex functions which do have this property
But in view of Corollary 2.16 (strong maximum principle) $u$ is then constant on the connected component of $\Omega$ containing $x_0$. But this implies that on the boundary of this connected component the value of $u$ is still $u(x_0)$, which implies

$$\sup_{\partial \Omega} u \geq u(x_0).$$

But this contradicts the assumption (2.18).

\[\square\]

**Remark 2.20.** A particular consequence of the strong maximum principle is the following. If for $\Omega \subset \subset \mathbb{R}^n$ we have $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying

$$\begin{cases}
\Delta u \geq 0 & \text{in } \Omega \\
u = g & \text{on } \partial \Omega
\end{cases}$$

for some $g \in C^0(\partial \Omega)$. Then the following (equivalent) statements are true:

- If $g \leq 0$ but $g \neq 0$ on $\partial \Omega$ then we have that $u < 0$ in all of $\Omega$.
- If $g \leq 0$ then either $u \equiv 0$ or $u < 0$ everywhere in $\Omega$.

Such a behaviour is special to the PDEs of order two. Even for

$$\Delta^2 u = \Delta(\Delta u) = 0 \quad \text{in } \Omega$$

the above statement may not hold (see e.g. [Gazzola et al., 2010]).

**Corollary 2.21** (Strong Comparison Principle). Let $\Omega \subset \subset \mathbb{R}^n$ open and connected. Assume that $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\Delta u_1 \geq \Delta u_2 \quad \text{in } \Omega.$$

If $u_1 \leq u_2$ on $\partial \Omega$, then exactly one of the following statements is true

1. either $u_1 \equiv u_2$
2. or $u_1(x) < u_2(x)$ for all $x \in \Omega$.

**Proof.** Let $w := u_1 - u_2$, then we have

$$\begin{cases}
\Delta w \geq 0 & \text{in } \Omega \\
w \leq 0 & \text{in } \partial \Omega
\end{cases}$$

The claim now follows from Remark 2.20. \[\square\]

The maximum principle is a great tool to get uniqueness for linear equations!

**Theorem 2.22** (Uniqueness for the Dirichlet problem). Let $\Omega \subset \subset \mathbb{R}^n$, $f \in C^0(\Omega)$ and $g \in C^0(\partial \Omega)$ be given. Then there is at most (!) one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of

$$\begin{cases}
\Delta u = f & \text{in } \Omega \\
u = g & \text{on } \partial \Omega
\end{cases}$$
Proof. Assume there are two solutions, \( u, v \) solving this equation. If we set \( w := u - v \) then \( w \) is a solution to the equation

\[
\begin{cases}
\Delta w = 0 & \text{in } \Omega \\
w = 0 & \text{on } \partial \Omega
\end{cases}
\]

In view of Corollary 2.18 we then have

\[
\sup_{\Omega} w \leq \sup_{\partial \Omega} w = 0.
\]

That is, \( w \leq 0 \) in \( \Omega \). But observe that \( -w \) solves the same equation, which implies that

\[
\sup_{\Omega} (-w) \leq \sup_{\partial \Omega} (-w) = 0,
\]

that is \( -w \leq 0 \) in \( \Omega \). But this readily implies that \( w \equiv 0 \) in \( \Omega \), that is \( v \equiv w \). \( \square \)

So comparison principles are a fantastic tool for obtaining uniqueness for PDEs. Let us also note that via the so-called \textit{Perron’s method} (which relies heavily on maximum principles) we also can obtain existence, Section 2.8. But first we need another comparison principle: Harnack inequality

2.7. Harnack Principle. Above we learned, e.g. in Corollary 2.16 of the strong maximum principle. Another type of maximum principle is the Harnack inequality.

\textbf{Theorem 2.23.} Let \( \Omega \subset \mathbb{R}^n \) open. For any open, connected, and bounded \( U \subset \subset \Omega \) there exists a constant \( C = C(U, \Omega) \) such that for any solution \( u \in C^2(\Omega) \) with \( u \geq 0 \) and such that

\[
\Delta u = 0 \quad \text{in } \Omega
\]

we have

\[
\sup_U u \leq C \inf_U u
\]

\textit{Proof.} The proof is based on the mean value formula, Theorem 2.14, namely for any \( x \in U \) and any \( r < \text{dist} (U, \partial \Omega) \) we have

\[
u(x) = \int_{B(x,r)} u(z) \, dz
\]

Let now \( R := \frac{1}{4} \text{dist} (U, \partial \Omega) \). For any \( x_0 \in U \) and any \( x \in B(x_0, R) \) we have (here we use \( u \geq 0 \) and that \( B(x, R) \subset B(y, 2R) \) for \( x, y \in B(x_0, R) \))

\[
u(x) = \int_{B(x,R)} u(z) \, dz \leq 2^n \int_{B(y,2R)} u(z) \, dz = 2^n u(y).
\]

Again, this holds for any \( x, y \in B(x_0, R) \). Taking the supremum for \( x \in B(x_0, R) \) and the infimum on \( y \in B(x_0, R) \) we get

\[
\sup_{B(x_0,R)} u \leq 2^n \inf_{B(x_0,R)} u.
\]
That is we have the Harnack principle on any Ball \( B(x_0, R) \). Since \( U \) is bounded, open and compactly contained in \( \Omega \) we can now cover all of \( U \) by finitely many balls \( (B_\ell)_{\ell=1}^N \) which lie inside \( \Omega \) centered at points in \( U \) and of radius \( R \).

Take any \( i_1, i_2 \in \{1, \ldots, N\} \) and assume that \( B_{i_1} \cap B_{i_2} \neq \emptyset \). Since then \( \inf_{B_{i_1}} u \leq \sup_{B_{i_2}} u \) Harnack’s principle on the ball \( B_{i_1} \) and the ball \( B_{i_2} \) implies
\[
\sup_{B_{i_1}} u \leq 2^n \inf_{B_{i_2}} u \quad \text{whenever } B_{i_1} \cap B_{i_2} \neq \emptyset.
\]

Repeating the same argument, assume now that \( i_1, i_2, i_3 \in \{1, \ldots, N\} \) such that \( B_{i_1} \cap B_{i_2} \neq \emptyset \) and \( B_{i_2} \cap B_{i_3} \neq \emptyset \). Then
\[
\sup_{B_{i_1}} u \leq 2^n \inf_{B_{i_2}} u \leq 2^n \sup_{B_{i_3}} u \leq 2^{4n} \inf_{B_{i_3}} u \quad \text{whenever } B_{i_1} \cap B_{i_2} \neq \emptyset \text{ and } B_{i_2} \cap B_{i_3} \neq \emptyset.
\]

By induction we readily conclude the following fact: Whenever we have \( i, j \in \{1, \ldots, N\} \) such that there are \( i_1, \ldots, i_K \in \{1, \ldots, N\} \) with \( i_1 = i \) and \( i_K = j \) and \( B_{i_1} \cap B_{i_{\ell+1}} \neq \emptyset \) for all \( \ell \) then we have
\[
\sup_{B_i} u \leq 2^{2nK} \inf_{B_j} u.
\]

Cf. Figure 2.2. Since \( U \) is connected and it is covered by \( N \) balls we conclude that
\[
\sup_U u \leq \sup_{i \in \{1, \ldots, N\}} \sup_{B_i} u \leq 2^{2nN} \inf_{j \in \mathbb{N}} \inf_{B_j} u \leq 2^{2nN} \inf_U u.
\]

Observe that \( N \) heavily depends on \( U \subset \subset \Omega \) – and the closer the boundary of \( U \) is to \( \Omega \), the larger \( N \) tends to be. Thus we have shown that
\[
\sup_U u \leq C(U, \Omega) \inf_U u.
\]

\[\square\]

We observe from the proof above that we can proof Harnack inequality on a ball with a uniform constant.

**Corollary 2.24.** For any dimension \( n \in \mathbb{N} \) there exists a constant \( C = C(n) \) such that the following holds:

Let \( B(x_0, R) \) be a ball. If \( u \in C^2(B(x_0, R)) \) with \( u \geq 0 \) in \( B(x_0, R) \) satisfies
\[
\Delta u = 0 \quad \text{in } B(x_0, R)
\]

then
\[
\sup_{B(x_0, R/2)} u \leq C \inf_{B(x_0, R/2)} u
\]

**Exercise 2.25.** Let \( \Omega \subset \mathbb{R}^n \) be any open set. Assume there is \( u \in C^0(\Omega) \) such that
\[
u \geq 0 \quad \text{in } \Omega
\]
Figure 2.2. From Harnack’s inequality on balls we can conclude Harnack’s inequality on any set $U \subset \subset \Omega$: Harnack’s principle repeatedly applied on balls implies $\sup_{B_{r_1}} u \leq 2^{2^{15}} \inf_{B_{r_15}} u$ (as long as each ball is small enough, so that e.g. twice the ball is in $\Omega$). Any set $U \subset \subset \Omega$ can be covered by finitely many such small balls. So we have $\sup_U u \leq C(U, \Omega) \inf_U u$.

and for some $\lambda \in (0, 1)$ and $\Lambda > 1$ we know that

$$u(x) \leq \Lambda \int_{B(x, r)} u$$

and

$$u(x) \geq \lambda \int_{B(x, r)} u$$

holds for all $x \in \Omega$ with $B(x, r) \subset \subset \Omega$.

Show that there exists a constant $C > 0$ only depending on $n, \lambda, \Lambda$ such that

$$\sup_{B(y, 2\rho)} u \leq C \inf_{B(y, \rho)} u$$

holds for all balls $B(y, 2\rho) \subset \Omega$.

An important consequence of Harnack inequality is that it implies Hölder continuity. This is of course more relevant if we do not a priori that $u \in C^2$ – but we still illustrate this, because this principles applies to many equations.
Example 2.26 (Harnack implies Hölder estimates). Assume
$$\Delta u = 0 \quad \text{in } \Omega$$
For $r > 0$ and any $x \Omega$ such that $B(x, 2r) \subset \Omega$.
$$M(x_0, r) := \sup_{B(x_0, r)} u, \quad m(x_0, r) := \inf_{B(x_0, r)} u.$$ (We assume both values are finite)
Then
$$\Delta(M(x_0, r) - u) = 0$$
and $M(x_0, r) - u \geq 0$ in $B(x_0, r)$ so we have from Harnack’s inequality Corollary 2.24 for a uniform constant $C$,
$$\sup_{B(x_0, r/2)} (M(x_0, r) - u) \leq C \inf_{B(x_0, r/2)} (M(x_0, r) - u),$$
and thus
$$M(x_0, r) - m(x_0, r/2) \leq C \left(M(x_0, r) - M(x_0, r/2)\right).$$
Similarly,
$$\sup_{B(x_0, r/2)} (u - m(x_0, r)) \leq C \inf_{B(x_0, r/2)} (u - m(x_0, r)),
and thus
$$M(x_0, r/2) - m(x_0, r) \leq C \left(m(x_0, r/2) - m(x_0, r)\right).$$
We add those two equations
$$M(x_0, r/2) - m(x_0, r) + M(x_0, r) - m(x_0, r/2) \leq C \left(m(x_0, r/2) - m(x_0, r) + M(x_0, r) - M(x_0, r/2)\right).$$
and thus
$$M(x_0, r/2) - m(x_0, r/2) \leq M(x_0, r/2) - m(x_0, r/2) + \underbrace{m(x_0, r)} + M(x_0, r) \geq 0 \leq C \left(M(x_0, r) - m(x_0, r) - (M(x_0, r/2) - m(x_0, r/2))\right).$$
And thus we have
$$M(x_0, r/2) - m(x_0, r/2) \leq C \left(M(x_0, r) - m(x_0, r) - (M(x_0, r/2) - m(x_0, r/2))\right).$$
which by absorbing becomes
$$(C + 1) \left(M(x_0, r/2) - m(x_0, r/2)\right) \leq C \left(M(x_0, r) - m(x_0, r)\right).$$
That is
$$(M(x_0, r/2) - m(x_0, r/2)) \leq \frac{C}{C + 1} \left(M(x_0, r) - m(x_0, r)\right).$$
Set
$$\gamma := \frac{C}{C + 1} < 1.$$If we then set the oscillation
$$\text{osc}_{B(x_0, r)} u := M(x_0, r) - m(x_0, r),$$
we have shown
\[ \text{osc}_{B(x_0,r/2)} u \leq \gamma \text{ osc}_{B(x_0,r)} u. \]

We can iterate this: for any \( k \in \mathbb{N} \) we have
\[ \text{osc}_{B(x_0,r/2^k)} u \leq \gamma^k \text{ osc}_{B(x_0,r)} u. \]

Now let \( \rho < r \), then there exists exactly one \( k \in \mathbb{N} \) such that \( \frac{\rho}{r} \in \left[ \frac{r}{2^{k-1}}, \frac{r}{2^k} \right) \). And we have (the oscillation is monotone, Exercise 2.28)
\[ \text{osc}_{B(x_0,\rho)} u \leq \text{osc}_{B(x_0,2^k \rho)} u \leq \gamma^k \text{ osc}_{B(x_0,r)} u. \]

Now observe that \( \gamma = 2^{-\sigma} \) for some \( \sigma > 0 \). So,
\[ \gamma^k = (2^k)^{-\sigma} \leq \left( \frac{r}{\rho} \right)^{-\sigma} = \frac{\rho^\sigma}{r^\sigma}. \]

Thus we have shown, for any \( \rho < r \)
\[ \text{osc}_{B(x_0,\rho)} u \leq \frac{\rho^\sigma}{r^\sigma} \text{ osc}_{B(x_0,r)} u. \]

If \( B(x_0,2r) \subset \Omega \) we in particular have
\[ \sup_{x_1 \in B(x_0,r)} \text{osc}_{B(x_0,r)} u \leq \frac{\rho^\sigma}{r^\sigma} \text{ osc}_{B(x_0,2r)} u. \]

This implies Hölder continuity, Exercise 2.27.

**Exercise 2.27.** Show that if for any \( \rho \in (0,r) \) we have
\[ \sup_{x_1 \in B(x_0,r)} \text{osc}_{B(x_1,\rho)} u \leq C \rho^\sigma, \]
then \( u \) is Hölder continuous, namely
\[ \sup_{x,y \in B(x_0,r)} \frac{|u(x) - u(y)|}{|x - y|^\sigma} < \infty. \]

**Exercise 2.28.** Show that if \( u \) is a bounded function then if we set
\[ \text{osc}_A u := \sup_A u - \inf_A u. \]

Show that if \( A \subset B \) then
\[ \text{osc}_A u \leq \text{osc}_B u. \]
2.8. Perron’s method (illustration). Comparison principles (weak, strong maximum principle, and Harnack) are not only great for estimates – they can also be used to show existence (for equations that have these comparison principles – which many have not).

To illustrate this we jump a little bit ahead, and recall that we can already solve the Laplace equation in a ball $B(x,R)$ (via the Green’s function method, Theorem 2.13). Namely, we shall accept that if $f \in C^0(\partial B_R(y))$ then for a certain constant $c_n > 0$ if we set

$$w(x) := \frac{R^2 - |x - y|^2}{c_n R^2} \int_{\partial B_R(y)} \frac{f(z)}{|z - x|^{n}} dz, \quad x \in B_R(y)$$

then $w \in C^0(B_R(y)) \cap C^2(B_R(y))$ and

$$\begin{cases} \Delta w = 0 & \text{in } B(y,r) \\ w = f & \text{on } \partial B(y,R). \end{cases}$$

For general open and bounded sets set with smooth boundary $\partial \Omega$, it is not so easy to get an explicit formula. But one can use Perron’s method and local replacements to show existence of solutions of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial \Omega \end{cases}$$

where $g \in C^0(\partial \Omega)$.

First we extend the notion of solution and subsolution to upper- and lowercontinuous functions.

**Definition 2.29.** Let $\Omega \subset \mathbb{R}^n$ open.

1. A function $f : \Omega \to (-\infty, \infty)$ is called subharmonic in $\Omega$ if it is continuous and for any $x \in \Omega, r > 0 B_r(x) \subset \Omega$ we have

$$f(x) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} f(y) dy.$$

2. A function $f : \Omega \to [-\infty, \infty)$ is called harmonic in $\Omega$ if it is continuous and for any $x \in \Omega, r > 0 B_r(x) \subset \Omega$

$$f(x) = \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} f(y) dy.$$

Similar to Theorem 2.15 one can show that if $u \in C^2$ then subharmonicity as defined above coincides with subharmonicity in the sense of $-\Delta u \leq 0$. One can show that any harmonic function as defined above must be $C^2$ and thus Theorem 2.15 says that indeed our notion of harmonicity coincides with the earlier one.

We now need a first important ingredient: Perron’s method works locally, so somehow one has to pass from the notion of local subsolutions to global subsolutions.
Lemma 2.30. Let $f : \Omega \to \mathbb{R}$ be continuous and assume that for any $x \in \Omega$ there exists $r = r(x)$ such that for any $r \in (0, r(x))$ we have

$$f(x) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} f(y) dy.$$ 

Then $f$ is subharmonic.

Proof. Denote $\rho(x)$ the maximal value such that

$$\rho(x) := \sup \{ \rho > 0 : f(x) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} f(y) dy \text{ for all } r \in (0, \rho) \}.$$ 

We observe that

$$f(x) \leq \frac{1}{|\partial B_{\rho(x)}|} \int_{\partial B_{\rho(x)}(x)} f(y) dy,$$

which follows from the continuity (for each fixed $x$ and $r > 0$)

$$r \mapsto \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} f(y) dy.$$

Observe also that

$$\lim_{r \to 0^+} \frac{1}{|\partial B_r|} \int_{\partial B(x,r)} f(y) dy = f(x).$$

Now we show that $\rho$ is lower semicontinuous, i.e.

$$\liminf_{\Omega \ni y \to x} \rho(y) \geq \rho(x).$$

Indeed, assume that there exists a sequence $y_k \to x_k$ and some $\varepsilon > 0$ such that $\rho(y_k) \leq \rho(x) - \varepsilon$ then there must be some $r_k \leq \rho(x) - \varepsilon$ such that

$$f(y_k) \geq \frac{1}{|\partial B_{r_k}|} \int_{\partial B_{r_k}(x)} f(z) dz + \varepsilon.$$

Clearly $r_k > 0$. Up to taking a subsequence we can assume that $r_k \to \bar{r} \in [0, \rho(x) - \varepsilon]$. Then we have (by continuity of $f$)

$$f(x) \geq \frac{1}{|\partial B_{\bar{r}}|} \int_{\partial B_{\bar{r}}(x)} f(z) dz + \varepsilon.$$

This is a contradiction, since $\bar{r} < \rho(x)$. The contradiction also holds if $\bar{r} = 0$, then the integral on the right-hand side would be replaced with $f(x)$.

So we indeed have

$$\liminf_{\Omega \ni y \to x} \rho(y) \geq \rho(x).$$

In particular on any compact subset $K \subset \Omega$, $\rho$ attains its global minimum in some $x_0 \in K$, and $\rho(x_0) > 0$. Call this minimum $\rho_{\min}$.

We need to show that $\rho(x) = \text{dist}(x, \partial \Omega)$ for all $x \in K$ (since $K \subset \subset \Omega$ is arbitrary this implies the claim.)
Assume that \( x \in K \) and \( \rho(x) < \text{dist}(x, \partial \Omega) \). Take \( \delta \in (0, \rho_{\text{min}}) \) such that \( R < \rho(x) + \delta < \text{dist}(x, \partial \Omega) \).

Let \( h \) be the solution to

\[
\begin{cases}
\Delta h = 0 & \text{in } B(x, R) \\
h = f & \text{on } \partial B(x, R).
\end{cases}
\]

We know that \( h \) exists, since we are in a ball and have the explicit Poisson formula. We then have that \( h \) satisfies the mean value equality, and thus

\[
h(x) = |\partial B(R)| \int_{\partial B(x, R)} h = |\partial B(R)| \int_{\partial B(x, R)} f.
\]

If only we could show that \( h(x) \geq f(x) \), we’d have that

\[
f(x) \leq |\partial B(R)| \int_{\partial B(x, R)} f \quad \forall R < \rho(x) + \delta,
\]

which contradicts the definition of \( \rho(x) \).

How do we show \( h(x) \geq f(x) \)? This is the maximum principle. Consider \( f - h \). We then have for any \( y \in B(x, R) \) and any \( r \leq \min\{\rho(y), B(x, R)\} \)

\[
(f - h)(y) \leq |\partial B(r)|^{-1} \int_{\partial B(r)} (f - h)(z) dz.
\]

This rules out that there is any local maximum of \( f - h \) anywhere in \( B(r) \), and thus there is no local maximum of \( f - h \) in \( B(x, R) \). In particular we have that

\[
f(x) - h(x) \leq \sup_{y \in B(x, R)} (f - h)(y) \leq \sup_{\partial B(x, R)} f - h = 0.
\]

Thus \( f(x) \leq h(x) \), thus we have shown

\[
f(x) \leq |\partial B(R)| \int_{\partial B(x, R)} f \quad \forall R < \rho(x) + \delta,
\]

a contradiction to \( \rho(x) \). Thus \( \rho(x) = \text{dist}(x, \partial \Omega) \) and we can conclude.

\( \Box \)

(Very roughly) the idea of Perron’s method is as follows.

**Perron: Step 1**

Consider the collection of *all* subsolutions (which is a nonempty set)

\[
S_g := \{ v \in C^0(\overline{\Omega}) : \quad v \leq g \quad \text{on } \partial \Omega, \quad v \text{ is subharmonic in } \Omega \}
\]

We need to show \( S_g \) is nonempty. This is easy. Take \( v := \min_{x \in \partial \Omega} g(x) \). Then \( v \) is constant, so \( -\Delta u = 0 \) (in particular \( v \) is subharmonic). And clearly \( v \leq g \) on \( \partial \Omega \).

**Perron: Step 2**
Here comes the trick: let $u$ be simply the largest subsolution, for $x \in \overline{\Omega}$

$$u(x) := \sup_{v \in S_g} v(x).$$

The idea is that since $u$ is the largest subsolution, then even locally there cannot be a larger one. However if locally $u$ was not harmonic, then we can use a harmonic replacement technique on a ball to get a contradiction.

First we need to ensure that $u$ is well-defined. Here we use the maximum principle, Corollary 2.18 and Corollary 2.16. Observe that these arguments were based on the continuity of a subsolution $v$ and the mean value formula so they still apply to our situation, and we have

$$v(x) \leq \sup_{\partial \Omega} v \quad \forall v \in S_g, \quad \forall x \in \overline{\Omega}.$$ 

This implies that for each $x \in \overline{\Omega}$ the family $\{v(x) : v \in S_g\}$ has an upper bound, so the supremum is well-defined. That is $u$ is well-defined.

Next we observe that (formally) $u$ is still subharmonic. Let $x \in \Omega$ and consider any ball $B_r(x) \subset \Omega$.

$$u(x) = \sup_{v \in S_g} v(x) \leq \frac{1}{|B_r|} \int_{B_r(x)} v(y)dy \leq \frac{1}{|B_r|} \int_{B_r(x)} \sup_{v \in S_g} v(y)dy \leq \frac{1}{|B_r|} \int_{B_r(x)} u(y)dy.$$ 

Alas the integral of $u$ may not exist (for all we know $u$ could be non-measurable!). That won’t happen, indeed we have

**Lemma 2.31.** $u$ is lower semicontinuous, that is

$$u(x) \leq \lim \inf_{y \to x} u(y).$$

Think of $u(t) := \sup_{r > 0} r^t$ for $t \in [0, 1]$ to see that $u$ may not be continuous!

**Proof.** Fix any $x \in \overline{\Omega}$ and let $\varepsilon > 0$. Then there must be some $\bar{v} \in S_g$ such that

$$u(x) \leq \bar{v}(x) + \varepsilon.$$ 

Since $\bar{v}$ is continuous, there exists $\delta > 0$ such that for any

$$|\bar{v}(x) - \bar{v}(y)| \leq \varepsilon \quad \forall y \in B(x, \delta) \cap \overline{\Omega}.$$ 

Consequently, for any $y \in \overline{\Omega}$,

$$u(x) - u(y) \leq \bar{v}(x) - \bar{v}(y) + \varepsilon \leq 2\varepsilon \quad \forall y \in B(x, \delta) \cap \overline{\Omega}.$$ 

Observe that we cannot do the same argument in the other direction, since $x$ is fixed and $y$ is variable. In any case, now we have

$$u(x) \leq u(y) + 2\varepsilon \forall y \in B(x, \delta) \cap \overline{\Omega}$$

which implies

$$u(x) \leq \lim \inf_{y \to x} u(y) + \varepsilon.$$
The above lemma makes \( u \) measurable, and since it is bounded

\[
\min_{\partial \Omega} g \leq u(x) \leq \sup_{\partial \Omega} g.
\]

\( u \) is integrable. But still it does not say that \( u \) is a subsolution (because we haven’t shown that \( u \) is continuous).

Fix now \( \bar{x} \in \Omega \). Then there must be a sequence of subharmonic \( \tilde{v}_n \in S_g \) such that \( \lim_{n \to \infty} \tilde{v}_n(\bar{x}) = u(\bar{x}) \). Set

\[
v_n(z) := \max\{\tilde{v}_1(x), \tilde{v}_2(x), \ldots, \tilde{v}_n(x), \min g\}.
\]

As a (finite) maximum of continuous functions \( v_n \in C_0^0(\Omega) \). As we did for \( u \) above, we can also easily check that \( v_n \) is still a subharmonic function. Moreover we have monotonicity

\[
v_n(x) \leq v_{n+1}(x) \quad \forall x \in \Omega,
\]

all while still ensuring \( \lim_{n \to \infty} \tilde{v}_n(\bar{x}) = u(\bar{x}) \).

Take now a ball \( B(\bar{x}, R) \subset \Omega \) (\( \bar{x} \) is in the interior of \( \Omega \)!). We now replace now \( v_n \) inside of \( B(\bar{x}, R) \) with its harmonic replacment, i.e. we set

\[
w_n(x) := \begin{cases} \frac{R^2-|x-\bar{x}|^2}{c_n R} \int_{\partial B_R(\bar{x})} \frac{v_n(z)}{|z-x|^n} \, dz & x \in B_R(\bar{x}) \\ v_n(x) & x \in \Omega \setminus B_R(\bar{x}). \end{cases}
\]

Then we have \( w_n \in C^0(\Omega) \). Since \( v_n \) was monotonically increasing, so is \( w_n \).

\[
w_n(x) \leq w_{n+1}(x) \quad \forall x \in \Omega.
\]

**Lemma 2.32.** We have the following properties

(1) \( w_n(x) \geq v_n(x) \) and

(2) \( w_n \in S_g \).

**Proof.**

(1) We have \( w_n \equiv v_n \) in \( \Omega \setminus B(\bar{x}, R) \). Since \( w_n \) is harmonic in \( B(\bar{x}, R) \) we have \( (v - w) \) is subharmonic in \( B(\bar{x}, R) \), and since \( v - w = 0 \) on \( \partial B(\bar{x}, R) \) the maximum principle implies \( v - w \leq 0 \) in \( B(\bar{x}, R) \), i.e.

\[
v(x) \leq w(x) \quad \forall x \in B(\bar{x}, R).
\]

(2) Since \( w_n \equiv v_n \) in \( \Omega \setminus B(\bar{x}, R) \) we have that \( w_n(x) \leq g(x) \) for all \( x \in \partial \Omega \). We have

\[
v_n(x) \leq \frac{1}{|B(r)|} \int_{B(x,r)} v_n(y) \, dy \leq \frac{1}{|B(r)|} \int_{B(x,r)} w_n(y) \, dy
\]

So for all \( x \in \mathbb{R}^n \setminus B(\bar{x}, R) \), \( v_n = w_n \) is subharmonic.
Let now \( x \in B(\bar{x}, R) \) (which is open). Since \( w_n \) is harmonic for all \( r < \text{dist}(x, \partial B(\bar{x}, R)) \) we have
\[
w_n(x) \leq \frac{1}{|B(r)|} \int_{B(x, r)} w_n(y).
\]
We conclude that \( w_n \) is subharmonic in \( \Omega \) by Lemma 2.30.

□

Since \( w_n \in S_g \) we conclude that
\[
v_n(x) \leq w_n(x) \leq u(x) \quad \forall x \in \Omega
\]
and thus in particular
\[
\lim_{n \to \infty} w_n(\bar{x}) = u(\bar{x}).
\]

**Lemma 2.33.** For \( x \in \overline{B(\bar{x}, R/2)} \) set
\[
w(x) := \lim_{n \to \infty} w_n(\bar{x}).
\]
(This exists since \( w_n \) is bounded by \( u \) and monotonicity). Then \( w \) is harmonic in \( B(\bar{x}, R/2) \) and \( w \leq u \) in \( B(\bar{x}, R/2) \).

**Proof.** For each \( n \in \mathbb{N} \) we know that \( w_n \) is harmonic in \( B(\bar{x}, R) \) (by definition).

So \( w_n - w_m \) for \( n, m \in \mathbb{N} \) is harmonic in \( B(\bar{x}, R) \). We want to apply Harnack’s inequality, Theorem 2.23, so let us assume \( n \geq m \), then we have \( w_n - w_m \geq 0 \), and thus
\[
\sup_{x \in B(\bar{x}, R/2)} (w_n(x) - w_m(x)) \leq C \inf_{y \in B(\bar{x}, R/2)} (w_n(y) - w_m(y)) \quad \forall n \geq m,
\]
i.e.
\[
\sup_{x \in B(\bar{x}, R/2)} |w_n(x) - w_m(x)| \leq C \inf_{y \in B(\bar{x}, R/2)} |w_n(y) - w_m(y)| \quad \forall n \geq m.
\]
In particular,
\[
\sup_{x \in B(\bar{x}, R/2)} |w_n(x) - w_m(x)| \leq C |w_n(\bar{x}) - w_m(\bar{x})| \xrightarrow{n,m \to \infty} 0.
\]
That is, \( w_n \) is a Cauchy sequence with respect to uniform convergence in \( \overline{B(\bar{x}, R/2)} \), and since \( w_n \) is continuous we conclude that there must be some \( w \in C^0(\overline{B(\bar{x}, R/2)}) \) such that \( w \) is the uniform limit of \( w_n \) in \( B(\bar{x}, R/2) \).

Since \( w_n \) is harmonic in \( B(\bar{x}, R/2) \) (in the sense of Definition 2.29(2)), so is \( w \) (by the uniform convergence).

Since \( w_n(x) \leq u(x) \) for all \( x \in \Omega \) (because \( w_n \in S_g \)), we conclude that \( w(x) = \lim_{n \to \infty} w_n(x) \leq u \) for all \( x \in \overline{B(\bar{x}, R/2)} \).

□

**Lemma 2.34.** Take \( w \) from Lemma 2.32. Then \( w = u \) in \( B(\bar{x}, R/2) \).
Proof. We already know $w \leq u$ from Lemma 2.32.

So assume that there is $\tilde{y} \in B(\bar{x}, R/2)$ such that $w(\tilde{y}) > u(\tilde{y})$.

Since $w(\tilde{y}) = \lim_{n \to \infty} w_n(\tilde{y})$ there must be some $n$ such that

$$w_n(\tilde{y}) > u(\tilde{y}).$$

But this is a contradiction since $w_n \in S_g$, and thus

$$u(\tilde{y}) = \sup_{v \in S_g} v(\tilde{y}) \geq w_n(\tilde{y}) > u(\tilde{y}).$$

We can conclude. \qed

Corollary 2.35. Let $u(x) := \sup_{v \in S_g} v(x)$. Then $u \in C^0(\Omega)$ and $\Delta u = 0$ in $\Omega$.

Proof. For every $\bar{x} \in \Omega$ there exists a small neighborhood $B(\bar{x}, R/2)$ where $u$ equals a harmonic function, Lemma 2.34. So $u$ must be harmonic and continuous around any point $x \in \Omega$. \qed

It remains to show that $u = g$ on $\partial \Omega$.

Lemma 2.36. Assume that $\partial \Omega \in C^\infty$ and $g$ is continuous in $\partial \Omega$. Let $u(x) := \sup_{v \in S_g} v(x)$, $u \in C^0(\Omega)$ (not yet up to the boundary!) be the harmonic function from before.

Then $u \in C^0(\bar{\Omega})$ and for any $\theta \in \partial \Omega$ we have

$$\lim_{x \to \theta} u(x) = g(\theta).$$

Proof. Since $u(x) := \sup_{v \in S_g} v(x)$ and $v \in S_g$ must satisfy $v \leq g$ on $\partial \Omega$ we conclude that $u \leq g$.

To see the other direction, we build what is called a barrier. A barrier at $\theta$ is a continuous function $b \in C^0(\bar{\Omega})$ which is superharmonic (i.e. $-b$ is subharmonic) in $\Omega$ and $b(x) \geq 0$ for all $x \in \bar{\Omega}$ and $b(x) = 0$ if and only if $x = \theta$.

Fix $\theta \in \partial \Omega$. Since $\partial \Omega$ is smooth, there exists (nontrivial exercise!) a ball $B(\bar{z}, R) \subset \mathbb{R}^n \setminus \Omega$ such that $\overline{B(\bar{z}, R)} \cap \bar{\Omega} = \{\theta\}$ (this is called the exterior sphere condition of $\partial \Omega$).

Here is our barrier function

$$b(x) := \begin{cases} R^{2-n} - |x - \bar{z}|^{2-n} & \text{if dimension } n \geq 3 \\ - \log(R) + \log(|x - \bar{z}|) & \text{if } n = 2. \end{cases}$$

Then $b \in C^\infty(\mathbb{R}^n \setminus \{\bar{z}\})$ and since it involves the fundamental solution we know that $\Delta b(x) = 0$ for all $x \in \mathbb{R}^n \setminus \{\bar{z}\}$. Since $\bar{z} \notin \bar{\Omega}$ we conclude that $\Delta b = 0$ in $\Omega$.

For $x \in \mathbb{R}^n \setminus \overline{B(\bar{z}, R)}$ we have $b(x) > 0$ (observe that $2 - n$ is a negative power!) and we have $b(\theta) = 0$. Since $\overline{B(\bar{z}, R)} \cap \bar{\Omega} = \{\theta\}$ this satisfies the barrier definition in $\Omega$. 

Fix $\varepsilon > 0$. Since $g$ is continuous on $\partial \Omega$ there exists $\delta > 0$ such that $$|g(x) - g(\theta)| < \varepsilon \quad \forall x \in \partial \Omega, |x - \theta| < \delta.$$ Set $$\lambda := \inf_{z \in \partial \Omega \setminus B(\theta, \delta)} \beta(z) > 0.$$ and $$\Lambda := 2 \sup_{\partial \Omega} |g| < \infty.$$ \[\bar{v}(x) := g(\theta) - \varepsilon - b(x) \frac{\Lambda}{\lambda}.\]

Then $\bar{v}$ is still harmonic in $\Omega$ (in particular it is subharmonic). Moreover for $x \in \partial \Omega$, if $|x - \theta| < \delta$ then
$$\bar{v}(x) - g(x) = g(\theta) - g(x) - \varepsilon - b(x) \frac{\Lambda}{\lambda} \leq |g(\theta) - g(x)| - \varepsilon \leq 0.$$ If on the other hand $|x - \theta| \geq \delta$ then
$$\bar{v}(x) - g(x) = g(\theta) - \varepsilon - g(x) - b(x) \frac{\Lambda}{\lambda} \leq \frac{\varepsilon}{1} \leq g(\theta) - g(x) - \lambda$$
$$\leq 2 \sup_{\partial \Omega} |g| - \lambda \leq 0.$$ So we have $\bar{v} \leq g$ on $\partial \Omega$, and thus $\bar{v} \in S_g$. Since $u = \sup_{v \in S_g} v$ we find that
$$u(\theta) \geq \bar{v}(\theta) = g(\theta) - \varepsilon.$$ Since this holds for all $\theta \in \partial \Omega$ we have shown
$$u(x) \geq g(x) - \varepsilon \quad \text{for all } x \in \partial \Omega.$$ This again holds for any $\varepsilon > 0$ so that we have
$$u(x) \geq g(x) \quad \text{for all } x \in \partial \Omega.$$ We conclude that $u(x) = g(x)$ for all $x \in \partial \Omega$ and we can conclude. \[\square\]

We finally can conclude

**Corollary 2.37.** Let $u(x) := \sup_{v \in S_g} v(x)$. Then $u \in C^0(\overline{\Omega})$ and $\Delta u = 0$ in $\Omega$ and $u = g$ on $\partial \Omega$.

Let us summarize some features of Perron’s method.

- Perron’s method shows existence of solutions via obtaining “a largest subsolution” (a “smallest supersolution” would work similarly).
- It relies on the ability to locally improve a subsolution to obtain a global solution (but observe that we worked hard to show that a local subsolution everywhere is a global subsolution).
• Perron’s method relies extremely on comparison principles. Take an equation without comparison principle (e.g. 4th order, or systems of equations), and there is essentially no hope of running this idea.
• Perron’s method likes to work with some form of continuity, not differentiability; in particular we need to define a notion of “weak” subsolution that makes sense for continuous functions. This works for second order equations with comparison principles often via theories like *Viscosity solutions*.

2.9. **Weak Solutions, Regularity Theory.** Now we look at our first encounter with *distributional solutions*. Let \( u \in L^1_{\text{loc}}(\Omega) \), that is \( u \) is a measurable function on \( \Omega \) which is integrable on every compactly contained set \( K \subset \Omega \), i.e.

\[
\int_K |u| < \infty.
\]

\( u \) certainly has no reason to be differentiable, it might not even be continuous. How on earth are we going to define

\[
\Delta u = 0 \quad \text{in } \Omega.
\]

The idea is that if \( u \in C^2(\Omega) \) then

\[
\Delta u = 0 \quad \text{in } \Omega
\]

is equivalent to saying that

\[
\int_\Omega u \Delta \varphi = 0 \quad \text{for all } \varphi \in C_\infty^0(\Omega).
\]

(Recall that \( C_\infty^0(\Omega) \) are those smooth functions that have compact support \( \text{supp } \varphi \subset \subset \Omega \)).

Indeed, for \( \varphi \in C_\infty^0(\Omega) \) and \( u \in C^2(\Omega) \) we have by integration by parts

\[
\int_\Omega u \Delta \varphi = \int_\Omega \Delta u \varphi.
\]

So for \( u \in C^2(\Omega) \) we clearly have that (2.21) is equivalent to

\[
\int_\Omega \Delta u \varphi = 0 \quad \text{for all } \varphi \in C_\infty^0(\Omega).
\]

Now if (2.20) holds then clearly (2.22) holds.

On the other hand assume that (2.22) holds, but (2.20) is false. That is assume there is \( x_0 \in \Omega \) such that (w.l.o.g.)

\[
\Delta u(x_0) > 0.
\]

Since \( u \in C^2(\Omega) \) we have \( \Delta u \in C^0(\Omega) \) and thus there exists a ball \( B(x_0, r) \subset \subset \Omega \) such that

\[
\Delta u > 0 \quad \text{on } B(x_0, r).
\]
Now let \( \varphi \in C^\infty_c(\Omega) \) a bump function (or cutoff function), namely a function \( \varphi \) such that \( \varphi \geq 1 \) in \( B(x_0, r/2) \) and \( \varphi \equiv 0 \) in \( \Omega \setminus B(x_0,r) \), and \( \varphi \geq 0 \) everywhere. These bump functions really exist: they can be build by essentially scaled and glued versions of

\[
\eta(x) := \begin{cases} 
  e^{-\frac{1}{1-|x|^2}} & \text{for } |x| < 1 \\
  0 & \text{for } |x| > 1
\end{cases}
\]

See Figure 2.9.

For this bump function \( \varphi \) we have from (2.23)

\[
\int_\Omega \varphi \Delta u > 0
\]

which contradicts (2.22). This proves the equivalence of (2.21) and (2.20) for \( C^2 \)-functions \( u \).

However, we notice that while (2.20) only makes sense for functions \( u \) that are twice differentiable, the statement (2.21) makes sense for all functions \( u \in L^1_{\text{loc}}(\Omega) \). This warrants the following definition:

**Definition 2.38** (Weak solutions of the Laplace equation). For a function \( u \in L^1_{\text{loc}}(\Omega) \) we say that (2.20) is satisfied in the weak sense (or in the distributional sense) if

\[
\int_\Omega u \Delta \varphi = 0 \quad \text{for all } \varphi \in C^\infty_c(\Omega).
\]

holds. The functions \( \varphi \) used to “test” the equation are for this very reason called test-functions.

To distinguish the notion of solution we used before, we say that if \( \Delta u = 0 \) in a differentiable function sense then \( u \) is a strong solution or classical solution.

Above, we already have shown the following statement

**Proposition 2.39.** Let \( u \in C^2(\Omega) \). Then the following two statements are equivalent:

1. \( u \) is a weak solution to the Laplace equation \( \Delta u = 0 \) in \( \Omega \)
2. \( u \) is a classical solution of \( \Delta u = 0 \) in \( \Omega \).
Weyl proved that this equivalence holds for $u \in L^1_{\text{loc}}$ (i.e. with no a priori differentiability at all) – this is our first result of regularity theory: showing that weak solutions which are a priori only integrable are actually differentiable. Observe: the reason this works here is that we have a homogeneous equation $\Delta u = 0$, and that $\Delta$ is a constant-coefficient linear elliptic operator (and one can spend much more time for proving similar results for more general linear elliptic operators). Having said that, in some sense, the regularity theory for elliptic equations is always somewhat based on the following Theorem, Theorem 2.40 (albeit in a hidden way).

**Theorem 2.40 (Weyl’s Lemma).** Let $u \in L^1_{\text{loc}}(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open. If $u$ is a weak solution of Laplace equation, i.e.

\begin{equation}
\int_{\Omega} u \Delta \varphi = 0 \quad \text{for all } \varphi \in C^\infty_c(\Omega).
\end{equation}

then $u \in C^\infty(\Omega)$ and $\Delta u$ in the classical sense.

Observe that this theorem (rightfully) does not say anything about $u$ on $\partial \Omega$, this is a purely interior result!

The proof of Theorem 2.40 exhibits the structure that many proofs in PDE have. First on obtains some *a priori estimates* (namely under the assumption that everything is smooth we find good estimates). Then we show that these estimates hold also for rough solutions by an approximation argument.

The a priori estimates for the Laplace equations are called the Cauchy estimates. These are truly amazing: They say that if we solve the Laplace equation we can estimate all derivatives, in pretty much any norm simply by the $L^1$-norm of the function.

**Lemma 2.41 (Cauchy estimates).** Let $u \in C^\infty(\Omega)$ be harmonic, $\Delta u = 0$ in $\Omega$. Then we have for any ball $B(x_0, r) \subset \Omega$ and for any multiindex $\gamma$ of order $|\gamma| = k$,

$$|\partial^\gamma u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0, r))}. $$

In particular we have for any $\Omega_2 \subset \subset \Omega$ that

$$\sup_{\Omega_2} |D^k u| \leq C(\text{dist} (\Omega_2, \Omega), k)\|u\|_{L^1(\Omega)} $$

**Proof of the Cauchy estimates, Lemma 2.41.** For $k = 0$ we argue with the mean value property for harmonic functions, Theorem 2.15. We have for any $\rho$ such that $B(x_0, \rho) \subset \Omega$ and any $x \in B(x_0, \rho/2)$,

$$|u(x)| = \left| \int_{B(x, \rho/2)} u(z) \, dz \right| \leq \frac{C}{\rho^n} \int_{B(x, \rho/2)} |u(z)| \, dz \leq \frac{C}{\rho^n} \int_{B(x_0, \rho)} |u(z)| \, dz. $$
That is, we have obtained that for if \( \Delta u = 0 \) on \( B(x_0, \rho) \) then

\[
(2.24) \quad \sup_{B(x_0, \rho/2)} |u| \leq \frac{C}{\rho^n} \|u\|_{L^1(B(x_0, \rho))}.
\]

This proves in particular the case \( k = 0 \) (taking \( \rho =: r \)).

For the case \( k = 1 \) we use a technique called “differentiating the equation” (and in more general situations where this is used in a discretized version we will study later is due to Nirenberg, cf. Section 5.2). Observe that \( \Delta u = 0 \) in \( \Omega \) implies

\[
\Delta \partial_i u = \partial_i \Delta u = 0 \quad \text{in} \ \Omega
\]

So if we set \( v := \partial_i u \) we have that \( \Delta v = 0 \) in \( \Omega \). For \( x \in B(x_0, \rho/4) \), again from the mean value property for harmonic functions, Theorem 2.15, we get with an additional integration by parts

\[
|\partial_i u(x)| = \left| \oint_{B(x, \rho/4)} \partial_i u(z) \, dz \right| = \frac{C}{\rho^n} \left| \int_{\partial B(x, \rho/4)} u(\theta) \nu^i \mathcal{H}^{n-1}(\theta) \right|
\leq \frac{C}{\rho^n} \rho^{n-1} \sup_{B(x, \rho/4)} |u|
\leq \frac{C}{\rho^n} \rho^{n-1} \sup_{B(x_0, \rho/2)} |u|
\]

Now in view of the estimates in the step \( k = 0 \), namely (2.24), we arrive at

\[
\sup_{B(x_0, \rho/4)} |\nabla u(x)| \leq \frac{C}{\rho^{n+1}} \|u\|_{L^1(B(x_0, \rho))}.
\]

Differentiating the equation again, we find by induction that (the constant changes in each appearance!)

\[
|\nabla^k u(x_0)| \leq \sup_{B(x_0, 4^{-k} \rho)} |\nabla^k u(x)| \leq \frac{C}{\rho^{n+1}} \|\nabla^{k-1} u\|_{L^1(B(x_0, 4^{1-k} \rho))} \leq \ldots \leq \frac{C}{\rho^{n+k}} \|u\|_{L^1(B(x_0, \rho))}.
\]

If we want to show the estimate on \( \Omega_2 \subset \subset \Omega \) we now pick \( \rho < \text{dist}(\Omega_2, \partial \Omega) \) and obtain the claim.

\[\square\]

**Proof of Weyl’s Lemma: Theorem 2.40.** We use a mollification argument, i.e. we approximate \( u \) with smooth functions \( u_\varepsilon \) that also solve (in the classical sense) the Laplace equation.

Let \( \eta \in C_c^\infty(B(0, 1)) \) be another bump function, this time with the condition \( \eta(x) = \eta(-x) \), i.e. \( \eta \) is even, \( \eta \geq 0 \) everywhere, and normalized such that

\[
\int_{\mathbb{R}^n} \eta = 1.
\]
We rescale $\eta$ by a factor $\varepsilon > 0$ and set
\[ \eta_\varepsilon(x) := \varepsilon^{-n} \eta(x/\varepsilon). \]

Then the convolution\(^4\) is defined as
\[ u_\varepsilon(x) := \eta_\varepsilon * u(x) := \int_{\mathbb{R}^n} \eta_\varepsilon(y - x) u(y) \, dy \]
Clearly this is not well-defined for all $x$, if $u \in L^1_{\text{loc}}(\Omega)$ only. But it is defined for all $x \in \Omega$ such that $\text{dist}(x, \partial \Omega) > \varepsilon$, since $\text{supp} \eta_\varepsilon(\cdot - x) \subset B(x, \varepsilon)$.

But observe that derivatives on $u_\varepsilon$ hit only the kernel $\eta_\varepsilon$ (which is smooth) (there is a dominated convergence to be used to show that, and for this we need $L^1_{\text{loc}}$!)
\[ \partial^\gamma u_\varepsilon(x) := \eta_\varepsilon * u(x) := \int_{\mathbb{R}^n} \partial^\gamma \eta_\varepsilon(y - x) u(y) \, dy \]
That is $u_\varepsilon \in C^\infty(\Omega_{-\varepsilon})$ where
\[ \Omega_{-\varepsilon} = \{ x \in \Omega, \text{dist}(x, \partial \Omega) > \varepsilon \} \]
The fun part (which we used above already) is that convolutions behave well with differential operators, namely we will show now that $\Delta u_\varepsilon = 0$ in $\Omega_{-\varepsilon}$.

For this let $\psi \in C_c^\infty(\Omega_{-\varepsilon})$ a testfunction, then we have
\[ \int_{\Omega_{-\varepsilon}} u^\varepsilon(x) \Delta \psi(x) \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(y) \eta_\varepsilon(x-y) \Delta \psi(x) \, dy \, dx = \int_{\mathbb{R}^n} u(y) \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) \Delta \psi(x) \, dx \, dy \]
Now, by integration by parts (for any fixed $y \in \mathbb{R}^n$)
\[ \int_{\mathbb{R}^n} \eta(x-y) \Delta \psi(x) \, dx = \int_{\mathbb{R}^n} \Delta_x \eta_\varepsilon(x-y) \psi(x) \, dx = \int_{\mathbb{R}^n} \Delta_y \eta_\varepsilon(x-y) \psi(x) \, dx = \Delta_y \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) \psi(x) \, dx \]
So if we set
\[ \varphi(y) := \eta_\varepsilon * \psi(y) \equiv \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) \psi(x) \, dx \]
then we have by the support condition on $\psi$ that $\varphi \in C^\infty_c(\Omega)$, and thus
\[ \int_{\Omega_{-\varepsilon}} u^\varepsilon(x) \Delta \psi(x) \, dx = \int_{\mathbb{R}^n} u(y) \Delta \varphi(y) \, dy \overset{(2.21)}{=} 0. \]
This argument works for any $\psi \in C^\infty_c(\Omega_{-\varepsilon})$, that is $u^\varepsilon$ is weakly harmonic in $\Omega_{-\varepsilon}$. But since $u_\varepsilon \in C^\infty_c(\Omega_{-\varepsilon})$ this implies in view of Proposition 2.39 that in the strong sense
\[ \Delta u_\varepsilon = 0 \quad \text{in} \quad \Omega_{-\varepsilon}. \]
So now $u_\varepsilon$ is a smooth solution to Laplace’s equation, so we use the a priori estimates of Lemma 2.41.

Fix $\Omega_2 \subset \subset \Omega$. Between $\Omega_2$ and $\Omega$ we can squeeze two more set $\Omega_3$, and $\Omega_4$,
\[ \Omega_2 \subset \subset \Omega_3 \subset \subset \Omega_4 \subset \subset \Omega. \]
\(^4\)we have seen this operation for the Fourier Transform argument above after (2.3), there we used a nonsmooth kernel $|\cdot|^{2-n}$ for the convolution
For any \( \varepsilon \) small enough, namely
\[ \varepsilon < \text{dist}(\Omega_3, \partial\Omega_4) \quad \text{and} \quad \varepsilon < \text{dist}(\Omega_3, \partial\Omega_4) \]
we have that \( \Delta u^\varepsilon = 0 \) in \( \Omega_3 \), so by the Cauchy estimates, Lemma 2.41, we have for any \( k \in \mathbb{N} \)
\[ \sup_{\Omega_2} |\nabla^k u_\varepsilon| \leq C(k, \Omega_2, \Omega_3) \|u_\varepsilon\|_{L^1(\Omega_3)}. \]

Now we estimate, by Fubini,
\[ \|u_\varepsilon\|_{L^1(\Omega_3)} \leq \int_{\Omega_3} \int_{\mathbb{R}^n} |\eta_\varepsilon(x - y)| |u(y)| \, dy \, dx = \int_{\mathbb{R}^n} |u(y)| \int_{\Omega_3} |\eta_\varepsilon(x - y)| \, dx \, dy. \]

Since \( \varepsilon \) is small enough we have that
\[ \text{supp} \left( \int_{\Omega_3} |\eta_\varepsilon(x - \cdot)| \, dx \right) \subset \Omega_4. \]

So we get
\[ \|u_\varepsilon\|_{L^1(\Omega_3)} \leq \|u\|_{L^1(\Omega_4)} \sup_{y \in \mathbb{R}^n} \int_{\Omega_3} |\eta_\varepsilon(x - y)| \, dx \leq \|u\|_{L^1(\Omega_4)} \int_{\mathbb{R}^n} |\eta_\varepsilon(z)| \, dz. \]

Now we use the definition of \( \eta_\varepsilon \) to compute via substitution\(^5\)
\[ \int_{\mathbb{R}^n} |\eta_\varepsilon(z)| \, dz = \varepsilon^{-n} \int_{\mathbb{R}^n} |\eta(\tilde{z})/\varepsilon| \, d\tilde{z} = \varepsilon^{-n} \int_{\mathbb{R}^n} |\eta(z)| \, dz = \int_{\mathbb{R}^n} |\eta(\tilde{z})| \, d\tilde{z} = 1. \]

The last equality is due to the normalization of \( \eta, \int \eta = 1 \).

That is, we have shown that for any \( k \in \mathbb{N} \cup \{0\} \)
\[ \sup_{\Omega_2} |\nabla^k u_\varepsilon| \leq C(k, \Omega_2, \Omega_3) \|u\|_{L^1(\Omega_4)}, \]
and the right-hand side is finite since \( u \in L^1_{\text{loc}}(\Omega) \) and \( \Omega_4 \subset \subset \Omega \).

This estimate holds for any \( \varepsilon > 0 \), so \( u_\varepsilon \) and all its derivative are uniformly equicontinuous (in \( \varepsilon \)). By Arzela-Ascoli (and a diagonal argument in \( k \)) we find a converging subsequence \( \varepsilon \to 0 \) and a function \( u_0 \in C^\infty(\Omega_2) \) such that for any \( k \in \mathbb{N} \cup \{0\} \)
\[ |\nabla^k u_\varepsilon(x) - \nabla^k u_0(x)| \xrightarrow{\varepsilon \to 0} 0 \quad \text{locally uniformly in} \ \Omega_2. \]

We claim that \( u = u_0 \) in almost every point (since \( u \) is an \( L^1_{\text{loc}} \)-function it is actually a the class of maps equal up to almost every point, \( u_0 \) is a continuous representative of the class \( u \)). Indeed, by the normalization \( \int \eta = 1 \) which implies \( \int \eta_\varepsilon = 1 \) we have
\[ |u_\varepsilon(x) - u(x)| = \left| \int \eta_\varepsilon(y - x)(u(y) - u(x)) \, dy \right| \leq C(\eta) \int_{B(x, \varepsilon)} |u(y) - u(x)| \, dy. \]

So, by the Lebesgue differentiation theorem, we have for almost every \( x \in \Omega_2 \),
\[ \lim_{\varepsilon \to 0} |u_\varepsilon(x) - u(x)| = 0, \]
\(^5\)observe for \( \tilde{z} = z/\varepsilon \) we have in \( n \) space dimensions \( d\tilde{z} = \varepsilon^{-n} dz \)
that is
\[ u_0 = u \text{ a.e. in } \Omega_2. \]
Thus \( u \in C^\infty(\Omega_2) \), and \( \Delta u = 0 \) in classical sense in \( \Omega_2 \).

Since this holds for any \( \Omega_2 \subset \Omega \) we have shown

\[ u \in C^\infty(\Omega), \text{ and } \Delta u = 0 \text{ in classical sense in } \Omega. \]

\[ \square \]

**Corollary 2.42** (Liouville). Let \( u \in C^2(\mathbb{R}^n) \) and \( \Delta u = 0 \) in all of \( \mathbb{R}^n \). If \( u \) is a bounded function then \( u \equiv \text{const} \).

**Proof.** Fix \( x_0 \in \mathbb{R}^n \). In view of Lemma 2.41 we have for such a function \( u \), for any radius \( r > 0 \),

\[ |Du(x_0)| \leq \frac{C}{r^{n+1}} \|u\|_{L^1(B(x_0,r))} \]

If \( u \) is bounded,

\[ \|u\|_{L^1(B(x_0,r))} \leq C r^n \sup_{\mathbb{R}^n} |u| < \infty \]

and thus

\[ |Du(x_0)| \leq C r^{-1} \sup_{\mathbb{R}^n} |u|. \]

This holds for any \( r > 0 \), so if we let \( r \to \infty \), we get

\[ |Du(x_0)| = 0, \]

which holds for any \( x_0 \in \mathbb{R}^n \). That is, \( Du \equiv 0 \), and by the fundamental theorem of calculus this means \( u \) is a constant. \( \square \)

2.10. **Methods from Calculus of Variations – Energy Methods.** As we have seen, comparison principles is a strong tool for uniqueness (and also existence). These arguments also work in some situations of nonlinear pdes, where the theory of distributional solutions does not work, but the theory of Viscosity solutions can be applied, see [Koike, 2004].

On the other hand, the comparison methods are (currently) restricted to first or second-order equations, and to scalar equations. For systems or higher-order PDEs they seem not to be that helpful.

In this section we have a short look on energy methods, which is a basic tool of distributional theory. They do not rely on any comparison principle, and they are often used for higher-order differential equations and systems. On the other hand for some fully nonlinear equations (“non-variational” equations, equations “not in divergence form”) they cannot be well applied.

The ideas should be reminiscent of the arguments we employed for the weak solutions in Theorem 2.40.
Assume that we have
\[
\begin{cases}
\Delta u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]  
(2.25)

We have seen before Theorem 2.40 that this equation is related to the integral equation
\[
\int_{\Omega} Du \cdot D\phi + f\phi = 0 \quad \forall \phi \in C^\infty_c(\Omega).
\]
The interesting point is that this expression is a Frechet-Derivative of a function acting on the map $u$ in direction $\phi$.

Indeed one can characterize solutions as minimizers of an energy functional. This is sometimes called the Dirichlet principle.

**Theorem 2.43** (Energy Minimizers are solutions and vice versa). Assume $f \in C^0(\Omega)$.

Denote the class of permissible functions
\[
X := \{ u \in C^2(\Omega), \quad u = 0 \quad \text{on } \partial \Omega \}
\]
and define the energy
\[
\mathcal{E}(u) := \int_{\Omega} \frac{1}{2} |Du|^2 + fu.
\]

Let $u \in X$ be a minimizer of $\mathcal{E}$ in $X$, i.e.
\[
\mathcal{E}(u) \leq \mathcal{E}(v) \quad \forall v \in X.
\]

Then $u$ solves (2.25).

Conversely, if $u \in X$ solves (2.25), then $u$ is a minimizer of $\mathcal{E}$ in the set $X$.

**Proof.** We compute what is called the Euler-Lagrange-equations of $\mathcal{E}$: Let $\phi \in C^\infty_c(\Omega)$, then certainly $u + t\phi \in X$ for all $t \in \mathbb{R}$. That is the minimizing property says that the function
\[
E(t) := \mathcal{E}(u + t\phi)
\]
has a minimum in $t = 0$. By Fermats theorem (one checks easily that $E$ is differentiable in $t$)
\[
\left. \frac{d}{dt} \right|_{t=0} E(t) = E'(0) = 0.
\]

Now observe that
\[
\left. \frac{d}{dt} \right|_{t=0} |Du + t\phi|^2 = 2\langle Du, D\phi \rangle
\]
and
\[
\left. \frac{d}{dt} \right|_{t=0} f(u + t\phi) = f\phi.
\]

Thus, we arrive at
\[
0 = \left. \frac{d}{dt} \right|_{t=0} E(t) = \int_{\Omega} Du \cdot D\phi + f\phi = 0.
\]
That is, \( u \) is a \textit{weak solution} of (2.25). But \( u \in C^2(\Omega) \), so we argue similar to the proof of Proposition 2.39:

By an integration by parts (for \( \varphi \in C^\infty_0(\Omega) \) there are no boundary terms), we thus have

\[
0 = \int_{\Omega} D u \cdot D \varphi + f \varphi = -\int_{\Omega} (\Delta u - f) \varphi.
\]

Since \( \Delta u - f \) is continuous, and the last estimate holds for any smooth \( \varphi \in C^\infty_0(\Omega) \) we get that (as for Proposition 2.39, or otherwise by the fundamental lemma of calculus of variations, Lemma 2.44,

\[
\Delta u - f = 0.
\]

That is the first claim is proven: minimizers are solutions.

For the converse assume \( u \) solves (2.25). Let \( w \) be any other map in \( X \). Then we have

\[
\int_{\Omega} (\Delta u - f)(u - w) = 0.
\]

Observe that \( u \) and \( w \) have the same boundary value 0 on \( \partial \Omega \). Thus, when we perform the following integration by parts we do not find boundary terms,

\[
(2.26) \quad 0 = -\int_{\Omega} \nabla u \cdot \nabla (u - w) + f(u - w) = 0.
\]

Now we compute (using Young’s inequality or Cauchy-Schwarz \( 2ab \leq a^2 + b^2 \))

\[
\int_{\Omega} |\nabla u|^2 + fu \overset{(2.26)}{=} \int_{\Omega} \nabla u \cdot \nabla w + fw \\
\leq \int_{\Omega} \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |\nabla w|^2 + fw \\
= \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \mathcal{E}(w)
\]

Subtracting \( \frac{1}{2} \int_{\Omega} |\nabla u|^2 \) from both sides in the estimate above we obtain

\[
\mathcal{E}(u) \leq \mathcal{E}(w).
\]

That is, we have shown: if \( u \) solves the equation, then \( u \) is a minimizer. \( \square \)

Above we have used the following statement for continuous functions. It is worth recording that this works also for locally integrable functions.

**Lemma 2.44** (Fundamental Lemma of the Calculus of Variations). Let \( \Omega \subset \mathbb{R}^n \) be any open set and assume \( f \in L^1_{\text{loc}}(\Omega) \), i.e. for any \( \Omega' \subset \subset \Omega \) we have

\[
\int_{\Omega'} |f| < \infty.
\]

1. If

\[
\int_{\Omega} f(x) \varphi(x) \geq 0 \quad \text{for all } \varphi \in C^\infty_c(\Omega) \text{ that are nonnegative, } \varphi \geq 0,
\]
then
\[ f \geq 0 \quad \text{almost everywhere in } \Omega. \]

(2) If
\[ \int_{\Omega} f(x) \varphi(x) = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega) \text{ that are nonnegative, } \varphi \geq 0, \]
then
\[ f \equiv 0 \quad \text{almost everywhere in } \Omega. \]

The proof is left as an exercise, it is a combination of convolution arguments as in Theorem 2.40 and the argument used for Proposition 2.39.

**Theorem 2.45** (Uniqueness). Assume \( f \in C^0(\overline{\Omega}) \cap L^1(\Omega) \)

Denote the class of permissible functions
\[ X := \{ u \in C^2(\overline{\Omega}), \ u = 0 \quad \text{on } \partial \Omega \} \]

Then there is at most one solution \( u \in X \) to (2.25)

**Proof.** Assume \( u, w \in X \) are two solutions, then
\[ \Delta (u - w) = 0. \]

Multiplying by \( u - w \) and integrating by parts (observe that there are no boundary terms since \( u = w \) on \( \partial \Omega \), we obtain
\[ \int_{\Omega} |\nabla (u - w)|^2 = 0. \]

But this implies \( \nabla (u - w) \equiv 0 \), so \( u - w \equiv \text{const} \). Since \( u = w \) on the boundary that constant is zero, and \( u \equiv w \). \( \square \)

**Exercise 2.46.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set with smooth boundary. Assume \( f \in C^0(\overline{\Omega}) \) and \( A \in C^2(\overline{\Omega}, \mathbb{R}^{n \times n}) \), \( A \) symmetric, and all eigenvalues strictly positive in \( \overline{\Omega} \), and let \( c \in C^0(\overline{\Omega}) \).

Denote the class of permissible functions
\[ X := \{ u \in C^2(\overline{\Omega}), \ u = 0 \quad \text{on } \partial \Omega \} \]

and define the energy
\[ E(u) := \int_{\Omega} \frac{1}{2} \langle ADu, Du \rangle_{\mathbb{R}^n} + \frac{1}{2} \int c|u|^2 + fu. \]

Let \( u \in X \) be a minimizer of \( E \) in \( X \), i.e.
\[ E(u) \leq E(v) \quad \forall v \in X. \]

Then \( u \) solves
\[ \begin{cases} \text{div} (A \nabla u) - cu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega \end{cases} \]
Conversely, if \( u \in X \) solves (2.27), then \( u \) is a minimizer of \( E \) in the set \( Y \).

These methods can be extended, e.g. for higher order differential equations (where no maximum principle holds), e.g. the Neumann boundary problem. Let \( \nu : \partial \Omega \rightarrow \mathbb{R}^n \) be the outwards facing unit normal. The \textit{Neumann problem} is the equation

\[
\begin{aligned}
\Delta u &= f \quad \text{in } \Omega \\
\partial_{\nu} u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\] (2.28)

**Exercise 2.47.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set with smooth boundary. Assume \( f \in C^0(\overline{\Omega}) \).

Denote the class of permissible functions

\[ Y := \{ u \in C^2(\overline{\Omega}) \} \]

and define the energy

\[ E(u) := \int_{\Omega} \frac{1}{2}|Du|^2 + fu. \]

Let \( u \in Y \) be a minimizer of \( E \) in \( Y \), i.e.

\[ E(u) \leq E(v) \quad \forall v \in Y. \]

Then \( u \) solves (2.28).

Conversely, if \( u \in Y \) solves (2.25), then \( u \) is a minimizer of \( E \) in the set \( Y \).

**Exercise 2.48** (Uniqueness modulo constants). Let \( \Omega \subset \mathbb{R}^n \) be a connected, bounded, open set with smooth boundary. Assume \( f \in C^0(\overline{\Omega}) \). Assume \( f \in C^0(\overline{\Omega}) \) Denote the class of permissible functions

\[ Y := \{ u \in C^2(\overline{\Omega}) \} \]

Then any two solutions \( u, v \in Y \) to (2.28) must satisfy \( u - v \equiv \text{constant} \).

2.11. \textbf{Linear Elliptic equations.} From now on we often use the \textit{Einstein summation convention}, often described as “summing over repeated indices”. We write

\[ a_{ij}\partial_{ij}u \iff \sum_{i,j} a_{ij}\partial_{ij}u. \]

\[ b_i\partial_i u \iff \sum_i b_i\partial_i u. \]

\[ b_i\partial_j u \not\iff \sum_{i,j} b_i\partial_j u. \]

In particular

\[ \Delta u \iff \partial_{ii}u. \]
Second order elliptic equations are a class of equations that in some sense are governed by the Laplacian operator.

**Definition 2.49** (Linear elliptic equations). (1) (“non-divergence form”) linear second order operators are defined to be operators of the form

\[ L := a_{ij} \partial_{ij} + b_i \partial_i + c \]

for coefficients \( a_{ij}, b_i, c : \Omega \rightarrow \mathbb{R} \). They act as follows on functions \( u \in C^2(\Omega) \)

\[ Lu(x) := a_{ij}(x)\partial_{ij}u(x) + b_i(x)\partial_iu(x) + c(x)u(x). \]

\( L \) is called a constant coefficient operator, if the coefficients \( a_{ij}, b_i \) and \( c \) are all constant.

(2) (“divergence form”) linear second order operators are defined to be operators of the form

\[ L := \partial_i (a_{ij} \partial_j) + b_i \partial_i + c \]

for coefficients \( a_{ij}, b_i, c : \Omega \rightarrow \mathbb{R} \). They act as follows on functions \( u \in C^2(\Omega) \)

\[ Lu(x) := \partial_i (a_{ij}(x)\partial_ju(x)) + b_i(x)\partial_iu(x) + c(x)u(x). \]

(3) Clearly, divergence on non-divergence form are very similar if \( a_{ij} \) is smooth enough, but they are different if \( a \) is not smooth (or, has happens often in applications: \( a \) depends on \( u \)).

(4) (divergence-form or non-divergence form) operators \( L \) are called elliptic (also often called uniformly elliptic and bounded) if there exists an ellipticity constants \( \Lambda > 0 \) such that

\[ \xi^T A \xi \equiv \xi^i a_{ij} \xi^j \geq \frac{1}{\Lambda} \]

and

\[ \sup_{\Omega} |a_{ij}|, |b_i|, |c| < \infty. \]

For simplicity, although this is not strictly necessary we will below always assume \( A \) is symmetric.

**Example 2.50.**

- The operator \( \Delta \) is clearly elliptic in the above sense, with

\[ a_{ij} = \delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases} \]

- Operators like \( \text{div} (|\nabla u|^{p-2} \nabla u) \) are not (uniformly elliptic), since \( |\nabla u| = 0 \) cannot be excluded. These operators are called degenerate elliptic.

**Definition 2.51.** \( u \in C^2(\Omega) \) is called a subsolution of \(-Lu = f\) for an elliptic operator \( L \), if

\[ -Lu \leq 0 \quad \text{in } \Omega \]

and a supersolution if

\[ -Lu \geq 0 \quad \text{in } \Omega. \]

\( u \in C^2(\Omega) \) is called a solution if it is both sub- and supersolution.
In the following we will restrict ourselves to elliptic non-divergence operators!

2.12. Maximum principles for linear elliptic equations. The first result is a generalization of the weak maximum principle for $\Delta$, Corollary 2.18.

**Theorem 2.52** (Weak maximum principle for $c = 0$). Let $\Omega \subset\subset \mathbb{R}^n$, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be an $L$-subsolution, i.e.

\begin{equation}
-Lu \leq 0 \quad \text{in } \Omega
\end{equation}

If $L$ is (non-divergence form) linear elliptic operator with $c \equiv 0$, then

$$
\sup_{\Omega} u = \sup_{\partial\Omega} u.
$$

If instead of (2.29) we have

\begin{equation}
-Lu \geq 0 \quad \text{in } \Omega
\end{equation}

then

$$
\inf_{\Omega} u = \inf_{\partial\Omega} u.
$$

**Proof.** First we assume instead of (2.29)

\begin{equation}
-Lu > 0 \quad \text{in } \Omega
\end{equation}

Clearly, by continuity of $u$ in $\overline{\Omega}$,

$$
\sup_{\Omega} u \geq \sup_{\partial\Omega} u
$$

If we had

$$
\sup_{\Omega} u > \sup_{\partial\Omega} u,
$$

then we would find the global (and thus a local) maximum $x_0 \in \Omega$, at which we have $Du(x_0) = 0$ and $D^2u(x_0) \leq 0$. But this implies (recall $c \equiv 0$)

$$
Lu(x_0) = a_{ij}(x_0)\partial_{ij}u(x_0) + b_i(x_0)\partial_iu(x_0)
$$

Since $a_{ij}(x_0)$ is elliptic, and $\partial_{ij}u(x_0) \geq 0$ we have

$$
a_{ij}(x_0)\partial_{ij}u(x_0) \geq 0.
$$

(This is a general Linear Algebra fact, if $A, B$ are symmetric, nonnegative matrices, then their Hilbert-Schmidt Scalar product $A : B := a_{ij}b_{ij} \geq 0$, Exercise 2.54.) That is, we have

$$
Lu(x_0) \geq 0
$$

which is a contradiction to (2.30).

We conclude that under the assumption (2.30) we have

$$
\sup_{\Omega} u = \sup_{\partial\Omega} u.
$$
In order to weak the assumption to (2.30) we consider, for some $\gamma > 0$, $v_\gamma(x) := e^{\gamma x_1}$, where $x_1$ is the first component of $x = (x_1, \ldots, x_n)$. Observe that

$$Lv_\gamma(x) = \left(a_{11}(x)\gamma^2 + b_1(x)\gamma\right)e^{\gamma x_1}$$

Since $L$ is elliptic we have $a_{11} \geq \frac{1}{\Lambda}$ and $b_1 \geq -\Lambda$, so

$$Lv_\gamma(x) = a_{11}(x)\gamma^2 + b_1(x)\gamma \geq e^{\gamma x_1}\left(\frac{1}{\Lambda}\gamma - \Lambda\right).$$

If we choose $\gamma = 3\Lambda$ we thus find

$$Lv_\gamma(x) > 0 \quad \text{in } \Omega.$$  

Consequently, under the assumption (2.29) we have for any $\varepsilon > 0$, for $w_\varepsilon := u + \varepsilon v_\gamma$,

$$Lw_\varepsilon(x) > 0 \quad \text{in } \Omega.$$  

and thus by the first step

$$\sup_{\Omega} w_\varepsilon = \sup_{\partial \Omega} w_\varepsilon$$

Since $w_\varepsilon = u + \varepsilon v_\gamma$ and $v_\gamma$ is continuous (and $\Omega$ is bounded) we have

$$\left|\sup_{\Omega} u - \sup_{\partial \Omega} u\right| \leq C(\Omega)\varepsilon.$$  

Letting $\varepsilon \to 0$ we obtain the claim.

The inf claim follows by taking $-u$ instead of $u$. \hfill \Box

**Exercise 2.53.** Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrices, i.e. $A^t = A$. Show that the following two conditions are equivalent

1. $A \geq 0$ in the sense of matrices, i.e.
   $$\xi^t A \xi \geq 0$$
2. all eigenvalues of $A$ are nonnegative.

**Exercise 2.54.** Let $A, B \in \mathbb{R}^{n \times n}$ be two symmetric matrices, i.e. $A^t = A$, $B^t = B$. Assume that $A, B \geq 0$ in the sense of matrices, i.e.

$$\xi^t A \xi \geq 0, \quad \xi^t B \xi \geq 0 \quad \forall \xi \in \mathbb{R}^n$$

Show that

$$A : B \equiv \sum_{i=1}^n \sum_{j=1}^n A_{ij} B_{ij} \geq 0.$$  

Also in the case $c \neq 0$ a type of weak maximum principle holds (essentially mimicking the above argument):
Theorem 2.55 (Weak maximum principle for \(c \leq 0\)). Let \(\Omega \subset \subset \mathbb{R}^n\), and consider
\[
L := a_{ij}(x) \partial_{ij} + b_i(x) \partial_i + c(x).
\]
where \(c \leq 0\) in \(\Omega\).

Assume \(u \in C^2(\Omega) \cap C^0(\overline{\Omega})\).

(1) If \(u\) solves
\[
-Lu \leq 0 \quad \text{in } \Omega
\]
Then
\[
\sup_{\Omega} u \leq \sup_{\partial \Omega} u_+,
\]
where \(u_+\) denotes the positive part of \(u\), namely
\[
u_+ = \max\{0, u\}.
\]

(2) If on the other hand \(u\) solves
\[
-Lu \geq 0 \quad \text{in } \Omega
\]
we have
\[
\inf_{\Omega} u \geq \inf_{\partial \Omega} (-u_-),
\]
where \(u_-\) denotes the positive part of \(u\), namely
\[
u_- = \max\{0, u\}, \quad u_- = -\min\{0, u\}
\]

(3) In particular, if \(Lu = 0\) then
\[
\sup_{\Omega} |u| \leq \sup_{\partial \Omega} |u|
\]

Proof. Let us assume \(-Lu \leq 0\). First we observe that if
\[
\sup_{\Omega} u \leq 0
\]
then there is nothing to show, since we have \(u_+ \geq 0\) by definition and thus
\[
\sup_{\Omega} u \leq 0 \leq \sup_{\partial \Omega} u_+.
\]
So w.l.o.g. we may assume that \(\sup_{\Omega} u > 0\). Set
\[
\Omega_+ := \{x \in \Omega : u(x) > 0\} \neq \emptyset
\]
Since \(u\) is continuous \(\Omega_+ = u^{-1}((0, \infty))\) is a nonempty, open set.

Define the elliptic operator \(L_0\) by
\[
L_0u := Lu - cu = a_{ij} \partial_{ij} u + b_i \partial_i u.
\]
Since \(-Lu \leq 0\) we have \(-L_0u \leq cu \leq 0\) in \(\Omega_+\) — since by assumption \(c \leq 0\). So, using the weak maximum principle for \(c \equiv 0\), Theorem 2.52,
\[
\sup_{\Omega} u \overset{u \leq 0: \Omega \setminus \Omega_+}{\leq} \sup_{\Omega_+} u \overset{\text{2.52}}{=} \sup_{\partial \Omega_+} u = \sup_{\partial \Omega_+} u_+ \leq \sup_{\partial \Omega} u_+.
\]
In the last step we used that $\partial \Omega_+ \subset \overline{\Omega}$ can be split into two parts: the part $\partial \Omega_+ \subset \Omega$ (on this part we have $u = u_+ = 0$), and the part $\partial \Omega_+ \subset \partial \Omega$ where $u_+ \geq 0$.

This settles the claim for $-Lu \leq 0$.

If we assume $-Lu \geq 0$ then $-u$ satisfies $-L(-u) \geq 0$, and we obtain the claim from the previous case

$$-\inf_{\Omega} u = \sup_{\Omega} (-u) \leq \sup_{\partial \Omega^+} (-u) = \sup_{\partial \Omega^-} u = -\inf_{\partial \Omega^-} (-u)$$

so

$$\inf_{\Omega} u \geq \inf_{\partial \Omega^-} (-u).$$

For the last case assume that $-Lu = 0$. By the arguments before we have then (observe that $|u| = u_+ + u_-$).

$$\sup_{\Omega} u \leq \sup_{\partial \Omega^+} u_+ \leq \sup_{\partial \Omega^+} |u|.$$

and

$$\inf_{\Omega} u \geq \inf_{\partial \Omega^-} (-u),$$

which can be rewritten as

$$-\inf_{\Omega} u \leq -\inf_{\partial \Omega^-} (-u) = \sup_{\partial \Omega^-} (u) \leq \sup_{\partial \Omega^-} |u|.$$

Now at least one of the following cases holds:

$$\sup_{\Omega} |u| = \sup_{\Omega} u, \quad \text{or} \quad \sup_{\Omega} |u| = -\inf_{\Omega} u$$

but in both cases the estimates above imply

$$\sup_{\Omega} |u| \leq \sup_{\partial \Omega} |u|$$

\[\square\]

Exercise 2.56 (Counterexample for $c \geq 0$). Consider

$$Lu = \Delta u + 5u$$

for $\Omega = (-1,1) \times (-1,1)$. Take

$$u = (1 - x^2) + (1 - y^2) + 1$$

Show that

1. $-Lu \leq 0$ in $\Omega$
2. $\sup_{\Omega} u \geq u(0) = 3$
3. $\sup_{\partial \Omega} u = 2$
4. Why is this no contradiction to Theorem 2.55?
Corollary 2.57 (Eigenvalues of $\Delta$). $\Delta$ with Dirichlet-boundary has no nonnegative eigenvalues. Namely there is no nontrivial solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ for $\lambda \geq 0$ to

$$\begin{cases}
\Delta u = \lambda u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

(Here, nontrivial means $u \not\equiv 0$).

Proof. The above equation is for $L := \Delta - \lambda$ equivalent to

$$\begin{cases}
-Lu = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

Since $\lambda \geq 0$, Theorem 2.55 is applicable, so for any solution to the above equation we’d have

$$\sup_{\Omega} |u| \leq \sup_{\partial \Omega} |u| = 0.$$ 

Thus $u \equiv 0$, i.e. $u$ is the trivial solution. \qed

As it was the case for the $\Delta$-operator, Theorem 2.22, the weak maximum principle implies uniqueness results.

Corollary 2.58 (Uniqueness for the Dirichlet problem). Let $L$ be as above a non-divergence form linear elliptic operator, $\Omega \subset\subset \mathbb{R}^n$ with smooth boundary, $c \leq 0$, $f \in C^0(\Omega)$, $g \in C^0(\partial \Omega)$. Then there exists at most one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of the Dirichlet boundary problem

$$\begin{cases}
Lu = f & \text{in } \Omega \\
u = g & \text{on } \partial \Omega
\end{cases}$$

Exercise 2.59. Prove Corollary 2.58.

Corollary 2.60 (Comparison principle). Let $L$ be a linear elliptic differential operator (non-divergence form), and assume that $c \leq 0$ in $\Omega \subset\subset \mathbb{R}^n$. Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy $-Lu \leq -Lv$ in $\Omega$. Then $u \leq v$ on $\partial \Omega$ implies $u \leq v$ in $\Omega$.

Exercise 2.61. Prove Corollary 2.60

Corollary 2.62 (Continuous dependence on data). Let $L$ be a linear elliptic differential operator (non-divergence form), and assume that $c \leq 0$ in $\Omega \subset\subset \mathbb{R}^n$.

Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\begin{cases}
-Lu = f & \text{in } \Omega \\
u = g & \text{on } \partial \Omega
\end{cases}$$

where $f \in C^0(\overline{\Omega})$ and $g \in C^0(\partial \Omega)$.
Then for some constant $C = C(a, b, c, \Omega)$ we have

$$\sup_{\Omega} |u| \leq C \left( \sup_{\partial \Omega} |g| + \sup_{\Omega} |f| \right).$$

**Exercise 2.63.** Prove Corollary 2.62.

**Hint:** Set $v_\lambda := u + \lambda e^{\mu|x-x_0|^2} \sup_{\Omega} |f|$ where $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$. Choose $\mu \gg 1$. Then choose $\lambda$ so that $Lv_\lambda \leq 0$ and use the weak maximum principle. Then choose $\lambda$ so that $Lv_\lambda \geq 0$, and again use the weak maximum principle.

Our next goal is the the strong maximum principle, for this we use the following result by Hopf:

**Lemma 2.64** (Hopf Boundary point Lemma). Let $B \subset \mathbb{R}^n$ be a ball, and let $L$ be as above. Let $u \in C^2(B) \cap C^0(\overline{B})$ and assume that for $x_0 \in \partial B$ we have

- $u(x) < u(x_0)$ for all $x \in B$
- $-Lu \leq 0$ in $B$.
- One of the following
  1. $c \equiv 0$
  2. $c \leq 0$ and $u(x_0) \geq 0$
  3. $u(x_0) = 0$

Then for $\nu$ the outwards facing normal of $B$ at $x_0$ (i.e. if $B = B(y_0, \rho)$ then for $\nu = \frac{y_0-x_0}{\rho}$

$$\partial_{\nu}u(x_0) > 0,$$

if that derivative exists.
An illustration of the setup of Lemma 2.64 is in Figure 2.4. Observe that \( \partial_\nu u(x_0) \geq 0 \) is clear, the Hopf-Lemma says this must be a strict inequality!

**Proof.** W.l.o.g. we may assume

\[
B = B(0,R), \quad c \leq 0, \quad u(x_0) = 0, \quad u < 0 \quad \text{in } B(0,R):
\]

Indeed, the condition \( B = B(0,R) \) can be assumed simply by shifting. As for the other conditions set (recall that \( c_+ = \max\{c, 0\} \))

\[
\tilde{L} := L - c_+.
\]

and

\[
\tilde{u} := u - u(x_0).
\]

Then in \( B \),

\[
-\tilde{L}\tilde{u} = -(L - c_+)(u - u(x_0)) = -Lu + c_+u + cu(x_0) - c_+(u - u(x_0)) + cu(x_0)
\]

If \( c \equiv 0 \) then we readily have \( -\tilde{L}\tilde{u} \leq 0 \).

If \( c \leq 0 \) we have \( c_+ \equiv 0 \), and again obtain \( -\tilde{L}\tilde{u} \leq 0 \).

If \( u(x_0) = 0 \) then \( c_+u \leq 0 \), since \( u \leq u(x_0) = 0 \) by assumption.

Since \( c - c_+ \leq 0 \) we observe that \( \tilde{L} \) is an operator that satisfies the missing conditions in (2.31). Thus, indeed, (2.31) can be assumed w.l.o.g.

So assume (2.31) from now on.

Set for some \( \alpha > 0 \)

\[
v_\alpha(x) := e^{-\alpha|x|^2} - e^{-\alpha R^2}.
\]

Clearly \( 0 \leq v_\alpha \leq 1 \) in \( B = B(0,R) \). Moreover

\[
v_\alpha \equiv 0 \quad \text{on } \partial B(0,R).
\]

For \( \rho \in (0,R) \) denote by \( A(\rho,R) \) the annulus \( B(0,R) \setminus B(0,\rho) \). We will show next

\[
(2.32) \quad \text{For any } \rho \in (0,R) \text{ there exists } \alpha > 0 \text{ such that } -Lv_\alpha < 0 \text{ in } A(\rho,R)
\]

For this we first compute

\[
(2.33) \quad \partial_i v_\alpha(x) = -2\alpha x_i e^{-\alpha |x|^2}.
\]

Next we compute

\[
\partial_{ij} v_\alpha(x) = \left(-2\alpha \delta_{ij} + 4\alpha^2 x_i x_j\right)e^{-\alpha |x|^2}
\]

so (using the ellipticity conditions, \( a_{ij}x_i x_j \geq \lambda |x|^2 \), and \( |a|, |b|, |c| \leq \Lambda \),

\[
-Lv(x) = -a_{ij} \partial_{ij} v - b_i \partial_i v - cv
\]

\[
= -a_{ij} \left(-2\alpha \delta_{ij} + 4\alpha^2 x_i x_j\right)e^{-\alpha |x|^2} - b_i \left(-2\alpha x_i e^{-\alpha |x|^2}\right) - ce^{-\alpha |x|^2} + ce^{-\alpha R^2}
\]

\[
\leq \left(2\alpha \Lambda - 4\alpha^2 \lambda |x|^2 + 2\alpha \Lambda |x| + \Lambda\right)e^{-\alpha |x|^2}.
\]
That is, for \( x \in A(\rho, R) \),
\[
-Lv(x) \leq \left( -4\lambda^2 \rho^2 + 2\alpha \Lambda + 2\alpha \Lambda R + \Lambda \right) e^{-\alpha |x|^2}
\]
\[\leq 0 \quad \text{for } \alpha \gg 1\]

If we take \( \alpha \) large, the (negative) \( \alpha^2 \)-term dominates, that is for \( \alpha \gg 1 \) (depending on \( \rho > 0, \Lambda, \lambda \) and \( R \)) we have (2.32).

Next, we consider the equation for \( u + \varepsilon v \), which in view of (2.32) becomes

\[-L(u + \varepsilon v) < 0 \text{ in } A(\rho, R).\]

The weak maximum principle, Theorem 2.55, implies

\[\sup_{A(\rho, R)} u + \varepsilon v \leq \sup_{\partial A(\rho, R)} (u + \varepsilon v)_+.\]

The boundary \( \partial A(\rho, R) \) is the union of \( \partial B(0, R) \) and \( \partial B(0, \rho) \).

On \( \partial B(0, R) \) we know \( v \equiv 0 \) and since \( u \) is continuous and \( u < 0 \) in \( B(0, R) \) we have \( u \leq 0 \) on \( \partial B(0, R) \). That is \( (u + \varepsilon v)_+ = 0 \) on \( \partial B(0, R) \).

On \( \partial B(0, \rho) \), since \( u < 0 \) on \( B(0, R) \) we have \( \sup_{\partial B(0, \rho)} u < 0 \), and consequently, since \( v \leq 1 \) we have for all \( 0 < \varepsilon < \varepsilon_0 := -\sup_{\partial B(0, \rho)} u \)

\[u + \varepsilon v < 0 \quad \text{on } \partial B(0, \rho)\]

That is (2.34) implies

\[u + \varepsilon v \leq 0 \quad \text{in } A(\rho, R).\]

Now fix \( \rho \in (0, R) \), choose \( \varepsilon, \alpha \) so that the above is true.

Denote \( \nu := \frac{x_0}{|x_0|} \) the outwards unit normal to \( \partial B \) at \( x_0 \in \partial B \). Observe that for all small \( 0 < t \ll 1 \) (depending on \( \rho \)) we have \( x_0 - t\nu \in A(\rho, R) \).

Recall that by assumption \( u(x_0) = 0 \), then (2.35) implies for any small \( t > 0 \),

\[u(x_0 - t\nu) + \varepsilon v(x_0 - t\nu) \overset{(2.35)}{\leq} 0 = u(x_0) + \varepsilon v(x_0).\]

This leads to (again: for all \( 0 < t \ll 1 \))

\[
\frac{u(x_0 - t\nu) - u(x_0)}{t} \leq -\varepsilon \frac{v(x_0 - t\nu) - v(x_0)}{t}
\]

Letting \( t \to 0^+ \) on both sides we obtain

\[\partial_{\nu} u(x_0) \leq \varepsilon \partial_{\nu} v(x_0).\]

Observe that (2.33) implies

\[\partial_{\nu} v(x_0) = \partial_i v(x_0) (\frac{x_0}{|x_0|}) = -2\alpha \frac{|x_0|^2}{R} e^{-\alpha R^2} < 0\]

That is (2.36) implies

\[-\partial_{\nu} u(x_0) < 0\]
which implies the claim.

The Hopf Lemma, Lemma 2.64 implies the strong maximum principle.

**Corollary 2.65** (Strong maximum principle). Let $\Omega \subset \mathbb{R}^n$ be an open and connected set, (but $\Omega$ may be unbounded). Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$-Lu \leq 0 \quad \text{in } \Omega.$$

Assume either

- $c \equiv 0$, or
- $c \leq 0$ and $\sup_{\Omega} u \geq 0$.

Then we have the following: If there exists $x_0 \in \Omega$ such that

$$u(x_0) = \sup_{\Omega} u$$

then $u \equiv u(x_0)$ in $\Omega$.

**Proof.** Assume the claim is false. Via the modification as in the proof of Lemma 2.64, we may assume w.l.o.g. $u \leq 0$ in $\Omega$ and $u(x_0) = 0$ for some $x_0 \in \Omega$, but $u \not\equiv 0$.

Let

$$\Omega_- := \{x \in \Omega : u(x) < 0\}.$$

Observe that $\Omega_-$ is open ($u$ is continuous) and $\Omega_- \neq \emptyset$ (because $u \leq 0$ and $u \not\equiv 0$).

Since $x_0 \in \Omega$ and $u(x_0) = 0$, the boundary of $\Omega_-$ cannot be contained in $\partial \Omega$, i.e. we have

$$\partial \Omega_- \cap \Omega \neq \emptyset.$$

**Indeed,** this follows from connectedness: Let $\gamma \subset \Omega$ be a continuous path from $x_0$ to a point in $\Omega_-$. Then there has to be a point on $\gamma$ where $\gamma$ leaves $\Omega_-$. This point lies in $\partial \Omega_-$ and in $\Omega$.

This means we can find a point $x_1 \in \Omega_-$ which is close to $\partial \Omega_-$ but not close to $\partial \Omega$, i.e.

$$x_1 \in \Omega_-, \quad \rho := \text{dist } (x_1, \partial \Omega_-) < 10 \text{dist } (x_1, \partial \Omega).$$

By definition of the distance

$$B(x_1, \rho) \subset \Omega_-, \quad \overline{B(x_1, \rho)} \setminus \Omega_- \neq \emptyset.$$

Let $x_2 \in \partial B(x_1, \rho) \setminus \Omega_-$. Since by construction $x_2 \in \partial \Omega_- \cap \Omega$ we have $u(x_2) = 0$ by continuity. Moreover $u < 0$ in $B(x_1, \rho) \subset \Omega_-.$

Since everything takes place well within $\Omega$, the conditions of the Hopf Lemma, Lemma 2.64, are satisfied and thus for $\nu$ the outwards facing normal at $x_2$ to $\partial B(x_1, \rho)$

$$\partial_\nu u(x_2) > 0.$$
But on the other hand \( x_2 \in \Omega \) is a local maximum for \( u \), so \( Du(x_2) = 0 \), which is a contradiction. The claim is then proven. \( \square \)

A consequence of the Hopf Lemma, Lemma 2.64, and the strong maximum principle, Corollary 2.65, is the uniqueness for the Neumann problem.

**Corollary 2.66** (Uniqueness for Neumann-boundary problem). Let \( \Omega \subset \mathbb{R}^n \) be open and connected. Moreover we assume a boundary regularity of \( \partial \Omega \), the interior sphere condition\(^6\):

Assume that for any \( x_0 \in \partial \Omega \) there exists a ball \( B \subset \Omega \) such that \( x_0 \in B \).

Then the following holds for any elliptic operator as above with \( c \equiv 0 \): For any given \( f \in C^0(\Omega) \) and any \( g \in C^0(\partial \Omega) \) there is at most one solution \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) of the Neumann boundary problem

\[
\begin{align*}
-Lu &= f \quad \text{in } \Omega \\
\partial_\nu u &= g \quad \text{on } \partial \Omega,
\end{align*}
\]

up to constant functions. That means, the difference of two solutions \( u, v \) is constant, \( u - v \equiv c \).

**Proof.** The difference of two solutions \( u, v, w := u - v \) satisfies\(^7\)

\[
\begin{align*}
-Lw &\leq 0 \quad \text{in } \Omega \\
\partial_\nu w &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

Firstly, assume that there exists \( x_0 \in \Omega \) such that \( \sup_{\Omega} w = w(x_0) \). Then, by the strong maximum principle, Corollary 2.65, we have \( w \equiv w(x_0) \) and the claim is proven. If this is not the case, then there must be \( x_0 \in \partial \Omega \) with \( w(x_0) > w(x) \) for all \( x \in \Omega \). If we take a ball from the interior sphere condition of \( \partial \Omega \) at \( x_0 \) then on this ball \( B \) we can apply Hopf Lemma, Lemma 2.64, which leads to \( \partial_\nu w(x_0) > 0 \), which is ruled out by the Neumann boundary assumption \( \partial_\nu w = 0 \). \( \square \)

### 3. Heat equation

3.1. **Again, sort of a physical motivation.** This is somewhat similar to Section 2.1.

The Laplacian \( \Delta u(x) \) describes the difference between the average value of a function around a point \( x \) and the value at the point \( x \) (cf. the mean value formula)

\[
\Delta u \approx \int_{\partial B(x,r)} u - u(x).
\]

If we think of \( u \) as a temperature, then \( \Delta u(x) > 0 \) means that the material surrounding \( x \) is hotter than \( u \), and \( \Delta u(x) < 0 \) means the surroundings are colder than \( u \). Heat will flow

---

\(^6\) This condition does not allow for outwards facing cusps. One can show that every set \( \Omega \) whose boundary \( \partial \Omega \) is a sufficiently smooth manifold satisfies the interior sphere condition

\(^7\) Actually we have \( = \) in the equation below, but the argument works for \( \leq \) as well
from the hotter areas to the lower areas, and the speed of this propagation is proportional to the difference in temperature (second law of thermodynamics). That is,

$$\partial_t u = c \Delta u$$

could describe the change in heat distribution over time (where \(c\) is a material property like conductivity). So if we solve

$$\begin{cases}
\partial_t u - \Delta u = 0 & \text{in } \Omega \times (0, T) \\
u(0, \cdot) = u_0 & \text{on } \Omega \\
u(x, t) = g(x, t) & \text{on } \partial \Omega \times (0, T)
\end{cases}$$

then \(u(x, t)\) describes the heat of the body at time \(t\) at the point \(x\) in the body \(\Omega\), of a system that started with the heat distribution \(u_0\) and heat source at \(\partial \Omega\) which is \(g(x, t)\).

The equation is thus called the heat equation, or it is said that \(u\) solves the heat flow.

We can believe that as time passes, there will be less and less change in the energy, so at \(T = \infty\) maybe we have that \(\partial_t u = 0\). That is at \(T = \infty\) the solution \(u(\infty, x)\) solves

$$-\Delta u = 0$$

that is stationary solutions (could be, this is not always true) appear as \(\lim_{t \to \infty}\) of flows.

3.2. Sort of an optimization motivation. We have discussed in Section 2.10 that we can solve the equation

$$\begin{cases}
-\Delta u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

by minimizing the energy

$$E(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 - uf$$

among functions with \(u = 0\) on \(\partial \Omega\) (to make this precise we need Sobolev spaces).

So, in some sense \(\nabla E\) (which we usually write as the variation \(\delta E\) corresponds to \(\Delta u + f\). (\(\delta E = 0\) means that we have found a minimizer of this convex functional.

What is the relation to

$$\begin{cases}
\partial_t u - \Delta u = f & \text{in } \Omega \times (0, T) \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

Well, this is

$$\partial_t u = -\delta E(u).$$

If \(u\) was a finite dimensional vector, then

$$\partial_t u = -\nabla E(u)$$

would be that \(u\) follows the steepest gradient descent.
3.3. **Fundamental solution and Representation.** We consider

\[
\begin{align*}
\partial_t u - \Delta u &= f \quad \text{in} \quad \mathbb{R}^{n+1}_+ \\
u(0, \cdot) &= g \quad \text{on} \quad \mathbb{R}^n.
\end{align*}
\]

(3.1)

If \( f = 0 \), then (3.1) is called **homogeneous heat equation**. For \( f \neq 0 \) it is called **inhomogeneous**.

Trivial solutions of the homogeneous equation constant maps \( u(x, t) \equiv c \), or (not completely trivial) time-independent harmonic functions \( u(x,t):=v(x) \) with \( \Delta v = 0 \) (these are called **stationary** solutions).

For elliptic equations we had the notion of a fundamental solution, Section 2.3; There exists a similar concept for the heat equation, the **heat kernel**, which we will (formally) derive now.

If we fix \( x \in \mathbb{R}^n \) and look at (3.1) as an equation in time \( t \) then it looks like an ODE, and naively the solution should be (Duhamel principle!)

\[
u(x,t) = e^{t\Delta}u(x,0) + \int_0^t e^{(t-s)\Delta} f(x,s) \, ds.
\]

Of course, \( e^{t\Delta} \) does not make any sense for now (it can be defined via **semi-group theory**).

To make (still formally, but more precise) sense of the “ODE argument”, we use the Fourier-transform (with respect to the variables \( x \in \mathbb{R}^n \)):

Let \( u \) be a solution of \( \partial_t u = \Delta u \). Taking the Fourier transform (in \( x \)) on both sides we find

\[
\frac{d}{dt} \hat{u}(\xi,t) = \hat{\partial_t u}(\xi,t) = \hat{\Delta u}(\xi,t) = -|\xi|^2 \hat{u}(\xi,t).
\]

(There should be a constant \( c \) in front of \( -|\xi|^2 \), but we ignore that for now)

Let \( \xi \) be fixed and let

\[
v(t) = \hat{u}(\xi,t).
\]

Then the above reads as

\[
\frac{d}{dt} v(t) = -|\xi|^2 \hat{v}(t).
\]

There is one solution to this ODE (starting from a given value \( v(0) \)):

\[
v(t) = e^{-t|\xi|^2} v(0).
\]

Observe that in particular \( v(\infty) = 0 \), \( \partial_t v(\infty) = 0 \), etc. (i.e. we have strong “decay at infinity”).

Ansatz: \( v(0) = 1 \), resp. \( u(0) = \delta_0 \). This means

\[
\hat{u}(\xi,t) = e^{-t|\xi|^2}.
\]
In this case we have
\[ u(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \]
which seems to be a special solution.

**Definition 3.1.**
\[ \Phi(x, t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, & t > 0, x \in \mathbb{R}^n \\ 0, & t < 0, x \in \mathbb{R}^n \end{cases} \]
is called fundamental solution or heat kernel.

One has
\[ \partial_t \Phi - \Delta \Phi = 0, \quad \text{for} \quad t > 0 \]
and
\[ \lim_{t \to 0} \Phi(x_0, t) = \begin{cases} 0, & x_0 \neq 0 \\ \infty, & x_0 = 0. \end{cases} \]

**Lemma 3.2.**
\[ \forall t > 0 : \int_{\mathbb{R}^n} \Phi(x, t) \, dx = 1. \]

**Proof.**
\[ \int_{\mathbb{R}^n} \Phi(x, t) \, dx = \hat{\Phi}(0, t) = 1. \]

Analogously to the fundamental solution for the Laplace equation, the heat kernel \( \Phi \) generates solutions to the heat equation. Indeed, if we set
\[ u(x, t) := \Phi(\cdot, t) * g(x) \]
\[ = \int_{\mathbb{R}^n} \Phi(x - y, t)g(y) \, dy \]
Then
\[ \hat{u}(\xi, t) = (\Phi(\cdot, t) * \hat{g})(\xi) = \hat{\Phi}(\xi, t)\hat{g}(\xi). \]
That is,
\[ \hat{u}(\xi, 0) = \hat{g}(\xi), \quad (\frac{d}{dt} + |\xi|^2)\hat{u}(\xi, t) = 0. \]
Revert the Fourier-transformation to obtain
\[ \begin{cases} (\partial_t - \Delta)u = 0 & \text{in} \ \mathbb{R}^{n+1}_+ \\ u(x, 0) = g(x) & x \in \mathbb{R}^n. \end{cases} \]
Motivated by this calculation we set
\[ u(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t)g(y) \, dy. \]
Theorem 3.3 (Potential representation). Let \( g \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \). Let \( u \) as in (3.3). Then \( u \) is defined in \( \mathbb{R}^n \) and there holds:

(i) \( u \in C^\infty(\mathbb{R}^{n+1}_+) \),

(ii) \( \partial_t u - \Delta u = 0 \) in \( \mathbb{R}^{n+1}_+ \) und

(iii) \[ \forall x_0 \in \mathbb{R}^n : \lim_{(x,t) \to (x_0,0)} u(x,t) = g(x_0). \]

Next we search a potential representation for

\[ (\partial_t - \Delta)u = f \quad \text{in} \quad \mathbb{R}^{n+1}_+ \]
\[ u(\cdot,0) = 0 \quad \text{on} \quad \mathbb{R}^n. \]

From the argument in the beginning, using the inverse Fourier transform, and Duhamel principle,

\[ u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s) \, dyds. \]  

Theorem 3.4. Let \( f \in C^2(\mathbb{R}^n \times [0,\infty)) \) with compact support and let \( u \) as in (3.3). Then

(i) \( u \in C^2(\mathbb{R}^n \times (0,\infty)) \),

(ii) \( (\partial_t - \Delta)u = f \) in \( \mathbb{R}^n \times (0,\infty) \)

(iii) \( \forall x_0 \in \mathbb{R}^n : \lim_{(x,t) \to (x_0,0)} u(x,t) = 0. \)

3.4. Mean-value formula. (cf. [Evans, 2010, Chapter 2.3])

Use the fundamental solution to construct a parabolic ball, or heat ball

\[ E(x,t;r) \subset \mathbb{R}^{n+1}. \]

Definition 3.5 (Heat ball). Let \( (x,t) \in \mathbb{R}^{n+1} \). Set

\[ E(x,t;r) = \left\{ (y,s) \in \mathbb{R}^{n+1} : s \leq t, \Phi(x-y,t-s) \geq \frac{1}{r^n} \right\}. \]

Cf. Figure 3.1.

Theorem 3.6 (mean value). Let \( X \subset \mathbb{R}^{n+1} \) be open and \( u \in C^2(X) \) solve \( (\partial_t - \Delta)u = 0 \) in \( X \). Then there holds

\[ u(x,t) = \frac{1}{4\pi^n} \int_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} \, dyds \]

for all \( E(x,t;r) \subset X \).
Figure 3.1. An illustration of the heatball $E(0,0;1)$ in $n = 1$ dimension taken from [user9464, ]. Observe that this heatball is “centered” at $(x,t) = (0,0)$, i.e. a heatball always goes backwards in time.

Figure 3.2. If $\Omega$ is an open set in $\mathbb{R}^n$ then $\Omega_T := \Omega \times (0,T]$ and the parabolic boundary is $\Gamma_T := \partial \Omega \times [0,T) \cup \Omega \times \{0\}$.

3.5. Maximum principle and Uniqueness.

Definition 3.7. Let $\Omega \subset \mathbb{R}^n$ be an open set and denote with $\Omega_T := \Omega \times (0,T]$ for some time $T > 0$. It is important to note that the top $\Omega \times \{T\}$ belongs to $\Omega_T$. The parabolic boundary $\Gamma_T$ of $\Omega_T$ is the boundary of $\Omega_T$ without the top,

$$\Gamma_T = \overline{\Omega_T \setminus \Omega} = \partial \Omega \times [0,T) \cup \Omega \times \{0\}.$$  

See Figure 3.2.

Theorem 3.8. Let $U$ be bounded and $u \in C^2(U_T) \cap C^0(\overline{U_T})$ be a solution of $\partial_t u = \Delta u$ in $U_T$. Then there holds

(1) the weak maximum principle:

$$\max_{\Omega_T} u = \max_{\Gamma_T} u$$  

(2) and the strong maximum principle: If $U$ is connected and if there is $(x_0,t_0) \in U_T$ (i.e. $t_0 \in (0,T]$, $x \in U$) with

$$u(x_0,t_0) = \max_{\underline{U_T}} u,$$

then $u$ is constant on all prior times, i.e.

$$u(x,t) = u(x_0,t_0) \quad \forall (x,t) \in U_{t_0}.$$

Exercise 3.9. Show that the strong maximum principle Theorem 3.8(2) implies the weak maximum principle Theorem 3.8(1).
Proof of Theorem 3.8 (2). Suppose there is \((x_0, t_0) \in U_T\) with
\[
u(x_0, t_0) = M = \max_{U_T} u.
\] (3.10)

Since \(t_0 > 0\), there exists a small heat ball \(E(x_0, t_0, r_0) \subset U_T\) and we have by Theorem 3.6
\[
M = u(x_0, t_0) = \frac{1}{4r_0^n} \int_{E(x_0, t_0, r_0)} u(y, s) \frac{|y - x|^2}{(t - s)^2} \, ds \, dy \leq M.
\] (3.11)

Hence \(u \equiv M\) in \(E(x_0, t_0; r_0)\).

Now we need to show \(u = M\) in all of \(U_{t_0}\). It suffices to show \(u \equiv M\) in any \(U_{t_1}\) for any \(t_1 < t_0\), by continuity \(u \equiv M\) in all of \(U_{t_0}\). So let \((x_1, t_1) \in U_{t_0}\), \(t_1 < t_0\). Then there exists a continuous path \(\gamma: [0, 1] \to U\) connecting \(x_0\) and \(x_1\). In the spacetime set
\[
\Gamma(r) = (\gamma(r), rt_1 + (1 - r)t_0).
\] (3.12)

Let
\[
\rho = \max\{r \in [0, 1]: u(\Gamma(r)) = M\}.
\] (3.13)

Show that \(\rho = 1\). Suppose \(\rho < 1\). Then we use the proof above to find a heat ball
\[
E = E(\Gamma(\rho), r'),
\] (3.14)

where \(u = M\). Since \(\Gamma\) crosses \(E\) (time parameter is decreasing along \(\Gamma\)), we obtain a contradiction to the maximality of \(\rho\). \(\square\)

Exercise 3.10. Use Theorem 3.8 to show the following infinite speed of propagation:

Assume \(u \in C^2(\overline{\Omega_T})\) satisfies
\[
\begin{cases}
\partial_t u - \Delta u = 0 & \text{in } \Omega_T \\
u = 0 & \text{on } \partial\Omega \times [0, T] \\
u = g & \text{in } \Omega \times \{0\}
\end{cases}
\]

(1) Show the following: if \(g \geq 0\) in \(\Omega\) but there exists any \(x_0 \in \Omega\) such that \(g(x_0) > 0\) then \(u(x, t) > 0\) in every point in \((x, t) \in \Omega_T\).

(2) Think about how this is a non-relativistic behaviour: any at an arbitrary point influences the whole universe instantaneously.

For general \(X \subset \mathbb{R}^{n+1}\) open we have a similar maximum principle:

Exercise 3.11. In Theorem 3.15 we learned of the strong maximum principle in parabolic Cylinders. Use this to obtain the strong maximum principle in general open sets \(X\):

let \(X \subset \mathbb{R}^{n+1}\) be a bounded, open set. Assume that \(u \in C^\infty(X)\) and
\[\partial_t u - \Delta u \text{ in } X.\]
Assume moreover that for some \((x_0, t_0) \in X\) we have

\[ M := u(x_0, t_0) = \sup_{(x,t) \in X} u(x,t). \]

(1) Describe (in words) in which set \(C\) the function is necessarily constant

\[ C := \{(x,t) \in X : u(x,t) = M\}. \]

(2) Assume the set \(X\) (grey) and the point \((x_0, t_0)\) are given in the picture. Draw (in orange) the set \(C\) from the question above.

**Theorem 3.12** (Uniqueness on bounded domains). Let \(U \subseteq \mathbb{R}^n\) bounded and \(g \in C^0(\Gamma_T), f \in C^0(U_T)\). Then there is at most one solution \(C^2(U_T) \cap C^0(U_T)\) to

\[
\partial_t u - \Delta u = f \quad \text{in } U_T \\
u = g \quad \text{on } \Gamma_T.
\]

(3.15)

**Exercise 3.13.** Prove Theorem 3.12.

**Theorem 3.14.** Let \(u \in C^2(\mathbb{R}^n \times (0,T]) \cap C^0(\mathbb{R}^n \times [0,T])\) be a solution of

\[
(\partial_t - \Delta) u = 0 \quad \text{in } \mathbb{R}^n \times (0,T) \\
u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}
\]

(3.16)

with the growth condition

\[
u(x,t) \leq Ae^{a|x|^2} \forall (x,t) \in \mathbb{R}^n \times [0,T]
\]

(3.17)
for some $a, A > 0$. Then there holds
\[ \sup_{\mathbb{R}^n \times [0,T]} u \leq \sup_{\mathbb{R}^n} g. \]

Proof. It suffices to show this estimate for small times, by splitting up the time interval into many small time steps. For this reason we assume first:
\[ 4aT < 1. \]

For $\varepsilon > 0$ and $\mu$ chosen below, let
\[ v(x,t) = u(x,t) - \frac{\mu}{(T + \varepsilon - t)^{n/2}} e^{\frac{|x|^2}{4(T + \varepsilon - t)}} \]
for some $\mu > 0$. Then $v_t - \Delta v = 0$ in $\mathbb{R}^n \times [0,T]$ (observe that $t$ appears in the negative above). Theorem 3.8 implies
\[ \forall U \subset \subset \mathbb{R}^n: \max_{\overline{U}_T} v \leq \max_{\overline{U}_T} \max(\sup_{\mathbb{R}^n} g, \max_{|x|=\sup_{\partial U \times [0,T]} v(x,t)}). \]

We have
\[ v(x,0) = g(x) - \frac{\mu}{(T + \varepsilon)^{n/2}} e^{\frac{|x|^2}{4(T + \varepsilon)}} \leq \sup_{\mathbb{R}^n} g. \]

Let $U = B_R(0)$, then
\[ \max_{B_R(0) \times [0,T]} v \leq \max \left( \sup_{\mathbb{R}^n} g, \max_{|x|=R, t \in [0,T]} v(x,t) \right). \]

For $|x| = R$ and $t \in (0,T)$
\[ v(x,t) = u(x,t) - \frac{\mu}{(T + \varepsilon - t)^{n/2}} e^{\frac{|x|^2}{4(T + \varepsilon - t)}} \leq Ae^{aR^2} - \frac{\mu}{(T + \varepsilon - t)^{n/2}} e^{\frac{R^2}{4(T + \varepsilon - t)}} \leq Ae^{aR^2} - \frac{\mu}{(T + \varepsilon)^{n/2}} e^{\frac{R^2}{4(T + \varepsilon)}} \]
Since $4aT < 1$, there exist $\varepsilon > 0, \gamma > 0$, such that
\[ a + \gamma = \frac{1}{4(T + \varepsilon)} \]
and hence
\[ v(x,t) \leq Ae^{aR^2} - \frac{\mu}{(T + \varepsilon)^{n/2}} e^{aR^2 + \gamma R^2}. \]
In particular, the right term dominates for $R >> 0$: in particular for all large $R > 0$ we have $v(x,t) \leq g(0)$. So for large $R$ and $|x| = R$ we have for all $t \in (0,T)$,
\[ v(x,t) \leq g(0) \leq \sup_{\mathbb{R}^n} g \]
and so
\[
\max_{(x,t) \in BR(0) \times (0,T]} v(x,t) \leq \sup_{\mathbb{R}^n} g \quad \forall R >> 1.
\]
Letting \( R \to \infty \) we find that
\[
\sup_{\mathbb{R}^n \times [0,T]} v(x,t) \leq \sup_{\mathbb{R}^n} g,
\]
i.e.
\[
\sup_{\mathbb{R}^n \times [0,T]} \left( u(x,t) - \frac{\mu}{(T + \varepsilon - t)^{\frac{\alpha}{2}}} e^{\frac{|x|^2}{4(T + \varepsilon - t)}} \right) \leq \sup_{\mathbb{R}^n} g
\]
This holds for any any \( \mu > 0 \).

Now fix \( \rho > 0 \). Then we have in particular
\[
\sup_{B(0,\rho) \times [0,T]} v(x,t) \leq \sup_{\mathbb{R}^n} g,
\]
and thus
\[
\sup_{B(0,\rho) \times [0,T]} u(x,t) - \mu \sup_{B(0,\rho) \times [0,T]} \frac{1}{(T + \varepsilon - t)^{\frac{\alpha}{2}}} e^{\frac{|x|^2}{4(T + \varepsilon - t)}} \leq \sup_{\mathbb{R}^n} g.
\]
Letting \( \mu \to 0 \) for fixed\(^8\) \( \rho \)
\[
\sup_{B(0,\rho) \times [0,T]} u(x,t) \leq \sup_{\mathbb{R}^n} g,
\]
Now we let \( \rho \to \infty \) to conclude
\[
\sup_{\mathbb{R}^n \times [0,T]} u(x,t) \leq \sup_{\mathbb{R}^n} g,
\]
i.e. we have the claim under the assumption that \( 4aT < 1 \).

If \( 4aT \geq 1 \), we can slice the time interval \((0, T]\) into parts \((0, T_1] \cup (T_1, T_2] \cup \ldots \cup (T_K, T]\) with \( 4a(T_{i+1} - T_i) < 1 \) for all \( i \). Using the estimate in each of these time intervals we conclude. \(\square\)

**Theorem 3.15.** Let \( g \in C^0(\mathbb{R}^n), \ f \in C^0(\mathbb{R}^n \times [0, T]) \). Then there is at most one solution \( u \in C^2(\mathbb{R}^n \times (0, T]) \cap C^0(\mathbb{R}^n \times [0, T]) \) of
\[
(\partial_t - \Delta)u = f \quad \text{in} \ \mathbb{R}^n \times (0, T)
\]
\[
u = g \quad \text{on} \ \mathbb{R}^n \times \{0\}
\]
with
\[
|u(x,t)| \leq Ae^{a|x|^2} \quad \forall (x,t) \in \mathbb{R}^n \times (0,T).
\]
\(^8\)This does not work if \( \rho = \infty \)!
Exercise 3.16. Prove Theorem 3.15

Without the assumption (3.31), Theorem 3.15 may fail. These solutions are sometimes called non-physical solutions, since they grow too fast.

Exercise 3.17. (cf. [John, 1991]) Define the following Tychonoff-function,
\[ u(x, t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}. \]

Here \( g^{(k)} \) denotes the \( k \)-th derivative of \( g \), given as
\[ g(t) := \begin{cases} e^{(-t-\alpha)} & t > 0 \\ 0 & t \leq 0. \end{cases} \]

(1) Show that \( u \in C^2(\mathbb{R}^n) \cap C^0(\mathbb{R} \times [0, \infty)) \).
(2) Show moreover that
\[ (\partial_t - \Delta) u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, T), \]
\[ u(x, 0) = 0 \quad \text{für} \quad x \in \mathbb{R}^n. \]

(3) Find a different solution \( v \neq u \) of (3.32).
(4) Why (without proof) does this not contradict 3.15?

3.6. Harnack’s Principle. In the parabolic setting an “immediate” Harnack principle is not true in general, to compare sup and inf of a function one needs to wait for an (arbitrary short) amount of time.

Theorem 3.18 (Parabolic Harnack inequality). Assume \( u \in C^2(\mathbb{R}^n \times (0, T)) \cap L^\infty(\mathbb{R}^n \times [0, T]) \) and solves
\[ \partial_t u - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, T) \]
and
\[ u \geq 0 \quad \text{in} \quad \mathbb{R}^n \times (0, T) \]

Then for any compactum \( K \subset \mathbb{R}^n \) and any \( 0 < t_1 < t_2 < T \) there exists a constant \( C \), so that
\[ \sup_{x \in K} u(x, t_1) \leq C \inf_{y \in K} u(y, t_2) \]

Proof. By the representation formula, Section 3.3, and uniqueness of the Cauchy problem
\[ u(x_2, t_2) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t_2)^{\frac{n}{2}}} e^{-\frac{|x_2-y|^2}{4t_2}} u_0(y) \, dy. \]

Now, for \( t_1 < t_2 \) whenever \( |x_1|, |x_2| \leq \Lambda < \infty \), there exists a constant \( C = C(|t_1 - t_2|, \Lambda) \) so that
\[ -\frac{|x_2 - y|^2}{4t_2} \geq -\frac{|x_1 - y|^2}{4t_1} - C. \quad \forall y \in \mathbb{R}^n \]
See Exercise 3.19.

Consequently,
\[
\begin{align*}
u(x_2, t_2) \geq & \left(\frac{t_1}{t_2}\right)^\frac{n}{2} e^{-C} \int_{\mathbb{R}^n} \frac{1}{(t_1)^\frac{n}{2}} e^{-\frac{|x_1-y|^2}{4t_1}} u_0(y) \, dy = \left(\frac{t_1}{t_2}\right)^\frac{n}{2} e^{-C} u(x_1, t_1).
\end{align*}
\]

\[\square\]

**Exercise 3.19.** Show the following estimate, which we used for Harnack-principle, Theorem 3.18:

If \( K \subset \mathbb{R}^n \) is compact and \( 0 < t_1 < t_2 < \infty \), then there exists a constant \( C > 0 \) depending on \( K \) and \( t_2, t_1 > 0 \), such that
\[
\frac{|x_1 - y|^2}{t_2} \leq \frac{|x_2 - y|^2}{t_1} + C \quad \forall x_1, x_2 \in K, \ y \in \mathbb{R}^n.
\]

**Exercise 3.20** (Counterexample Harnack). (1) Let \( u_0 : \mathbb{R}^n \to [0, \infty) \) a smooth function with compact support such that \( u_0(0) = 1 \). Set
\[
u(x, t) := \int_{\mathbb{R}^n} \Phi(x - y, t) u_0(y) \quad t > 0
\]
Show that
\[
\inf_{x \in \mathbb{R}^n} u(x, t) = 0 \quad \text{for all} \ t > 0.
\]

However
\[
\sup_{x \in \mathbb{R}^n} u(x, t) > 0 \quad \text{for all} \ t > 0.
\]

Why does this not contradict Harnack’s principles, Theorem 3.18?

(2) Let us consider one space-dimension. Let \( \xi \in \mathbb{R} \) be given and \( u \) defined as
\[
u_\xi(x, t) := (t + 1)^{-\frac{1}{2}} e^{\frac{|x + \xi|^2}{4(t+1)}}.
\]
Show that \( u \) is a solution of \((\partial_t - \Delta)u = 0\) in \( \mathbb{R} \times (0, \infty) \).

Moreover show for each fixed \( t > 0 \) there is no constant \( C = C(t) > 0 \) such that
\[
\sup_{x \in [-1,1]} u_\xi(x, t) \leq C \inf_{y \in [-1,1]} u_\xi(y, t) \quad \forall \xi \in \mathbb{R}^n.
\]

Why does this not contradict Harnack’s principles, Theorem 3.18?

**Hint:** Choose \( x = -\frac{\xi}{|\xi|} \) and \( y = 0 \). What happens if \( |\xi| \to \infty \)?

### 3.7. Parabolic scaling

While we will not use it in this (short) section, let us introduce the notion of parabolic scaling.

**Exercise 3.21.** Assume that \( \Omega \subset \mathbb{R}^n \) is an open set and \( u \in C^2(\Omega), \ f \in C^0(\Omega) \) solve
\[
\Delta u = f \quad \text{in} \ \Omega
\]

Let \( r > 0 \) and set
\[
u_r(x) := u(rx), \quad f_r := f(rx).
\]
Show that
\[ \Delta u_r = r^2 f_r \quad \text{in} \quad \frac{1}{r} \Omega \]
where
\[ \frac{1}{r} \Omega = \left\{ \frac{1}{r} y : y \in \Omega \right\}. \]

If we try to get the same for the equation \((\partial_t - \Delta)u = f\) we have to use a parabolic scaling.

Exercise 3.22. Assume that \(\Omega \subset \mathbb{R}^n, T > 0\) is an open set and \(u \in C^2(\Omega)\) solves
\[ (\partial_t - \Delta)u = f \quad \text{in} \quad \Omega \times (0,T] \]
Let \(r > 0\) and set
\[ u_r(x,t) := u(rx, r^2 t), \quad f_r := f(rx, r^2 t). \]
Show that
\[ (\partial_t - \Delta)u_r = r^2 f_r \quad \text{in} \quad \frac{1}{r} \Omega \times \left(0, \frac{T}{r^2}\right] \]
where
\[ \frac{1}{r} \Omega = \left\{ \frac{1}{r} y : y \in \Omega \right\}. \]


Theorem 3.23 (Smoothness). Let \(u \in C^2(\Omega_T)\) satisfy
\[ (3.33) \quad \partial_t u = \Delta u \quad \text{in} \quad \Omega_T. \]
Then \(u \in C^\infty(\text{int}(\Omega_T)).\)

Proof. The main idea is to transform the equation in \(\Omega_T\) into an equation in \(\mathbb{R}^n \times [0,T]\) and use the representation from Theorem 3.4. This is a very common and very useful technique:

Fix some \((x_0, t_0) \in \Omega_T\), i.e. \(x_0 \in \Omega\) and \(t \in (0,T]\).
We will use the parabolic regions
\[ C(x,t;r) = \{ (y,s) : |x-y| \leq r, t-r^2 \leq s \leq t \}. \] (3.34)

We set
\[ C_1 = C(x_0,t_0;r), \quad C_2 = C(x_0,t_0;\frac{3}{4}r), \quad C_3 = C(x_0,t_0;\frac{r}{2}) \] (3.35)

for some suitably small \( r > 0 \) such that \( C_1 \subset \Omega_T \). Cf. Figure 3.3.

We now choose a cut-off function \( \eta \in C^\infty(\mathbb{R}^n \times [0,t_0]) \) with \( 0 \leq \eta \leq 1, \eta|_{C_2} \equiv 1, \eta \equiv 0 \) around \( \mathbb{R}^n \times [0,t_0] \setminus C_1 \).

Set
\[ v(x,t) = \eta(x,t)u(x,t) \quad \forall (x,t) \in \mathbb{R}^n \times (0,t_0]. \] (3.37)

This is well defined for all \( (x,t) \in \mathbb{R}^n \times (0,t_0] \) because \( \eta \equiv 0 \) where \( u \) is not defined!

Then in \( \mathbb{R}^n \times (0,t_0] \) we have the following equation (using that \( (\partial_t - \Delta)u = 0 \) in any point where \( \eta \neq 0 \).

\[
\partial_t v - \Delta v = \eta \partial_t u + \partial_t \eta u - \eta \Delta u - u \Delta \eta - 2 \langle \nabla u, \nabla \eta \rangle \\
= \partial_t (\eta u) - u \Delta \eta - 2 \langle \nabla u, \nabla \eta \rangle \\
=: f(x,t).
\] (3.38)

Observe, \( v \in C^2(\mathbb{R}^n \times [0,t_0]) \) and \( f \in C^1(\mathbb{R}^n \times [0,t_0]). \)

By Theorem 3.4
\[
v(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s) \, dy \, ds \\
= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) \left( u(y,s) \partial_t \eta(y,s) - u(y,s) \Delta \eta(y,s) \\
- 2 \langle \nabla u(y,s), \nabla \eta(y,s) \rangle \right) \, dy \, ds
\] (3.39)

If we assume \( (x,t) \in C_3 \) we see that \( \partial_t \eta(y,s), \Delta \eta(y,s), \nabla \eta(y,s) \equiv 0 \) around \( y = x \) and \( s = t \).

That is for any \( (x,t) \in C_3 \) we have
\[
v(x,t) = \int_{\mathbb{R}^n} K(x,y,s,t) \, u(y,s) \, dy \, ds
\] (3.40)

where \( K(x,\cdot,s,t) \) has uniformly compact support in \( \mathbb{R}^n \) and is \( K(\cdot) \) is smooth in all variables.
Thus $v$ is smooth and so is $u \equiv v$ around $(x_0, t_0)$.

We can make the above more precise

**Theorem 3.24 (Cauchy estimates).** For all $k, l \in \mathbb{N}$ there exists $C > 0$ such that for all $u \in C^{2,1}(U_T)$ ($u \in L^1_{\text{loc}}$ will be sufficient), solving

\begin{equation}
(\partial_t - \Delta) u = 0,
\end{equation}

there holds

\begin{equation}
\max_{C(x_0, t_0; r/2)} |D^k_x \partial^l_t u| \leq C \frac{r^{k+2l+n+2}}{r^{k+l+n+2}} \|u\|_{L^1(C(x_0, t_0; r))}
\end{equation}

for all $C(x_0, t_0; r) \subset U_T$.

**Proof.** Suppose first $(x_0, t_0) = (0, 0)$ and $r = 1$. Set

\begin{equation}
C(1) = C(0, 0; 1).
\end{equation}

Then as in the proof of Theorem 3.23 we have

\begin{equation}
u(x, t) = \int_{C(1)} K(x, t, y, s) u(y, s) \, dy \, ds \quad \forall (x, t) \in \mathcal{C}(\frac{1}{2}).
\end{equation}

Then

\begin{equation}
D^k_x \partial^l_t u(x, t) = \int_{C(1)} (D^k_x \partial^l_t K(x, t, y, s)) u(y, s) \, dy \, ds
\end{equation}

and hence

\begin{equation}
|D^k_x \partial^l_t u(x, t)| \leq C_{k,l} \|u\|_{L^1(C(1))} \quad \forall (x, t) \in \mathcal{C}(\frac{1}{2}).
\end{equation}

Thus the claim is proven for $r = 1$. For $r > 0$ and $(x_0, t_0) \in \mathbb{R}^{n+1}$ set

\begin{equation}
v(x, t) = u(x_0 + rx, t_0 + r^2t).
\end{equation}

Then

\begin{equation}
\max_{C(\frac{1}{2})} |D^k_x \partial^l_t v| \leq C_{k,l} \|v\|_{L^1(C(1))}.
\end{equation}

Hence

\begin{equation}
\max_{C(x_0, t_0; r/2)} |D^k_x \partial^l_t u| r^{k+2l} \leq C_{k,l} r^{-(n+2)} \|u\|_{L^1(C(1))}.
\end{equation}

\qed
3.9. **Variational Methods.** Consider

\[
\begin{cases}
\partial_t u - \Delta u = f & \text{in } \Omega \in (0, T)
\\
u = g & \text{on } \Omega \times \{0\}, \partial\Omega \times [0, T),
\end{cases}
\]

and we want to discuss uniqueness – but for some reason we don’t want to use maximum principles.

Assume there is another solution of the same problem, let’s call it \(v\). Then set \(w := u - v\), then that would solve

\[
\begin{cases}
\partial_t w - \Delta w = 0 & \text{in } \Omega \in (0, T)
\\
w = 0 & \text{on } \Omega \times \{0\}, \partial\Omega \times [0, T),
\end{cases}
\]

As in the Laplace equation case, we multiply this equation by \(w\), and we find

\[
\partial_t \int_{\Omega} |w|^2 = -2 \int_{\Omega} |Dw|^2
\]

Observe the right-hand side is negative (unless \(w\) is constant, then it is zero – this is called the *energy decay*). Anyways, integrating this equation we obtain

\[
\int_{\Omega} |w(t)|^2 - \int_{\Omega} |w(0)|^2 = -2 \int_{\Omega} |Dw|^2.
\]

That implies that if \(w(0) = 0\) (which it is by assumption), then \(w(T) = 0\). That is \(w(t) \equiv 0\), i.e. \(u = v\).

4. **Wave Equation**

The wave equation is written as

\[\partial_{tt} u - \Delta u = 0.\]

Alternatively we can think of it as

\[\partial_{tt} u = \Delta u.\]

In this form, we can consider it as Newton’s law: Force equals mass times acceleration. The mass is set to 1. If we think about \(u(x, t)\) as the dilation of a surface from an equilibrium state (if \(x\) is one dimensional, then height of string) then \(\Delta u(x, t)\) is proportional to the stress that this dilation exacts on the surface, i.e. the force. By Newton’s law, this force \(\Delta u\) is equal to the acceleration \(\partial_{tt} u\) – and this is the wave equation.

In one space dimension

\[\partial_{tt} u - \partial_{xx} u = (\partial_t - \partial_x) (\partial_t + \partial_x) u.\]

So we could hope by solving the one-dimension wave equation by considering solutions of

\[\partial_t u \pm u_x = 0.\]

This is a transport equation which could be solved via the method of characteristics.
In more than one space dimension this is more complicated, because $Du$ is a vector, so
\[ \partial_{tt}u - \Delta u = (\partial_t - D)(\partial_t + D) \]
does not really make sense. What would make sense it so
\[ \partial_{tt}u - \Delta u = (\partial_t - i\sqrt{-\Delta})(\partial_t + i\sqrt{-\Delta})u \]
if only we understood $\sqrt{-\Delta}$ (we can e.g. via Fourier transform). This is called the halfwave decomposition.

4.1. **Global Solution via Fourier transform.** We want to consider the wave equation
\[
\left\{ \begin{array}{ll}
(\partial_{tt} - \Delta)u = 0 & \text{in } \mathbb{R}^n \times \mathbb{R} \\
u(0, x) = u_0(x) & \text{in } \mathbb{R}^n \\
\partial_t u(0, x) = v_0(x) & \text{in } \mathbb{R}^n
\end{array} \right.
\]
If we again take the point of view that this is an ODE in time then this is a second order ODE, so the initial value problem should depend on $u(0)$ and $\partial_t u(0)$.

Let us take the Fourier transform in space, then the above becomes
\[
\left\{ \begin{array}{ll}
\partial_{tt}u(\hat{\xi}, t) + c|\xi|^2 u(\hat{\xi}, t) = 0 & \text{in } \mathbb{R}^n \times \mathbb{R} \\
u(0, x) = u_0(x) & \text{in } \mathbb{R}^n \\
\partial_t u(0, x) = v_0(x) & \text{in } \mathbb{R}^n
\end{array} \right.
\]
This is an equation of the type
\[ g''(t) = -cg(t) \]
Fundamental solutions to this equation are $\sin(\sqrt{ct})$ and $\cos(\sqrt{ct})$ – which gets messy. It is more convenient to use complex notation: For some $A \in \mathbb{C}$,
\[ g(t) = A e^{i\sqrt{c}|\xi|t} + B e^{-i\sqrt{c}|\xi|t} \]
and we must choose $A, B \in \mathbb{C}$ so that
\[ \hat{u}_0(\xi) = g(0) = A + B \]
and
\[ \hat{v}_0(\xi) = g'(0) = i\sqrt{c}|\xi|(A - B). \]
or equivalently (unless $|\xi| = 0$)
\[ \frac{\hat{v}_0(\xi)}{i\sqrt{c}|\xi|} = (A - B). \]
We add the equation for $A + B$ to the equation for $A - B$ and find
\[ A = \frac{1}{2} \hat{u}_0(\xi) + \frac{1}{2i\sqrt{c}|\xi|} \hat{v}_0(\xi) \]
and subtracting the equation for $A - B$ from the equation for $A + B$ we have

$$B = \frac{1}{2} \hat{u}_0(\xi) - \frac{1}{i \sqrt{c|\xi|}} \hat{v}_0(\xi)$$

Together we have found that

$$g(t) = \hat{u}_0(\xi) \left( \frac{1}{2} e^{i \sqrt{c|\xi|} t} + \frac{1}{2} e^{-i \sqrt{c|\xi|} t} \right) + \frac{1}{2i \sqrt{c|\xi|}} \hat{v}_0(\xi) \left( e^{i \sqrt{c|\xi|} t} - e^{-i \sqrt{c|\xi|} t} \right)$$

If we call suggestively

$$e^{it \sqrt{-\Delta}} f := \mathcal{F}^{-1} \left( e^{i \sqrt{c|\xi|} \mathcal{F} f} \right)$$

we have the semigroup representation

$$u(x, t) = \frac{e^{it \sqrt{-\Delta}} + e^{-it \sqrt{-\Delta}}}{2} u_0(x) + \frac{e^{it \sqrt{-\Delta}} - e^{-it \sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} v_0(x)$$

We next discuss the **Duhamel principle**:

If we want to consider

$$\begin{cases}
(\partial_{tt} - \Delta_x) u = f & \text{in } \mathbb{R}^n \times \mathbb{R} \\
u(0, x) = u_0(x) & \text{in } \mathbb{R}^n \\
\partial_t u(0, x) = v_0(x) & \text{in } \mathbb{R}^n 
\end{cases}$$

$$u(x, t) = \frac{e^{it \sqrt{-\Delta}} + e^{-it \sqrt{-\Delta}}}{2} u_0(x) + \frac{e^{it \sqrt{-\Delta}} - e^{-it \sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} v_0(x)$$

$$+ \int_0^t e^{i(t-s) \sqrt{-\Delta}} - e^{-i(t-s) \sqrt{-\Delta}} \sqrt{-\Delta}^{-1} f(x, s) ds.$$ 

Indeed,

$$\left. \int_0^t \frac{e^{i(t-s) \sqrt{-\Delta}} - e^{-i(t-s) \sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x, s) ds \right|_{t=0} = 0$$

$$\partial_t \left|_{t=0} \int_0^t \frac{e^{i(t-s) \sqrt{-\Delta}} - e^{-i(t-s) \sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x, s) ds \right. = 0$$
and

\[
\begin{aligned}
\partial_t \int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}} - e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,s) ds \\
= \partial_t \left( \frac{e^{i0\sqrt{-\Delta}} - e^{-i0\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,t) + \int_0^t \partial_t \frac{e^{i(t-s)\sqrt{-\Delta}} - e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,s) ds \right) \\
= \partial_t \left( \int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}} - e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,s) ds \right) \\
= \int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}} + e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,t) \\
+ \int_0^t \partial_t \frac{e^{i(t-s)\sqrt{-\Delta}} - e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,s) ds \\
= \sqrt{-\Delta} \sqrt{-\Delta}^{-1} f(x,t) + \sqrt{-\Delta} \sqrt{-\Delta}^{-1} f(x,t) \\
= f(x,t) + \Delta \int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}} - e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,s) ds \\
\end{aligned}
\]

or, in other words,

\[
(\partial_t - \Delta) \int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}} - e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,s) ds = f(x,t)
\]

4.2. Energy methods. Cf. [Evans, 2010, 2.4.3].

Consider solutions to the inhomogeneous wave equation.

\[
\begin{aligned}
\left\{ \begin{array}{ll}
(\partial_t - \Delta) u = f & \text{in } \Omega \times (0,T) \\
u = g & \text{on } \Omega \times \{0\} \cup \partial \Omega \times (0,T) \\
\partial_t u = h & \text{on } \Omega \times \{0\}
\end{array} \right.
\]

\[
(4.1)
\]

**Theorem 4.1** (Uniqueness). There exist at most one function \( u \in C^2(\bar{\Omega} \times [0,T]) \) which solves (4.1).
Proof. Assume there are two solutions \( u, v \in C^2(\overline{\Omega} \times [0, T]) \). Then we can consider \( w := u - v \) which solves

\[
\begin{cases}
(\partial_t - \Delta)w = 0 & \text{in } \Omega \times (0, T) \\
w = 0 & \text{on } \Omega \times \{0\} \cup \partial\Omega \times (0, T) \\
\partial_t w = 0 & \text{on } \Omega \times \{0\}
\end{cases}
\]  

(4.2)

For \( t \in [0, T) \) define

\[
E(t) := \frac{1}{2} \int_{\Omega} |\partial_t w(x, t)|^2 \, dx + \int_{\Omega} |Dw(x, t)|^2 \, dx.
\]

We compute the derivative of \( E \) (which we can do since \( w \in C^2 \),

\[
\dot{E}(t) = \int_{\Omega} \partial_t w(x, t) \partial_t w(x, t) \, dx + \int_{\Omega} Dw(x, t) D\partial_t w(x, t) \, dx
\]

\[
= \int_{\Omega} \partial_t w(x, t) \partial_t w(x, t) \, dx - \int_{\Omega} \text{div} (Dw(x, t)) \partial_t w(x, t) \, dx
\]

\[
= \int_{\Omega} \partial_t w(x, t) (\partial_t - \Delta)w(x, t) \, dx \tag{4.2}
\]

\[
= \int_{\Omega} \partial_t w(x, t) 0 \, dx = 0.
\]

That is we have \( \dot{E}(t) = 0 \) for all \( t \in (0, T) \)

\[
E(t) = E(0) \tag{4.2} = 0
\]

In particular \( Dw \equiv 0 \), so \( w \) is constant, and because of the boundary conditions in (4.2) we conclude \( w \equiv 0 \). Thus \( u \equiv v \).

\( \square \)

5. Black Box – Sobolev Spaces


Definition 5.1. (1) Let \( 1 \leq p \leq \infty \), \( k \in \mathbb{N} \) and \( \Omega \subset \mathbb{R}^n \) open, nonempty. The Sobolev space \( W^{k,p}(\Omega) \) is the set of functions

\[
u \in L^p(\Omega)
\]

such that for any multiindex \( \gamma, |\gamma| \leq k \) we find a function (the distributional \( \gamma \)-derivative or weak \( \gamma \)-derivative) “\( \partial^\gamma u \)” \( \in L^p(\Omega) \) such that

\[
\int_{\Omega} u \partial^\gamma \varphi = (-1)^{|\gamma|} \int_{\Omega} ”\partial^\gamma u” \varphi \quad \forall \varphi \in C_c^\infty(\Omega).
\]

Such \( u \) are also sometimes called Sobolev-functions.

(2) For simplicity we write \( W^{0,p} = L^p \).
(3) The norm of the Sobolev space $W^{k,p}(\Omega)$ is given as

$$
\|u\|_{W^{k,p}(\Omega)} = \sum_{|\gamma| \leq k} \|\partial^\gamma u\|_{L^p(\mathbb{R}^n)}
$$

or equivalently (exercise!)

$$
\|u\|_{W^{k,p}(\Omega)} = \left( \sum_{|\gamma| \leq k} \|\partial^\gamma u\|_{L^p(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}.
$$

(4) We define another Sobolev space $H^{k,p}(\Omega)$ as follows

$$
H^{k,p}(\Omega) = \overline{C^\infty(\Omega)}_{\|\cdot\|_{W^{k,p}(\Omega)}}.
$$

that is the (metric) closure or completion of the space $(C^\infty(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$. In yet other words, $H^{k,p}(\Omega)$ consists of such functions $u \in L^p(\Omega)$ such that there exist approximations $u_k \in C^\infty(\Omega)$ with

$$
\|u_k - u\|_{W^{k,p}(\Omega)} \xrightarrow{k \to \infty} 0.
$$

We will later see that $H^{k,p}$ is the same as $W^{k,p}$ locally, or for nice enough domains; and use the notation $H$ or $W$ interchangeably.

(5) Now we introduce the Sobolev space $H^{k,p}_0(\Omega)$

$$
H^{k,p}_0(\Omega) = \overline{C^\infty_c(\Omega)}_{\|\cdot\|_{W^{k,p}(\Omega)}}.
$$

We will later see that this space consists of all maps $u \in H^{k,p}(\Omega)$ that satisfy $u, \nabla u, \ldots, \nabla^{k-1} u \equiv 0$ on $\partial \Omega$ in a suitable sense (the trace sense, for a precise formulation see Theorem 5.26). – Again, later we see that $H = W$ and thus, $W^{k,p}_0(\Omega) = H^{k,p}_0(\Omega)$ for nice sets $\Omega$.

Observe that $L^p(\Omega) = W^{0,p}(\Omega) = W^{0,p}_0(\Omega)$.

(6) The local space $W^{k,p}_{loc}(\Omega)$ is similarly defined as $L^p_{loc}(\Omega)$. A map belongs to $u \in W^{k,p}_{loc}(\Omega)$ if for any $\Omega' \subset \subset \Omega$ we have $u \in W^{k,p}(\Omega')$.

**Remark 5.2.** Some people write $H^{k,p}(\Omega)$ instead of $W^{k,p}(\Omega)$. Other people use $H^k(\Omega)$ for $H^{k,2}$ – notation is inconsistent...

Some people claim that $W$ stand for Weyl, and $H$ for Hardy or Hilbert.

**Example 5.3.** For $s > 0$ let

$$
f(x) := |x|^{-s}.
$$

Observe that $f$ is only defined for $x \neq 0$, but since measurable functions need only be defined outside of a null-set this is still a reasonable function.

We have already seen, when working with fundamental solutions, that $f \in L^p_{loc}(\mathbb{R}^n)$ for any $1 \leq p < \frac{n}{s}$.

We can compute for $x \neq 0$ that

$$
\partial_i f(x) = -s |x|^{-s-2}x^i
$$

(5.1)
and by the same argument as above we could conjecture that $\partial_i f \in L^q_{\text{loc}}(\mathbb{R}^n)$ for any $1 \leq q < \frac{n}{s+1}$.

**Exercise 5.4.** Show that

(1) (5.1) holds in the distributional sense, i.e. that if $n \geq 2$ and $0 < s < n - 1$ then for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$
\int_{\mathbb{R}^n} f(x) \partial_i \varphi(x) \, dx = \int_{\mathbb{R}^n} s |x|^{-s-2} x^i \varphi(x) \, dx.
$$

(2) to conclude that $f \in W^{1,q}_{\text{loc}}(\mathbb{R}^n)$ for any $1 \leq q < \frac{n}{s+1}$.

However pointwise a.e. derivatives and the Sobolev/distributional derivative does not necessarily coincide always:

**Exercise 5.5.** Let $\Omega = (-1, 1)$ and consider the Heaviside function

$$
f(x) = \begin{cases} 
-0 & x < 0 \\
1 & x \geq 0
\end{cases}
$$

Show that

(1) $f'(x) = 0$ for a.e. $x \in (-1, 1)$

(2) $f' \notin L^1((-1, 1))$ in the sense of Sobolev spaces – i.e. $f \notin W^{1,1}((-1, 1))$.

**Hint:** You can first show

$$
\int_{\Omega} f \varphi'(x) = \varphi(0).
$$

**Exercise 5.6.** Let

$$
f(x) := \log |x|.
$$

One can show that $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ for any $1 \leq p < \infty$, and $f \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ for all $p \in [1, n)$, if $n \geq 2$.

**Exercise 5.7.** Let

$$
f(x) := \log \log 2 |x| \text{ in } B(0, 1)
$$

One can show that for $n \geq 2$, $f \in W^{1,n}(B(0, 1))$.

Moreover, for $n = 2$, in distributional sense

$$
-\Delta f = |Df|^2
$$

Observe that this serves as an example for solutions to nice differential equations that are not continuous!

**Proposition 5.8** (Basic properties of weak derivatives). Let $u, v \in W^{k,p}(\Omega)$ and $|\gamma| \leq k$. Then
\( \partial^\gamma u \in W^{k-|\gamma|,p}(\Omega) \).

(2) Moreover \( \partial^\alpha \partial^\beta u = \partial^\beta \partial^\alpha u = \partial^{\alpha+\beta} u \) if \( |\alpha| + |\beta| \leq k \).

(3) For each \( \lambda, \mu \in \mathbb{R} \) we have \( \lambda u + \mu v \in W^{k,p}(\Omega) \) and \( \partial^\alpha (\lambda u + \mu v) = \lambda \partial^\alpha u + \mu \partial^\alpha v \).

(4) If \( \Omega' \subset \Omega \) is open then \( u \in W^{k,p}(\Omega') \).

(5) For any \( \eta \in C^\infty_c(\Omega) \), \( \eta u \in W^{k,p}(\Omega) \) and (if \( k \geq 1 \)), and we have the Leibniz formula (aka product rule)

\[ \partial_i (\eta u) = \partial_i \eta \ u + \eta \partial_i u. \]

(6) If \( f : \mathbb{R} \to \mathbb{R} \) is Lipschitz and bounded, and \( u \in W^{1,p}(\Omega) \) then \( f(u) \in W^{1,p}(\Omega) \), and \( Df(u) = Df(u) \cdot Du \).

**Proposition 5.9.** \((W^{k,p}(\Omega), \| \cdot \|_{W^{k,p}(\Omega)}), (H^{k,p}(\Omega), \| \cdot \|_{W^{k,p}(\Omega)}), (H^{k,p}_0(\Omega), \| \cdot \|_{W^{k,p}(\Omega)})\) are all Banach spaces.

For \( p = 2 \) they are Hilbert spaces, with inner product

\[ \langle u, v \rangle = \sum_{|\gamma| \leq k} \int_{\Omega} \partial^\gamma u \partial^\gamma v. \]

For \( p \in (1, \infty) \) (not \( p = 1 \) and not \( p = \infty \)), \( W^{k,p}(\Omega) \) and \( W^{k,p}_0(\Omega) \) are reflexive. In particular we have the following consequence of Banach-Alaoglu:

**Theorem 5.10 (Weak compactness).** Let \( 1 < p < \infty \), \( k \in \mathbb{N} \), \( \Omega \subset \mathbb{R}^n \) open. Assume that \((f_i)_{i \in \mathbb{N}}\) is a bounded sequence in \( W^{k,p}(\Omega) \), that is

\[ \sup_{i \in \mathbb{N}} \| f_i \|_{W^{k,p}(\Omega)} < \infty. \]

Then there exists a function \( f \in W^{k,p}(\Omega) \) and a subsequence \( f_{i_j} \) such that \( f_{i_j} \) weakly \( W^{k,p} \)-converges to \( f \), that is for any \( |\gamma| \leq k \) and any \( g \in L^{p'}(\Omega) \), where \( p' = \frac{p}{p-1} \) is the Hölder dual of \( p \), we have

\[ \int_{\Omega} \partial^\gamma f_{i_j} g \xrightarrow{i \to \infty} \int_{\Omega} \partial^\gamma f g. \]

We have lower semicontinuity of the norm,

\[ \| f \|_{W^{k,p}(\Omega)} \leq \liminf_{i \to \infty} \| f_i \|_{W^{k,p}(\Omega)}. \]

The same statement holds when we replace \( W^{k,p}(\Omega) \) with \( W^{k,p}_0(\Omega) \).

5.1. **Approximation by smooth functions.** It is often ok to think of Sobolev maps as (essentially) smooth functions with bounded \( W^{k,p} \)-norm. The reason is approximation:

**Proposition 5.11 (Local approximation by smooth functions).** Let \( \Omega \) be open, \( u \in W^{k,p}(\Omega) \), \( 1 \leq p < \infty \). Set

\[ u_\varepsilon(x) := \eta_\varepsilon * u(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(y-x) u(y) \, dy \quad x \in \Omega_{-\varepsilon}. \]
Here $\eta(\varepsilon) = \varepsilon^{-n} \eta(z/\varepsilon)$ for the usual bump function $\eta \in C^\infty_c(B(0,1),[0,1]), \int_{B(0,1)} \eta = 1$. Then

\begin{enumerate}
  \item $u_\varepsilon \in C^\infty(\Omega_{-\varepsilon})$, where as before \\
  $\Omega_{-\varepsilon} := \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \varepsilon \}$ for each $\varepsilon > 0$ such that $\Omega_{-\varepsilon} \neq \emptyset$.
  \item Moreover for any $\Omega' \subset\subset \Omega$,
  $$\|u_\varepsilon - u\|_{W^{k,p}(\Omega')} \xrightarrow{\varepsilon \to 0} 0.$$ 
  We call this $W^{k,p}_{\text{loc}}$-approximation.
\end{enumerate}

If we want to approximate $W^{k,p}(\Omega)$ with functions $u \in C^\infty(\overline{\Omega})$ we need regularity of $\Omega$.

**Theorem 5.12** (Smooth approximation for Sobolev functions). Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and $\partial \Omega \in C^1$. For any $u \in W^{k,p}(\Omega)$ there exist a smooth approximating sequence $u_i \in C^\infty(\overline{\Omega})$ such that 

$$\|u_i - u\|_{W^{k,p}(\Omega)} \xrightarrow{i \to \infty} 0.$$ 

On $\mathbb{R}^n$ approximation is much easier, indeed we can approximate with respect to the $W^{k,p}$-norm any $u \in W^{k,p}(\mathbb{R}^n)$ by functions $u_k \in C^\infty_c(\mathbb{R}^n)$. That is, $W^{k,p}(\mathbb{R}^n) = W^{k,p}_0(\mathbb{R}^n)$. We could describe this as “$u \in W^{k,p}(\mathbb{R}^n)$ implies that $u$ and $k-1$-derivatives of $u$ all vanish at infinity”.

**Proposition 5.13.**  

\begin{enumerate}
  \item Let $u \in W^{k,p}(\Omega), \ p \in [1,\infty)$. If $\text{supp } u \subset\subset \mathbb{R}^n$ then there exists $u_k \in C^\infty_c(\Omega)$ such that 

  $$\|u - u_k\|_{W^{k,p}(\Omega)} \xrightarrow{k \to \infty} 0.$$ 

  \item Let $u \in W^{k,p}(\mathbb{R}^n), \ p \in [1,\infty)$. Then there exists $u_k \in C^\infty_c(\mathbb{R}^n)$ such that 

  $$\|u - u_k\|_{W^{k,p}(\mathbb{R}^n)} \xrightarrow{k \to \infty} 0.$$ 

  That is $W^{k,p}(\mathbb{R}^n) = W^{k,p}_0(\mathbb{R}^n)$

  \item Let $u \in W^{k,p}(\mathbb{R}^n_+) = \mathbb{R}^{n-1} \times (0,\infty)$. Then there exists $u \in C^\infty_c(\mathbb{R}^{n-1} \times [0,\infty))$ (i.e. $u$ may not be zero on $(x',0)$ for small $x'$) such that 

  $$\|u - u_k\|_{W^{k,p}(\mathbb{R}^n_+)} \xrightarrow{k \to \infty} 0.$$ 
\end{enumerate}

5.2. Difference Quotients. We used above, e.g. for the Cauchy estimates, Proof of Lemma 2.41 the method of differentiating the equation (e.g. that if $\Delta u = 0$ then also for $v := \partial_i u$ we have $\Delta v = 0$ – so we can easier estimates for $\partial_i u$). In the Sobolev space category this is also a useful technique. Sometimes, the “first assume that everything is smooth, then use mollification”-type argument as for Lemma 2.41 is difficult to put into
practice. In this case, a technique developed by Nirenberg, is discretely differentiating the equation (which does not require the function to be a priori differentiable):

\[ \Delta u = 0 \Rightarrow v(x) := (\Delta^e_h u)(x) := \frac{u(x + he_i) - u(x)}{h} : \Delta v = 0 \]

For this to work, we need some good estimates. Recall that (by the fundamental theorem of calculus), for $C^1$-functions $u$,

\[ \|\Delta^e_h u\|_{L^\infty} \leq \|\partial_\ell u\|_{L^\infty}. \]

This also holds in $L^p$ for $W^{1,p}$-functions $u$, which is a result attributed to Nirenberg, see Proposition 5.15.

One important ingredient is that the fundamental theorem of calculus holds for Sobolev functions:

**Lemma 5.14.** Let $u \in W^{1,1}_{\text{loc}}(\Omega)$. Fix $v \in \mathbb{R}^n$. Then for almost every $x \in \Omega$ such that the path $[x, x + v] \subset \Omega$ we have

\[ u(x + v) - u(x) = \int_0^1 \partial_\alpha u(x + tv)v^\alpha \, dt. \]

**Proposition 5.15.** (1) Let $k \in \mathbb{N}$, (i.e. $k \neq 0$), and $1 \leq p < \infty$. Assume that $\Omega' \subset\subset \Omega$ are two open (nonempty) sets, and let $0 < |h| < \text{dist}(\Omega', \partial\Omega)$. For $u \in W^{k,p}(\Omega)$ we have

\[ \|\Delta^e_h u\|_{W^{k-1,p}(\Omega')} \leq \|\partial_\ell u\|_{W^{k-1,p}(\Omega)}. \]

Moreover we have

\[ \|\Delta^e_h u - \partial_\ell u\|_{W^{k-1,p}(\Omega')} \xrightarrow{h \to 0} 0. \]

(2) Let $u \in W^{k-1,p}(\Omega)$, $1 < p \leq \infty$. Assume that for any $\Omega' \subset\subset \Omega$ and any $\ell = 1, \ldots, n$ there exists a constant $C(\Omega')$ such that

\[ \sup_{|h| < \text{dist}(\Omega', \partial\Omega)} \|\Delta^e_h u\|_{W^{k-1,p}(\Omega')} \leq C(\Omega', \ell) \]

Then we have $u \in W^{k,p}_{\text{loc}}(\Omega)$, and for any $\Omega' \subset \Omega$ we have

\[ \|\partial_\ell u\|_{W^{k-1,p}(\Omega')} \leq \sup_{|h| < \text{dist}(\Omega', \partial\Omega)} \|\Delta^e_h u\|_{W^{k-1,p}(\Omega')} . \]

If $p = \infty$ we even have $u \in W^{k,\infty}(\Omega)$ with the estimate

\[ \|\partial_\ell u\|_{W^{k-1,\infty}(\Omega)} \leq \sup_{\Omega' \subset \Omega} \sup_{|h| < \text{dist}(\Omega', \partial\Omega)} \|\Delta^e_h u\|_{W^{k-1,\infty}(\Omega')} . \]
5.3. $W^{1,\infty}$ is Lipschitz. Let $\Omega$ be a bounded set with smooth boundary $\partial \Omega \in C^\infty$ (this is not optimal).

From our definition we have $f \in W^{1,\infty}(\Omega)$ if and only if $f \in L^\infty(\Omega)$ and $Df \in L^\infty(\Omega, \mathbb{R}^n)$.

Assume that $f$ is Lipschitz, i.e. $f$ is continuous (so $f \in L^\infty(\Omega)$) and we have

$$|f(x) - f(y)| \leq \Lambda |x - y| \quad x, y \in \Omega.$$  

From Proposition 5.15, we conclude that $f \in W^{1,\infty}(\Omega)$.

The other direction is also true. Assume that $f \in W^{1,\infty}(\Omega)$. Then in particular $f \in W^{1,2}(\Omega)$, and thus by mollification we can approximate

$$f(x) = \lim_{\varepsilon \to 0} f * \eta_\varepsilon(x) \quad \text{in } \Omega' \subset \subset \Omega.$$  

The right-hand side is smooth, and we observe

$$\sup_\varepsilon \|D(f * \eta_\varepsilon)\|_{L^\infty(\Omega')} \lesssim \|Df\|_{L^\infty(\Omega)}.$$  

That is, $f * \eta_\varepsilon$ all have a Lipschitz constant which is uniformly bounded. By Arzela Ascoli we get that (up to subsequence) $f * \eta_\varepsilon \xrightarrow{\varepsilon \to 0} g$ uniformly in $\Omega'$. Thus $g$ is Lipschitz. Is $g = f$? Yes, but only almost everywhere!

Since $f * \eta_\varepsilon$ converges to $f$ in $L^2(\mathbb{R}^d)$, we have (up to subsequence) that $f * \eta_\varepsilon$ converges to $f$ almost everywhere. Thus $f = g$ a.e.

So what we have is the following

**Theorem 5.16.** Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with smooth boundary $\partial \Omega \in C^\infty$.

Then the following are equivalent

1. $u \in W^{1,\infty}(\Omega)$
2. There exists a representative $\tilde{u} \in C^{0,1}(\overline{\Omega})$ such that $u = \tilde{u}$ almost everywhere.

One can also show now that $D\tilde{u}$ (from Rademeacher’s theorem) equals a.e. $Du$ (from Sobolev space definition) – in particular we can prove Rademacher’s theorem like this: any Lipschitz function has a.e. a derivative.

In the same vein we can identify $W^{k,\infty}(\Omega)$ with $C^{k-1,1}(\overline{\Omega})$.

5.4. Embedding Theorems.

**Theorem 5.17** (Rellich-Kondrachov). Let $\Omega \subset \subset \mathbb{R}^n$, $\partial \Omega \subset C^{0,1}$, $1 \leq p \leq \infty$. Assume that $(u_k)_{k \in \mathbb{N}} \in W^{1,p}(\Omega)$ is bounded, i.e.

$$\sup_{k \in \mathbb{N}} \|u_k\|_{W^{1,p}(\Omega)} < \infty.$$  

Then there exists a subsequence $k_i \to \infty$ and $u \in L^p(\Omega)$ such that $u_{k_i}$ is (strongly) convergent in $L^p(\Omega)$, moreover the convergence is pointwise a.e..
Theorem 5.18 (Poincaré). Let $\Omega \subset \subset \mathbb{R}^n$ be open and connected, $\partial \Omega \in C^{0,1}$, $1 \leq p \leq \infty$.

Let $K \subset W^{1,p}(\Omega)$ be a closed (with respect to the $W^{1,p}$-norm) cone with only one constant function $u \equiv 0$. That is, let $K \subset W^{1,p}(\Omega)$ be a closed set such that

1. $u \in K$ implies $\lambda u \in K$ for any $\lambda \geq 0$.
2. if $u \in K$ and $u \equiv \text{const}$ then $u \equiv 0$.

Then there exists a constant $C = C(K, \Omega)$ such that

$$
\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in K.
$$

In one dimension this is an easy consequence of the fundamental theorem, and sometimes called Wirtinger’s inequality. Denote $(u) := \frac{1}{2} \int_{-1}^{1} u$ the mean value. Then

$$
\|u - (u)\|_{L^p((-1,1))} = \int_{(-1,1)} |u(x) - (u)|^p \leq \frac{1}{2} \int_{(-1,1)} \int_{(-1,1)} |u(x) - u(z)|^p \, dx \, dz
$$

$$
\leq \frac{1}{2} \int_{(-1,1)} \int_{(-1,1)} \int_{x}^{z} u'(y)dy |^p \, dx \, dz
$$

$$
\leq 2^{p-2} \int_{(-1,1)} \int_{(-1,1)} \int_{-1}^{1} |u'(y)|^p \, dy \, dx \, dz
$$

$$
= 2^p \int_{-1}^{1} |u'(y)|^p \, dy.
$$

Corollary 5.19 (Poincaré type lemma). Let $\Omega \subset \subset \mathbb{R}^n$ be open, connected, and $\partial \Omega \in C^{0,1}$.

1. There exists $C = C(\Omega)$ such that for all $u \in W^{1,p}(\Omega)$ we have

$$
\|u - (u)\|_{L^p(\Omega)} \leq C(\Omega) \|\nabla u\|_{L^p(\Omega)}
$$

2. For any $\Omega' \subset \subset \Omega$ open and nonempty there exists $C = C(\Omega, \Omega')$ such that for all $u \in W^{1,p}(\Omega)$ we have

$$
\|u - (u)\|_{L^p(\Omega)} \leq C(\Omega, \Omega') \|\nabla u\|_{L^p(\Omega)}
$$

3. There exists $C = C(\Omega)$ such that for all $u \in W^{1,p}_0(\Omega)$

$$
\|u\|_{L^p(\Omega)} \leq C(\Omega) \|\nabla u\|_{L^p(\Omega)}
$$

If $\Omega = B(x, r)$ (and in the second claim $\Omega' B(x, \lambda r)$) then $C(\Omega) = C(B(0,1)) r$ (and for the second claim: $C(\Omega, \Omega') = C(B(0,1), B(0,\lambda)) r$).

Exercise 5.20. Let $B(x_0, r) \subset \mathbb{R}^n$ a ball. Show that there exists a constant $C > 0$ independent of $r$ and $x_0$ such that the following holds

1. $\|u - (u)_{B(x_0,r)}\|_{L^p(B(x_0,r))} \leq C r \|\nabla u\|_{L^p(B(x_0,r))}$ for all $u \in W^{1,p}(B(x_0, r))$. (Here, as before $(u)_{B(x_0,r)} = |B(x_0,r)|^{-1} \int_{B(x_0,r)} u$).
2. $\|u\|_{L^p(B(x_0,r))} \leq C r \|\nabla u\|_{L^p(B(x_0,r))}$ for all $u \in W^{1,p}_0(B(x_0, r))$. 


**Theorem 5.21** (Sobolev inequality). Let \( p \in [1, \infty) \) such that \( p^* := \frac{np}{n-p} \in (1, \infty) \) (equivalently: \( p \in [1, n) \)). Then \( W^{1,p}(\mathbb{R}^n) \) embeds into \( L^{p^*}(\mathbb{R}^n) \). That is, if \( u \in W^{1,p}(\mathbb{R}^n) \) then \( u \in L^{p^*}(\mathbb{R}^n) \) and we have\(^9\)
\[
\| u \|_{L^{p^*}(\mathbb{R}^n)} \leq C(p, n) \| Du \|_{L^p(\mathbb{R}^n)}.
\]

**Corollary 5.22** (Sobolev-Poincaré embedding). Let \( u \in W^{1,p}(\mathbb{R}^n), 1 \leq p < n \). For any \( q \in [p, p^*] \) we have \( u \in L^q(\mathbb{R}^n) \) with the estimate
\[
\| f \|_{L^q(\mathbb{R}^n)} \leq C(q, n) \left( \| f \|_{L^p(\mathbb{R}^n)} + \| Df \|_{L^{p^*}(\mathbb{R}^n)} \right).
\]

**Corollary 5.23** (Sobolev-Poincaré embedding on domains). Let \( \Omega \subset \mathbb{R}^n \) and \( \partial \Omega \) be \( C^1 \). For \( 1 \leq p < n \) we have for any \( u \in W^{1,p}(\Omega) \),
\[
\| u \|_{L^{p^*}(\Omega)} \leq C(\Omega) \left( \| u \|_{L^p(\Omega)} + \| Du \|_{L^p(\Omega)} \right)
\]
Also, for any \( q \in [p, p^*] \)\(^10\)
\[
\| u \|_{L^q(\Omega)} \leq C(\Omega, q, \| u \|_{W^{1,p}(\Omega)}).
\]

If moreover \( \Omega \subset \subset \mathbb{R}^n \) and \( u \in W^{1,p}_0(\Omega) \) then
\[
\| u \|_{L^{p^*}(\Omega)} \leq C(\Omega) \| Du \|_{L^p(\Omega)}.
\]
Lastly, if \( 1 \leq p < \infty \) and \( \Omega \subset \subset \mathbb{R}^n \), \( u \in W^{1,p}(\Omega) \) then for any \( q \in [1, p^*] \) (if \( p < n \)) or for any \( q \in [1, \infty) \) (if \( p \geq n \))
\[
\| u \|_{L^q(\Omega)} \leq C(\Omega, q, p, n) \| u \|_{W^{1,p}(\Omega)}.
\]

**Theorem 5.24** (Sobolev Embedding). Let \( \Omega \subset \subset \mathbb{R}^n \) be open, \( \partial \Omega \in C^{0,1} \), \( k \geq \ell \) for \( k, \ell \in \mathbb{N} \cup \{0\} \), and \( 1 \leq p, q < \infty \) such that
\[
(5.5) \quad k - \frac{n}{p} \geq \ell - \frac{n}{q}.
\]
Then the identity is a continuous embedding \( W^{k,p}(\Omega) \hookrightarrow W^{\ell,q}(\Omega) \). That is,
\[
(5.6) \quad \| u \|_{W^{\ell,q}(\Omega)} \leq C(\| u \|_{W^{k,p}(\Omega)})
\]
If \( k > \ell \) and we have the strict inequality
\[
(5.7) \quad k - \frac{n}{p} > \ell - \frac{n}{q},
\]
then the embedding above is compact. That is, whenever \( (u_i)_{i \in \mathbb{N}} \subset W^{k,p}(\Omega) \) such that
\[
\sup_i \| u_i \|_{W^{k,p}(\Omega)} < \infty
\]

\(^9\)The optimal constant \( C(p, n) \) has actually a geometric meaning, and is related to the isoperimetric inequality, cf. [Talenti, 1976].

\(^10\)This means the following: For any \( \Lambda > 0 \) there exists a constant \( C(\Omega, q, \Lambda) \) such that
\[
\| u \|_{L^q(\Omega)} \leq C(\Omega, q, \Lambda) \quad \forall u : \| u \|_{W^{1,p}(\Omega)} \leq \Lambda.
\]
then there exists a subsequence \((u_{i_j})_{j \in \mathbb{N}}\) such that \((u_{i_j})_{j \in \mathbb{N}}\) is convergent in \(W^{\ell,q}(\Omega)\).

**Theorem 5.25** (Morrey Embedding). Let \(\Omega \subset \subset \mathbb{R}^n\) with \(\partial \Omega \in C^k\), \(k \in \mathbb{N}\). Assume that for \(p \in (1, \infty)\), \(\alpha \in (0, 1)\) and \(\ell < k\) we have

\[
k - \frac{n}{p} \geq \ell + \alpha.
\]

Then the embedding \(W^{k,p}(\Omega) \hookrightarrow C^{\ell,\alpha}(\Omega)\) is continuous.

If \(k - \frac{n}{p} > \ell + \alpha\) then the embedding is compact.

5.5. **Trace Theorems.** Let \(\Omega\) be a smoothly bounded domain, i.e. \(\partial \Omega \subset \subset \mathbb{R}^n\) is a smooth (compact) manifold.

For \(s \in (0, 1)\), \(p \in [1, \infty)\) and for \(u \in C^\infty(\partial \Omega)\) we set (one of) the fractional Sobolev space norm, often called Gagliardo-seminorm or Sobolev-Slobodeckij-norm as

\[
[u]_{W^{s,p}(\partial \Omega)} := \left( \int_{\partial \Omega} \int_{\partial \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n-s}} \, dx \, dy \right)^{\frac{1}{p}}.
\]

We say \(u \in W^{s,p}(\partial \Omega)\) if \(u \in L^p(\partial \Omega)\) and \([u]_{W^{s,p}(\partial \Omega)} < \infty\).

These spaces are sometimes called trace space, because of the following property: they describe the trace of \(W^{1,p}(\Omega)\)-function

**Theorem 5.26** (Trace theorem). Let \(\Omega\) be open, bounded domain with smooth boundary \(\partial \Omega\) and \(p \in (1, \infty)\). Then

- \(W^{1,p}(\Omega) \hookrightarrow W^{1-\frac{1}{p},p}(\partial \Omega)\) in the following sense.
  - For every \(u \in W^{1,p}(\Omega)\), if we restrict \(u \big|_{\partial \Omega}\) (in the right way), then
    \[
    [u]_{W^{1-\frac{1}{p},p}(\partial \Omega)} \lesssim \|\nabla u\|_{L^p(\Omega)}
    \]
    and
    \[
    \|u\|_{L^p(\partial \Omega)} + [u]_{W^{1-\frac{1}{p},p}(\partial \Omega)} \lesssim \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.
    \]
  - “In the right way” means that the restriction operator \(T : u \in C^\infty(\bar{\Omega}) \to C^\infty(\partial \Omega), u \mapsto u\big|_{\partial \Omega}\) has the above estimates. By density this operator than is defined for any \(u \in W^{1,p}(\Omega)\).

- For any \(u \in W^{1-\frac{1}{p},p}(\partial \Omega)\) there exists \(U \in W^{1,p}(\Omega)\) and
  \[
  \|\nabla U\|_{L^p(\Omega)} \lesssim_{p,\Omega} [u]_{W^{1-\frac{1}{p},p}(\partial \Omega)}
  \]
  and
  \[
  \|U\|_{L^p(\Omega)} + \|\nabla U\|_{L^p(\Omega)} \lesssim [u]_{W^{1-\frac{1}{p},p}(\partial \Omega)}.
  \]
The statement above holds also for \( p = \infty \) (if we recall that \( W^{1,\infty} \) are simply Lipschitz maps. For \( p = 1 \) there are versions in the spirit of the above trace (observe \( 1 - 1/1 = 0 \))

6. Existence and \( L^2 \)-regularity theory for Laplace Equation

In this section we want to discuss the basic existence and regularity theory, namely we want to show existence and regularity for the following model equation for an elliptic equation.

We will later extend this to more complicated linear equations Section 6.6.

Let \( \Omega \subset \mathbb{R}^n \) be an open set (and we shall for now always assume \( \Omega \) to be bounded and \( \partial \Omega \in C^\infty \)). We want to find a solution to

\[
\begin{cases}
-\text{div} (A \nabla u) = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

which we equivalently write as

\[
\begin{cases}
\sum_{\alpha, \beta=1}^n \partial_\alpha (A_{\alpha\beta} \partial_\beta u) = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

and using the Einstein summation convention

\[
\begin{cases}
\partial_\alpha (A_{\alpha\beta} \partial_\beta u) = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

Here \( f \) is a given datum (we discuss below what assumptions we need). \( A \in C^\infty(\overline{\Omega}, \mathbb{R}^{n \times n}) \) are symmetric matrices which are (uniformly) elliptic and bounded. It is important that each and any of the previous assumptions can be relaxed, and leads to interesting, and possibly very challenging theories – we focus on a simple, basic model case.

**Uniform boundedness** means that there exists \( \Lambda > 0 \) such that

\[
\sup_{x \in \Omega} |A| \leq \Lambda < \infty.
\]

**Uniform ellipticity** means that all eigenvalues of \( A(x) \) (which is symmetric) are positive and bounded from below. Equivalently, there exits \( \lambda > 0 \) such that

\[
\inf_{|v|=1, x \in \Omega} \langle A(x)v, v \rangle \geq \lambda > 0.
\]

The above will be our standing assumptions below (again, some assumptions can be weakened).

The following theorems describe a very typical way of action in order of finding solutions to the above equation:
Theorem 6.1 (Existence in $W^{1,2}_0$). Let $f \in \left(W^{1,2}_0(\Omega)\right)^*$, that is $f$ is a linear bounded functional on $W^{1,2}_0(\Omega)$ such that
\[ |f[\varphi]| \lesssim \|f\|_{\left(W^{1,2}_0(\Omega)\right)^*} \|\varphi\|_{W^{1,2}(\Omega)} \quad \forall \varphi \in C^\infty_c(\Omega). \]
Then there exists a solution $u \in W^{1,2}(\Omega)$ to
\[
\begin{aligned}
- \text{div} (A\nabla u) &\equiv \sum_{\alpha,\beta} \partial_\alpha (A_{\alpha\beta} \partial_\beta u) = f &\text{in } \Omega \\
u &= 0 &\text{on } \partial \Omega,
\end{aligned}
\]
in the sense that $u \in W^{1,2}_0(\Omega)$ and
\[
\int_\Omega A_{\alpha\beta} \partial_\alpha u \partial_\beta \varphi = f[\varphi] \quad \forall \varphi \in C^\infty_c(\Omega).
\]
Our particular solution $u$ satisfies
\[ \|u\|_{W^{1,2}(\Omega)} \leq C(\Omega) \|f\|_{\left(W^{1,2}(\Omega)\right)^*}. \]

Exercise 6.2. Show that
(1) if $f \in L^2(\Omega)$ then $f \in \left(W^{1,2}_0(\Omega)\right)^*$, via the identification
\[ f[\varphi] := \int_\Omega f \varphi, \quad \varphi \in W^{1,2}_0(\Omega). \]
Show that
\[ \|f\|_{\left(W^{1,2}(\Omega)\right)^*} \leq C(\Omega) \|f\|_{L^2(\Omega)}. \]
Hint: Poincaré inequality.
(2) For some $g \in L^2(\Omega)$ and $\alpha \in \{1, \ldots, n\}$ define
\[ f[\varphi] := \int_\Omega g \partial_\alpha \varphi \]
Show that $f \in (W^{1,2}(\Omega))^*$ and
\[ \|f\|_{(W^{1,2}(\Omega))^*} \leq \|g\|_{L^2(\Omega)}. \]

Theorem 6.3 (Uniqueness in $W^{1,2}_0$). For fixed $f \in \left(W^{1,2}_0(\Omega)\right)^*$ there is at most one solution $u \in W^{1,2}(\Omega)$ that solves in the above sense
\[
\begin{aligned}
- \text{div} (A\nabla u) &\equiv \sum_{\alpha,\beta} \partial_\alpha (A_{\alpha\beta} \partial_\beta u) = f &\text{in } \Omega \\
u &= 0 &\text{on } \partial \Omega,
\end{aligned}
\]
That is, if $u$ and $v$ are both solutions, then $u = v$ a.e.

Theorem 6.4 (Interior $L^2$-regularity). Let $f \in L^2(\Omega)$ and assume $u \in W^{1,2}(\Omega)$ solves
\[
\begin{aligned}
- \text{div} (A\nabla u) &= f &\text{in } \Omega \\
u &= 0 &\text{on } \partial \Omega,
\end{aligned}
\]
in the above sense. (Observe we assume nothing on the boundary).
Then $u \in W^{2,2}_{\text{loc}}(\Omega)$, and we have for any $\Omega' \subset \subset \Omega$

$$\|D^2 u\|_{L^2(\Omega')} \leq C(A,\Omega',\Omega) \left( \|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} \right).$$

More generally, if $f \in W^{k,2}(\Omega)$ then $u \in W^{k+2,2}_{\text{loc}}(\Omega)$ and for any $\Omega' \subset \subset \Omega$ we have

$$\|D^{k+2} u\|_{L^2(\Omega')} \leq C(A,k,\Omega',\Omega) \left( \|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} \right).$$

In particular if $f \in C^\infty$ then $u \in C^\infty$ – and the equation holds in the classical sense\(^{11}\).

**Remark 6.5.** In the estimate above one can replace the $\|u\|_{W^{1,2}(\Omega)}$-term on the right-hand side with $\|u\|_{L^2(\Omega)}$, cf. Exercise 6.7, e.g. get an estimate of the form

$$\|D^{k+2} u\|_{L^2(\Omega')} \leq C(\Omega',\Omega) \left( \|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right).$$

**Theorem 6.6 (Global $L^2$-regularity).** Assume $\Omega$ is a bounded open set with smooth boundary $\partial \Omega \in C^\infty$. Let $f \in L^2(\Omega)$ and assume $u \in W^{1,2}_0(\Omega)$ solves

$$\begin{cases}
-\text{div} \ (A \nabla u) = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

in the above sense.

Then $u \in W^{2,2}(\Omega)$, and we have

$$\|D^2 u\|_{L^2(\Omega)} \leq C(\Omega) \left( \|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} \right).$$

As for regularity theory: there is also $L^p$-theory, but this is way more involved, and called Calderon-Zygmund theory. We can also obtain regularity theory in the sense of Hölder spaces, which is called Schauder theory.


The PDE is in divergence form, which means its variational – which essentially means that the direct method of Calculus of Variations works:\(^{12}\)

**Proof of Theorem 6.1.** We use what is called the direct method of Calculus of Variations: Set

$$\mathcal{E}(u) := \frac{1}{2} \int_{\Omega} A_{\alpha\beta} \partial_\alpha u \partial_\beta u + f[u].$$

As in Section 2.10 one can check that there is at most one minimizer in $W^{1,2}_0(\Omega)$ of this functional, that any minimizer is a solution to (6.1) and that any solution is a minimizer.

So all that is needed to show is the existence of a minimizer with the claimed estimate.

\(^{11}\text{this last part follows from the Morrey embedding, Theorem 5.25, and then by convolution}\)

\(^{12}\text{we did something very similar in the variational methods section, Section 2.10, but we did not have the tools to show existence of a minimizer}\)
Let \( u_k \in W^{1,2}_0(\Omega) \) be a sequence that approximates \( \inf \mathcal{E} \) (this exists by the very definition of \( \inf \)),

\[
\lim_{k \to \infty} \mathcal{E}(u_k) = \inf_{W^{1,2}_0(\Omega)} \mathcal{E}.
\]

In particular, we can assume that \( \mathcal{E}(u_k) \leq \mathcal{E}(0) = 0 \) for all \( k \in \mathbb{N} \). Now observe that by ellipticity\(^{13}\),

\[
\lambda |Du|^2 \leq A_{\alpha \beta} \partial_\alpha u \partial_\beta u.
\]

Thus,

\[
\frac{\lambda}{2} \|Du_k\|_{L^2(\Omega)}^2 = \mathcal{E}(u_k) - f[u_k] \leq \mathcal{E}(0) + \|f\|_{(W^{1,2}(\Omega))^*} \|u_k\|_{W^{1,2}(\Omega)}.
\]

That is, by Poincaré inequality, Corollary 5.19,

\[
\|u_k\|_{W^{1,2}(\Omega)}^2 \leq C \|f\|_{(W^{1,2}(\Omega))^*} \|u_k\|_{W^{1,2}(\Omega)}.
\]

Dividing both sides by \( \|u_k\|_{W^{1,2}(\Omega)} \) we get

\[
\sup_k \|u_k\|_{W^{1,2}(\Omega)} \leq C \|f\|_{W^{1,2}(\Omega)}.
\]

That is, \( u_k \) is uniformly bounded in \( W^{1,2}(\Omega) \).

The property we have just shown (of the energy \( \mathcal{E} \) and the space \( W^{1,2}_0(\Omega) \)) is called \textit{coercivity}: sequences \( u_k \in W^{1,2}_0(\Omega) \) with bounded \( \sup_k \mathcal{E}(u_k) < \infty \) must satisfy \( \sup_k \|u_k\|_{W^{1,2}(\Omega)} < \infty \).

By the weak compactness theorem, Theorem 5.10, we can thus (up to taking a subsequence) assume \( u_k \) weakly converging to \( u \in W^{1,2}_0(\Omega) \), which in particular implies

\[
f[u_k] \xrightarrow{k \to \infty} f[u].
\]

We also have by symmetry of \( A^{14} \)

\[
0 \leq \int_\Omega A_{\alpha \beta} \partial_\alpha (u - u_k) \partial_\beta (u - u_k)
\]

\[
= - \int_\Omega A_{\alpha \beta} \partial_\alpha u \partial_\beta u + \int_\Omega A_{\alpha \beta} \partial_\alpha u_k \partial_\beta u_k - 2 \int_\Omega A_{\alpha \beta} \partial_\alpha (u_k - u) \partial_\beta u.
\]

Thus,

\[
\int_\Omega A_{\alpha \beta} \partial_\alpha u \partial_\beta u \leq \int_\Omega A_{\alpha \beta} \partial_\alpha u_k \partial_\beta u_k - 2 \int_\Omega A_{\alpha \beta} \partial_\alpha (u_k - u) \partial_\beta u.
\]

This holds for any \( k \in \mathbb{N} \), so taking the liminf on both sides we have, using weak convergence,

\[
\int_\Omega A_{\alpha \beta} \partial_\alpha u \partial_\beta u \leq \liminf_{k \to \infty} \int_\Omega A_{\alpha \beta} \partial_\alpha u_k \partial_\beta u_k + 2 \liminf_{k \to \infty} \left| \int_\Omega A_{\alpha \beta} \partial_\alpha (u_k - u) \partial_\beta u \right|
\]

\[
= 0 \text{ since } u_k - u \rightharpoonup 0
\]

\(^{13}\text{Einstein summation!!!}\)

\(^{14}\text{we could also use more abstractly that norms are weakly lower semicontinuous}\)
Thus, we conclude

\[ E(u) \leq \liminf_{k \to \infty} E(u_k) = \inf_{W_0^{1,2}(\Omega)} E. \]

This property (again of the \textit{energy} \( E \) and the \textit{topology of weak convergence} \( W^{1,2}(\Omega) \)) is called \textit{lower semicontinuity}: if \( u_k \) converges to \( u \) \text{ w.r.t. weak convergence } \( W^{1,2}(\Omega) \) then \( E(u) \leq \liminf_{k \to \infty} E(u_k) \).

We can now conclude: since \( u \in W_0^{1,2}(\Omega) \) we also have

\[ E(u) \geq \inf_{W_0^{1,2}(\Omega)} E, \]

and thus

\[ E(u) = \inf_{W_0^{1,2}(\Omega)} E. \]

That is we have found a minimizer of \( E \). \( \square \)

\textbf{Remark:} The above is a \textit{variational technique} (Direct Method of Calculus of Variations). Other possible techniques are: \textit{Lax-Milgram}. More advanced methods are fixed point theorems (Banach, or Leray-Schauder/Schafer). Fredholm-alternative, Closed Range theorem.

6.2. \textbf{Uniqueness: Proof of Theorem 6.3.} There are two proofs of uniqueness which are both very useful (here both work – in general this might not be the case):

\textit{Proof of Theorem 6.3 by convexity.} Any solution to

\[
\begin{cases}
-\text{div}(A\nabla u) \equiv \sum_{\alpha\beta} A_{\alpha\beta} \partial_\alpha(A_{\alpha\beta} \partial_\beta u) = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

is a minimizer in \( W_0^{1,2}(\Omega) \) of

\[ E(u) := \frac{1}{2} \int_\Omega (A_{\alpha\beta} \partial_\alpha u \partial_\beta u + f[u]). \]

We have already discussed one direction: if \( u \) is a minimizer then \( u \) solves the PDE.

For the other direction assume that \( u \) solves the PDE and \( v \) is any other map in \( W_0^{1,2}(\Omega) \). Then we have (recall that \( A \) is symmetric)

\[
A_{\alpha\beta} \partial_\alpha v \partial_\beta v - A_{\alpha\beta} \partial_\alpha u \partial_\beta u \\
= A_{\alpha\beta} \partial_\alpha (v - u) \partial_\beta u + A_{\alpha\beta} \partial_\alpha u \partial_\beta (v - u) + A_{\alpha\beta} \partial_\alpha (v - u) \partial_\beta (v - u) \\
= 2A_{\alpha\beta} \partial_\alpha u \partial_\beta (v - u) + A_{\alpha\beta} \partial_\alpha (v - u) \partial_\beta (v - u) \\
\geq 2A_{\alpha\beta} \partial_\alpha u \partial_\beta (v - u) + \lambda |D(v - u)|^2 \\
\geq 2A_{\alpha\beta} \partial_\alpha u \partial_\beta (v - u).
\]
Thus,
\[ \mathcal{E}(v) - \mathcal{E}(u) \geq \frac{2}{\mu} \int_{\Omega} (A_{\alpha\beta} \partial_\alpha u \partial_\beta (v - u) + f[v - u] = 0, \]
by the PDE. That is \( u \) is a minimizer.

Now we observe that \( u \mapsto \mathcal{E}(u) \) is strictly convex. Indeed, let \( u \neq v \) and \( \mu \in (0,1) \) then
\[
A_{\alpha\beta} \partial_\alpha (\mu u + (1 - \mu)v) \partial_\beta (\mu u + (1 - \mu)v) \\
= \mu^2 A_{\alpha\beta} \partial_\alpha u \partial_\beta u + (1 - \mu)^2 A_{\alpha\beta} \partial_\alpha v \partial_\beta v + 2\mu(1 - \mu) A_{\alpha\beta} \partial_\alpha u \partial_\beta v \\
= \mu A_{\alpha\beta} \partial_\alpha u \partial_\beta u - \mu(1 - \mu) A_{\alpha\beta} \partial_\alpha u \partial_\beta v \\
+ (1 - \mu) A_{\alpha\beta} \partial_\alpha v \partial_\beta v - (1 - \mu) \mu A_{\alpha\beta} \partial_\alpha v \partial_\beta v \\
+ 2\mu(1 - \mu) A_{\alpha\beta} \partial_\alpha u \partial_\beta v \\
= \mu A_{\alpha\beta} \partial_\alpha u \partial_\beta u + (1 - \mu) A_{\alpha\beta} \partial_\alpha v \partial_\beta v \\
- \mu(1 - \mu) A_{\alpha\beta} (\partial_\alpha u \partial_\beta u + \partial_\alpha v \partial_\beta v - 2\partial_\alpha u \partial_\beta v) \\
= \mu A_{\alpha\beta} \partial_\alpha u \partial_\beta u + (1 - \mu) A_{\alpha\beta} \partial_\alpha v \partial_\beta v \\
- \mu(1 - \mu) A_{\alpha\beta} (\partial_\alpha (u - v) \partial_\beta (u - v)) \\
\leq \mu A_{\alpha\beta} \partial_\alpha u \partial_\beta u + (1 - \mu) A_{\alpha\beta} \partial_\alpha v \partial_\beta v - \lambda \mu (1 - \mu) |D(u - v)|^2.
\]
That is, whenever \( D(u - v) \neq 0 \) (i.e. \( u - v \) not constant, which by the same boundary data means \( u \neq v \)), and \( \mu \in (0,1) \) we have
\[
\mathcal{E}(\mu u + (1 - \mu)v) \leq \mu \mathcal{E}(u) + (1 - \mu) \mathcal{E}(v) - \lambda \mu (1 - \mu) \|D(u - v)\|^2_{L^2(\Omega)} < \mu \mathcal{E}(u) + (1 - \mu) \mathcal{E}(v),
\]
which is strict convexity.

Now, strictly convex functions have at most one global minimizer. Indeed assume that \( u, v \) are both global minimizer (thus \( \mathcal{E}(u) = \mathcal{E}(v) \leq \mathcal{E}(w) \) for any competitor \( w \)). We set \( w := \frac{1}{2}(u + v) \). Unless \( u \equiv v \) we’d then have
\[
\mathcal{E}(w) < \frac{1}{2} \mathcal{E}(u) + \frac{1}{2} \mathcal{E}(v) = \mathcal{E}(u) \leq \mathcal{E}(w),
\]
a contradiction. \( \square \)

Uniqueness by testing. Assume we have two solutions \( u \) and \( v \) then (by linearity) \( w := u - v \) solves the equation
\[
\begin{cases}
-\text{div} \left( A \nabla w \right) = 0 & \text{in } \Omega \\
w = 0 & \text{on } \partial \Omega,
\end{cases}
\]
We can use \( w \) as the test function of this PDE and have
\[
\lambda \|Dw\|^2_{L^2} \leq \int_{\Omega} A_{\alpha\beta} \partial_\alpha w \partial_\beta w = 0.
\]
Thus \( \|Dw\|^2_{L^2} = 0 \) and since \( w \in W^{1,2}_0(\Omega) \) we have \( w \equiv 0 \). \( \square \)
6.3. Interior regularity theory: Proof of Theorem 6.4. Many techniques in regularity theory of PDE are based on using (a version of) the solution \( u \) as a test function, or colloquially “multiply by \( u \) and integrate by parts”.

Let us illustrate this (without getting any good estimate). Assume \( u \) solves

\[-\text{div} (A \nabla u) = f \quad \text{in} \ \Omega\]

The basic idea by using \( u \) as a test function, we obtain good estimates.

Formally, we could the equation with \( u \) and (if we ignore the boundary data) integrating we obtain

\[\lambda \int_{\Omega} |\nabla u|^2 \leq \int_{\Omega} A_{\alpha \beta} \partial_\alpha u \partial_\beta u = \int f u \lesssim \|f\|_{L^2} \|u\|_{L^2}.\]

Now we have an estimate of \( u \) in terms of the \( W^{1,2} \)-function of \( u \) (assuming that \( f \in L^2! \)).

But we made a mistake! \( u \) is not a permissible test-function, since \( u \) is not zero at the boundary (and we cannot ignore the boundary data – which we actually do not know).

But don’t despair. Pick any \( \Omega' \subset \subset \Omega \). We can find a cutoff function \( \eta \in C^\infty_c(\Omega, [0,1]) \) such that \( \eta \equiv 1 \) in \( \Omega' \). It is enough to find a good estimate for \( \eta u \). So let us compute the equation of \( \eta u \).

\[\quad -\text{div} (A \nabla (\eta u)) = -\text{div} (\eta A \nabla u) - \text{div} (A (\nabla \eta) u) = \eta f - (\partial_\beta \eta) A_{\alpha \beta} \partial_\beta u - \text{div} (A (\nabla \eta) u).\]

Observe that even for \( u \in L^2 \), the right-hand side belongs to \((W_0^{1,2})^*\) in a nice way, e.g.

\[\quad \int (\partial_\beta \eta) A_{\alpha \beta} \partial_\beta u \varphi = -\int u \partial_\beta ((\partial_\beta \eta) A_{\alpha \beta} \partial_\beta \varphi) \lesssim \|u\|_{L^2} \|\varphi\|_{W^{1,2}}.\]

So, we can use existence, Theorem 6.1 to show that there exists \( v \in W_0^{1,2}(\Omega) \)

\[\quad -\text{div} (A \nabla v) = \eta f - (\partial_\beta \eta) A_{\alpha \beta} \partial_\beta u - \text{div} (A (\nabla \eta) u).\]

and we have

\[\|v\|_{W^{1,2}} \lesssim \|\eta f - (\partial_\beta \eta) A_{\alpha \beta} \partial_\beta u - \text{div} (A (\nabla \eta) u)\|_{(W_0^{1,2}(\Omega))^*} \lesssim \|f\|_{L^2} + \|u\|_{L^2}.\]

But on the other hand we have uniqueness, Theorem 6.3, so \( \eta u = v \) and we have found an estimate, cf. Exercise 6.7.

Exercise 6.7. Let \( u \in W^{1,2}(\Omega) \) solves the equation

\[\text{div} (A \nabla u) = f\]

Show that for any \( \Omega_1 \subset \subset \Omega \)

\[\|u\|_{W^{1,2}(\Omega_1)} \lesssim \|f\|_{(W_0^{1,2}(\Omega))^*} + \|u\|_{L^2(\Omega)},\]

where the constants in \( \lesssim \) depends on \( A \) and \( \Omega_1 \).

Hint: Use the argument that lead to Equation (6.2) and Theorem 6.3.
We want to apply this idea to the derivative, i.e. we compute the PDE for \( \eta \partial_\gamma u \).

\[
- \text{div} (A \nabla (\eta \partial_\gamma u)) = - \text{div} (A \nabla (\partial_\gamma (\eta u))) + \text{div} (A \nabla ((\partial_\gamma \eta) u))
\]

Now as above\(^{15}\) we have that this implies

\[
\| - \text{div} (A \nabla (\eta \partial_\gamma u)) \|_{(W_0^{1,2}(\Omega))^*} \lesssim \| u \|_{W^{1,2}(\Omega)}.
\]

So, again using existence, Theorem 6.1, to find \( v \in W_0^{1,2}(\Omega) \) such that

\[
- \text{div} (A \nabla (\eta \partial_\gamma u)) = - \text{div} (A \nabla v) \quad \text{in } \Omega
\]

\( v \) is is unique -- but here is the \textit{problem}: uniqueness, Theorem 6.3, is in \( W_0^{1,2} \) and \( \eta \partial_\gamma u \in L^2(\Omega) \supseteq W_0^{1,2}(\Omega) \). So all we get are \textit{a priori estimates}

**Lemma 6.8** (A priori estimates). Assume that \( u \in W^{2,2}(\Omega) \) solves

\[
- \text{div} (A \nabla u) = f \quad \text{in } \Omega
\]

Then in any open \( \Omega' \subset \subset \Omega \) we have the estimate

\[
\int_{\Omega'} |D^2u|^2 \leq C(\lambda, \Lambda, \Omega, \Omega') \left( \|f\|_{L^2(\Omega)}^2 + \|u\|_{W^{1,2}(\Omega)}^2 \right)
\]

**Proof.** Fix \( \gamma \in \{1, \ldots, n\} \), \( \Omega' \subset \subset \Omega \) and let \( \eta \in C^\infty(\Omega) \) such that \( \eta \equiv 1 \) in \( \Omega' \). Take \( v \in W_0^{1,2}(\Omega) \) solving

\[
- \text{div} (A \nabla (\eta \partial_\gamma u)) = - \text{div} (A \nabla v) \quad \text{in } \Omega,
\]

which by the above argument satisfies

\[
\|v\|_{W^{1,2}(\Omega)} \lesssim \| - \text{div} (A \nabla (\eta \partial_\gamma u)) \|_{(W_0^{1,2}(\Omega))^*} \lesssim \| f \|_{L^2(\Omega)} + \| u \|_{W^{1,2}(\Omega)}.
\]

Since \( u \in W^{2,2}(\Omega) \) we have that \( \eta \partial_\gamma u \in W_0^{1,2}(\Omega) \), so we have uniqueness, Theorem 6.3, which implies that \( v = \eta \partial_\gamma u \). Thus we have

\[
\| \eta \partial_\gamma u \|_{W^{1,2}(\Omega)} \lesssim \| f \|_{L^2(\Omega)} + \| u \|_{W^{1,2}(\Omega)}.
\]

Since \( \eta \equiv 1 \) in \( \Omega \),

\[
\| \partial_\gamma u \|_{W^{1,2}(\Omega')} \leq \| \eta \partial_\gamma u \|_{W^{1,2}(\Omega)} \lesssim \| f \|_{L^2(\Omega)} + \| u \|_{W^{1,2}(\Omega)}.
\]

Since this holds for any \( \gamma \in \{1, \ldots, n\} \) we have

\[
\| u \|_{W^{2,2}(\Omega')} \leq \| \eta \partial_\gamma u \|_{W^{1,2}(\Omega)} \lesssim \| f \|_{L^2(\Omega)} + \| u \|_{W^{1,2}(\Omega)}.
\]

We can conclude.

\(^{15}\)We will do this computation in details below, for the discrete differentiation.
So how do we show that \( u \in W^{2,2} \)? There are different ways to do this, the one we illustrate here is using difference quotients (i.e. discrete differentiation of the PDE):

**Proposition 6.9.** Assume that \( u \in W^{1,2}(\Omega) \) solves

\[
-\text{div} (A \nabla u) = f \quad \text{in} \ \Omega
\]

Then \( u \in W^{2,2}_{\text{loc}}(\Omega) \) (and thus we have the estimate from the previous theorem).

We will use **discrete differentiation** (which in view of Proposition 5.15 we can relate do Sobolev space estimates) for which we will use some properties

**Exercise 6.10** (Discrete differentiation). For \( h \in \mathbb{R}^n \setminus \{0\} \) denote by \( \delta_h f(x) := f(x+h) - f(x) \).

1. **Show the discrete product rule**: \( \delta_h (fg)(x) = \delta_h f(x)g(x) + f(x)\delta_h g(x) + \delta_h f(x)\delta_h g(x) \)
2. **Let \( \Omega \) be an open set and \( \Omega_2 \subset \subset \Omega \). Assume \( |h| \leq \frac{1}{2} \text{dist} (\Omega_2, \partial \Omega) \). Show the discrete integration by parts formula

\[
\int_{\Omega} f \delta_h g = \int_{\Omega} \delta_{-h} fg \quad \text{if supp} \ f \subset \Omega_2 \text{ or supp} \ g \subset \Omega_2.
\]
3. **Use the discrete integration by parts formula to show the usual integration by parts formula**

\[
\int_{\Omega} f \partial_{\gamma} g = - \int_{\Omega} \partial_{\gamma} fg
\]

for all smooth functions \( f \) and \( g \) which vanish in a neighborhood of \( \partial \Omega \), where \( \gamma \in \{1, \ldots, n\} \).

**Proof of Proposition 6.9.** Fix \( \Omega' \subset \subset \Omega_2 \subset \subset \Omega \). Let \( \eta \in C^\infty_c(\Omega_2, [0, 1]) \) with \( \eta \equiv 1 \) in \( \Omega' \).

Fix \( h \in \mathbb{R}^n \) with \( |h| \leq \frac{1}{100} \text{dist} (\text{supp} \ \eta, \partial \Omega_2) \).

Denote by \( \delta_h u(x) := u(x+h) - u(x) \). We now do what we did above, but with \( \partial_{\gamma} \) replaced by \( \delta_h \).

We observe that we have a discrete analogue of the product rule and the discrete integration by parts, see Exercise 6.10.

So let us compute

\[
g := -\text{div} (A \nabla (\eta \delta_h u)) \quad \text{in} \ \Omega.
\]

and show that

\[
\|g\|_{W^{1,2}(\Omega)} \lesssim |h| \|u\|_{W^{1,2}(\Omega)}.
\]

The computation is quite involved and long.

We have

\[
-\text{div} (A \nabla (\eta \delta_h u)) = -\text{div} (A \nabla \delta_h (\eta u)) + \text{div} (A \nabla (\delta_h \eta \ u)) - \text{div} (A \nabla (\delta_h \eta \delta_h u))
\]
Set
\[ \Gamma_1 := \text{div} \left( A \nabla (\delta_h \eta u) \right), \quad \Gamma_2 := -\text{div} \left( A \nabla (\delta_h \eta \delta_h u) \right) \]
Both \( \Gamma_1 \) and \( \Gamma_2 \) satisfy the estimate in (6.3). Indeed, for \( \varphi \in C^\infty_c(\Omega) \) we have
\[
|\Gamma_1[\varphi]| = \left| \int_\Omega A_{\alpha \beta} \partial_\beta (\delta_h \eta u) \partial_\alpha \varphi \right|
\lesssim |A|_{L^\infty} \| \partial_\beta (\delta_h \eta u) \|_{L^2(\Omega)} \| D\varphi \|_{L^2(\Omega)}
\lesssim |A|_{L^\infty} \left( \| \delta_h (D\eta) u \|_{L^2(\Omega)} + \| \delta_h \eta \|_{L^2(\Omega)} \right) \| D\varphi \|_{L^2(\Omega)}
\lesssim |A|_{L^\infty} \left( |h| \| D^2 \eta \|_{L^\infty} \| u \|_{L^2(\Omega_2)} + |h| \| D\eta \|_{L^\infty} \| Du \|_{L^2(\Omega_2)} \right) \| D\varphi \|_{L^2(\Omega)}
\leq C(\Lambda, \eta) [h| \| u \|_{W^{1,2}(\Omega_2)}] \| \varphi \|_{W^{1,2}(\Omega_2)}.
\]
Thus,
\[
\| \Gamma_1 \|_{(W^{1,2})^*(\Omega)} \leq C(\Lambda, \eta) [h| \| u \|_{W^{1,2}(\Omega_2)} \leq C(\Lambda, \eta) [h| \| u \|_{W^{1,2}(\Omega)}).
\]
We argue the same way for \( \Gamma_2 \) and have
\[
\| \Gamma_2 \|_{(W^{1,2})^*(\Omega)} \leq C(\Lambda, \eta) [h| \| \delta_h u \|_{W^{1,2}(\Omega_2)}
\lesssim C(\Lambda, \eta) [h| \left( \| u \|_{W^{1,2}(\Omega_2)} + \| u \|_{W^{1,2}(\Omega_2 + h)} \right)
\lesssim 2C(\Lambda, \eta) [h| \| u \|_{W^{1,2}(\Omega)}
\]
So we have shown
\[
-\text{div} \left( A \nabla (\eta \delta_h u) \right) = -\text{div} \left( A \nabla \delta_h \left( \eta u \right) \right) + \Gamma_1 + \Gamma_2
\]
where \( \Gamma_1 \) and \( \Gamma_2 \) satisfy the estimate we want, (6.3). Since there will be many \( \Gamma_i \) we are going to call \( \Gamma \) any “good term” that satisfies the estimate (6.3), i.e. whenever
\[
\| \Gamma \|_{(W^{1,2})^*(\Omega)} \lesssim [h| \| u \|_{W^{1,2}(\Omega)}.
\]
- and \( \Gamma \) will change from line to line. For now we have
\[
-\text{div} \left( A \nabla (\eta \delta_h u) \right) = -\text{div} \left( A \nabla \delta_h \left( \eta u \right) \right) + \Gamma
= -\delta_h \text{div} \left( A \nabla \left( \eta u \right) \right) + \delta_h A \nabla \left( \eta u \right) + \delta_h \nabla \left( \eta u \right) + \Gamma
\]
Set
\[
\Gamma_3 := \text{div} \left( \delta_h A \nabla \left( \eta u \right) \right), \quad \Gamma_4 := \text{div} \left( \delta_h A \delta_h \nabla \left( \eta u \right) \right).
\]
We check that \( \Gamma_1 \) and \( \Gamma_3 \) are of the type \( \Gamma \): For \( \varphi \in C^\infty_c(\Omega) \) we have
\[
\text{div} \left( \delta_h A \nabla \left( \eta u \right) \right)[\varphi] = -\int_\Omega \delta_h A_{\alpha \beta} \partial_\beta (\eta u) \partial_\alpha \varphi
\lesssim |\delta_h A_{\alpha \beta}| \| \partial_\beta (\eta u) \|_{L^2} \| D\varphi \|_{L^2}
\lesssim |h| \| DA \|_{L^\infty} C(\eta) \| u \|_{W^{1,2}(\Omega_2)} \| \varphi \|_{W^{1,2}(\Omega)}
\]
The estimate for \( \Gamma_4 \) is similar, with the same adaptations as for \( \Gamma_2 \) (using that \( \eta \in C^\infty_c(\Omega_2) \) localizes everything to \( \Omega_2 \)). Thus, we have shown (for a new \( \Gamma \) but still satisfying (6.3))
\[
-\text{div} \left( A \nabla (\eta \delta_h u) \right) = -\delta_h \text{div} \left( A \nabla \left( \eta u \right) \right) + \Gamma
= -\delta_h \text{div} \left( A \left( \eta \nabla u \right) \right) - \delta_h \text{div} \left( A \left( \nabla \eta \ u \right) \right) + \Gamma
\]
We show that
\[ \Gamma_5 := -\delta_h \text{div} (A (\nabla \eta u)) \]
is of type \( \Gamma \). Let \( \varphi \in C_c^\infty(\Omega) \) then (using discrete integration by parts, again \( \eta \) localizes everything to \( \Omega_2 \))
\[
\Gamma_5[\varphi] = -\int_\Omega \text{div} (A (\nabla \eta u)) \delta_{-h} \varphi
\leq \| \text{div} (A (\nabla \eta u)) \|_{L^2(\Omega)} \| \delta_{-h} \varphi \|_{L^2(\Omega_2)}
\leq h \| \| D A \|_{L^\infty} \| D \eta \|_{L^\infty} \| u \|_{L^2(\Omega)} + \| A \|_{L^\infty} \| D (\nabla \eta u) \|_{L^2(\Omega)} + \| \varphi \|_{W^{1,2}(\Omega)}
\leq C(\eta, A) \| u \|_{W^{1,2}(\Omega)} \| \varphi \|_{W^{1,2}(\Omega)}.
\]
Thus, \( \Gamma_5 \) is of type \( (\Gamma) \), and we have
\[
-\text{div} (A \nabla (\eta \delta_h u)) = -\delta_h \text{div} (A (\eta \nabla u)) + \Gamma
= -\delta_h (\eta \text{div} (A \nabla u)) - \delta_h (\partial_\alpha \eta A_{\alpha\beta} \partial_\beta u) + \Gamma
= -\delta_h (\eta f) - \delta_h (\partial_\alpha \eta A_{\alpha\beta} \partial_\beta u) + \Gamma
\]
So we finally set
\[ \Upsilon := -\delta_h (\eta f), \quad \Gamma_6 := -\delta_h (\partial_\alpha \eta A_{\alpha\beta} \partial_\beta u). \]
We first show that \( \Gamma_6 \) is of type \( \Gamma \). Let \( \varphi \in C_c^\infty(\Omega) \), then by an discrete integration by parts (again: the integral is actually in a strict subset of \( \Omega \) because of \( \eta \in C_c^\infty(\Omega_2) \) and \( |h| \ll 1 \)),
\[
\Gamma_6[\varphi] = \int_\Omega -\delta_h (\partial_\alpha \eta A_{\alpha\beta} \partial_\beta u) \varphi
= -\int_\Omega (\partial_\alpha \eta A_{\alpha\beta} \partial_\beta u) \delta_{-h} \varphi
\leq \| D \eta \|_{L^\infty} \| A \|_{L^\infty} \| D u \|_{L^2(\Omega)} \| \delta_{-h} \varphi \|_{L^2(\Omega_2)}
\leq C(\eta, A) \| u \|_{W^{1,2}(\Omega)} \| \varphi \|_{W^{1,2}(\Omega)}.
\]
So \( \Gamma_6 \) is of type \( \Gamma \).

Lastly we need to show an estimate for \( \Upsilon \) – and here is the first and only time we use that \( f \in L^2(\Omega) \): Let \( \varphi \in C_c^\infty(\Omega) \)
\[
\Upsilon[\varphi] = -\int_\Omega \eta f \delta_{-h} \varphi \leq \| f \|_{L^2(\Omega)} \| \delta_{-h} \varphi \|_{L^2(\Omega_2)} \leq \| h \| \| f \|_{L^2(\Omega)} \| \varphi \|_{W^{1,2}(\Omega)}.
\]
In conclusion, we have shown
\[ -\text{div} (A \nabla (\eta \delta_h u)) = \Upsilon + \Gamma, \]
and we have
\[ \| \mathcal{Y} \|_{(W^{1,2}_0(\Omega))^*} \lesssim |h| \| f \|_{L^2(\Omega)} \]
and
\[ \| \Gamma \|_{(W^{1,2}_0(\Omega))^*} \lesssim |h| \| u \|_{W^{1,2}(\Omega)} \]
On the other hand from Theorem 6.1 there exists some \( v = v_h \in W^{1,2}_0(\Omega) \) such that
\[ -\text{div} \, (A \nabla v) = \mathcal{Y} + \Gamma, \]
and \( v \) comes with the estimate
\[ \| v \|_{W^{1,2}(\Omega)} \lesssim \| \mathcal{Y} \|_{(W^{1,2}_0(\Omega))^*} + \| \Gamma \|_{(W^{1,2}_0(\Omega))^*} \lesssim |h| \left( \| f \|_{L^2(\Omega)} + \| u \|_{W^{1,2}(\Omega)} \right). \]
By Theorem 6.3 \( v \) is unique in \( W^{1,2}_0(\Omega) \) and we observe that \( (\eta \delta_h u) \in W^{1,2}_0(\Omega) \). So we actually have \( v = (\eta \delta_h u) \) and thus
\[ \| \eta \delta_h u \|_{W^{1,2}(\Omega)} \lesssim |h| \left( \| f \|_{L^2(\Omega)} + \| u \|_{W^{1,2}(\Omega)} \right). \]
Since \( \eta \equiv 1 \) in \( \Omega' \) we conclude that
\[ |h|^{-1} \| \delta_h u \|_{W^{1,2}(\Omega')} \lesssim \left( \| f \|_{L^2(\Omega)} + \| u \|_{W^{1,2}(\Omega)} \right). \]
This holds for any \( |h| \ll 1 \). So by Proposition 5.15 we conclude that
\[ \| u \|_{W^{2,2}(\Omega')} \leq C(A, \Omega', \Omega) \left( \| f \|_{L^2(\Omega)} + \| u \|_{W^{1,2}(\Omega)} \right). \]
This argument works for any \( \Omega' \subset \subset \Omega \), so we conclude that \( u \in W^{2,2}_{\text{loc}}(\Omega) \). \( \square \)

We are essentially done, now we argue by induction to get

**Proof of Theorem 6.4.** We claim that for all \( k \in \mathbb{N} \cup \{0\} \), whenever \( v \in W^{1,2}_{\text{loc}}(U) \) and \( g \in W^{k,2}(\Omega) \) solves
\[ \text{div} \, (A \nabla v) = g \quad \text{in } U \]
then for any \( U_1 \subset \subset U_2 \subset \subset U \) we have
\[ \| v \|_{W^{k+2,2}(U_1)} \lesssim C(U_1, U_2, A) \left( \| g \|_{W^{k,2}(\Omega_2)} + \| v \|_{W^{1,2}(U_2)} \right). \]
For \( k = 0 \) this is proven in Proposition 6.9 (nevermind the “loc” in \( v \in W^{1,2}_{\text{loc}}(U) \) – the equation is then satisfied in \( U_2 \) and we have \( u \in W^{1,2}(U_2) \)).

Now fix \( k \geq 1 \) and assume the claim is shown already for \( k - 1 \).

Let \( u \in W^{1,2}_{\text{loc}}(\Omega) \) and assume \( f \in W^{k,2}(\Omega) \) solves
\[ \text{div} \, (A \nabla u) = f \quad \text{in } \Omega. \]
Then from the induction hypothesis
\[ \| u \|_{W^{k+1,2}(\Omega_1)} \lesssim C(\Omega_1, \Omega_2, A) \left( \| f \|_{W^{k-1,2}(\Omega_2)} + \| u \|_{W^{1,2}(\Omega_2)} \right). \]
Since \( k \geq 1 \) we thus already know \( u \in W^{2,2}_{loc}(\Omega) \) so \( \partial_\gamma u \in W^{1,2}_{loc}(\Omega) \) for each fixed \( \gamma \in \{1, \ldots, n\} \). We then can differentiate the equation

\[
\text{div} (A \nabla \partial_\gamma u) = \partial_\gamma f + \text{div} (\partial_\gamma A \nabla u)
\]

Applying the induction hypothesis to \( v := \partial_\gamma u \in W^{k,2}_{loc}(\Omega) \subset W^{1,2}_{loc}(\Omega) \) we find for any \( \Omega_1 \subset \subset \Omega_2 \subset \subset \Omega_3 \subset \subset \Omega \)

\[
\|\partial_\gamma u\|_{W^{k+1,2}(\Omega)} \leq C(\Omega_1, \Omega_2, A) \left( \|\partial_\gamma f\|_{W^{k-1,2}(\Omega_2)} + \|\text{div} (\partial_\gamma A \nabla u)\|_{W^{k-1,2}(\Omega_2)} + \|\partial_\gamma u\|_{W^{1,2}(\Omega_2)} \right)
\]

Applying once more the induction hypothesis (to \( u \) this time) we have

\[
\|u\|_{W^{k+2,2}(\Omega_2)} \leq C(\Omega_2, \Omega_3, A) \left( \|f\|_{W^{k-1,2}(\Omega_3)} + \|u\|_{W^{1,2}(\Omega_3)} \right)
\]

So we have shown

\[
\|u\|_{W^{k+2,2}(\Omega_1)} \leq \max\{\gamma \in \{1, \ldots, n\}\} \|\partial_\gamma u\|_{W^{k+1,2}(\Omega_1)}
\]

This holds for any \( \Omega_1 \subset \subset \Omega_3 \) (we can always find a suitable \( \Omega_2 \)), so we have shown the induction step and can conclude.

\[ \square \]

**Exercise 6.11.** Prove the statement in Remark 6.5.

### 6.4. Global/Boundary regularity theory: Proof of Theorem 6.6. Assume \( u \) is a solution to

\[
\begin{cases}
-\text{div}(A \nabla u) = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

Since \( \partial \Omega \) is a smooth manifold, for any point \( x_0 \in \partial \Omega \) there exists a small radius \( r(x_0) > 0 \) and a diffeomorphism

\[ \Phi : B(0, 1) \to \mathbb{R}^n \]

with \( \Phi(0) = x_0, \Phi(B(0, 1) \cap \mathbb{R}^n_+) \subset \Omega, \Phi(B(0, 1) \cap \mathbb{R}^n_-) \subset \mathbb{R}^n \setminus \Omega \) and \( \Phi(B(0, 1)) \supset B(x_0, r) \).
Take \( \eta \in C_c^\infty(B(x_0, r)) \) such that \( \eta \equiv 1 \) in \( B(x_0, r/2) \). It suffices to show that \( \eta u \in W^{2,2}(\Omega) \) and
\[
\| \eta u \|_{W^{2,2}(\Omega)} \lesssim \| u \|_{W^{1,2}(\Omega)} + \| f \|_{L^2(\Omega)}.
\]
Indeed, if we can do that we can find finitely many balls \( (B(x_i, r_i))_{i=1}^N \) such that \( B(x_i, r_i/2) \) covers a neighborhood of \( \partial \Omega \) (because \( \partial \Omega \) is compact). Set \( \Omega_0 := \Omega \setminus \bigcup_i B(x_i, r_i/4) \subset \subset \Omega \). Then we take a partition of unity \( \eta_i \) of \( \Omega \): \( \eta_i \in C_c^\infty(B(x_i, r_i)) \), \( \eta_i \equiv 1 \) in \( B(x_i, r_i/2) \), and \( \eta_0 \in C_c^\infty(\Omega_0) \) such that
\[
\sum_{i=0}^N \eta_i \equiv 1 \quad \text{in} \quad \Omega.
\]
From the interior theory we have already the estimate
\[
\| \eta_0 u \|_{W^{2,2}(\Omega)} \lesssim \| u \|_{W^{1,2}(\Omega)} + \| f \|_{L^2(\Omega)}.
\]
If we have (6.4) then we’d get
\[
\| u \|_{W^{2,2}(\Omega)} \lesssim \sum_{i=0}^N \sum \| \eta_i u \|_{W^{2,2}(\Omega)} \lesssim N \left( \| u \|_{W^{1,2}(\Omega)} + \| f \|_{L^2(\Omega)} \right).
\]
Observe that the choice of \( \eta_i \) and \( B(x_i, r_i) \) and \( N \) only depends on the set \( \Omega \).
As before,
\[
\text{div} \left( A \nabla (\eta u) \right) = \eta f + \text{div} (A(\nabla \eta) u) + \partial_\alpha \eta A_{\alpha\beta} \nabla u \quad \text{in} \quad \Omega
\]
If we set
\[
g := \eta f + \text{div} (A(\nabla \eta) u) + \partial_\alpha \eta A_{\alpha\beta} \nabla u,
\]
we see that
\[
\| g \|_{L^2(\Omega)} \lesssim C(A, \eta) \left( \| f \|_{L^2(\Omega)} + \| u \|_{W^{1,2}(\Omega)} \right).
\]
So if we set \( \tilde{u} := (\eta u) \) we actually have to consider \( W^{2,2} \) estimates the equation
\[
\begin{cases}
-\text{div} (A \nabla \tilde{u}) = g & \text{in} \; B(x_0, r) \cap \Omega \\
\tilde{u} = 0 & \text{on} \; \partial (B(x_0, r) \cap \Omega)
\end{cases}
\]
Now we proceed with a method called \textit{flattening the boundary}.
We set \( B(0, 1)^+ := B(0, 1) \cap \mathbb{R}_+^n \) and define
\[
v(x) := \tilde{u} \circ \Phi \in W^{1,2}_0(B(0, 1) \cap \mathbb{R}_+^n).
\]
We then have (also in the distributional sense which is proven by approximation)
\[
\partial_\alpha v(x) = (\partial_\gamma \tilde{u})(x) \partial_\alpha \Phi^\gamma(x).
\]
Denote the matrix \( (D\Phi(x))_{\alpha\beta} := \partial_\alpha \Phi^\gamma(x) \). This matrix is invertible, and we call this inverse \( (D\Phi)^{-1}(x) \). Observe that by chain rule
\[
(D\Phi)^{-1}(x) = D(\Phi^{-1})(\Phi(x)).
\]
Set
\[
\tilde{A}(x) := \begin{pmatrix} |\det(D\Phi)| (D\Phi)^{-1} A D\Phi^{-1} \end{pmatrix}(\Phi(x)) : B(0, 1)^+ \to \mathbb{R}^{n \times n}.
\]
This is a smooth, symmetric, elliptic matrix-valued map (using heavily that $\Phi$ is a diffeomorphism!). Moreover we have for any $\varphi \in C_c^\infty(B(0,1)^\circ)$, setting $\psi := \varphi \circ \Phi^{-1} \in C_c^\infty(B(x_0, r) \cap \Omega)$

$$
\int_{B(0,1)^\circ} \partial_\beta \tilde{v}(x) \tilde{A}_{\alpha\beta}(x) \partial_\alpha \varphi(x) \, dx 
= \int_{B(0,1)^\circ} \partial_\beta (\tilde{u} \circ \Phi)(x) \tilde{A}_{\alpha\beta}(x) \partial_\alpha (\psi(\Phi))(x) \, dx 
= \int_{\Omega \cap B(x_0, r)} \partial_\beta (\tilde{u} \circ \Phi)(\Phi^{-1}(z)) \tilde{A}_{\alpha\beta}(\Phi^{-1}(z)) \partial_\alpha (\psi(\Phi))(\Phi^{-1}(z)) \, | \det(D\Phi^{-1}(z))| \, dz 
= \int_{\Omega \cap B(x_0, r)} \partial_\beta \tilde{u}(z) \left( |\det(D\Phi)|^{-1} D\Phi' \tilde{A} D\Phi \right)_{\alpha\beta}(\Phi^{-1}(z)) \partial_\alpha \psi(z) \, dz 
= \int_{\Omega \cap B(x_0, r)} g \psi \, dz 
= \int_{B(0,1)^\circ} g \circ \Phi \, |\det(D\Phi)| \, \varphi
$$

We have reduced Theorem 6.6 to the following

**Proposition 6.12.** Let $u \in W_0^{1,2}(\mathbb{R}_+^n)$ solve the equation

$$
-\text{div}(A \nabla u) = f \quad \text{in } \mathbb{R}_+^n.
$$

Also assume that $u \equiv 0$ in $\mathbb{R}_+^n \setminus B(0,1)$.

If $f \in L^2(\mathbb{R}_+^n)$ then $u \in W^{2,2}(\mathbb{R}_+^n)$ with the estimate

$$
\|u\|_{{W^{2,2}(\mathbb{R}_+^n)}} \lesssim \|f\|_{L^2(\mathbb{R}_+^n)} + \|u\|_{W^{1,2}(\mathbb{R}_+^n)}.
$$

We sketch the idea of the proof.

**Sketch of the proof of Proposition 6.12.** The idea is – yet again – differentiation of the equation. If we consider $i \in \{1, \ldots, n-1\}$ then

$$
-\text{div}(A \nabla \partial_i u) = \partial_i f - \text{div}(\partial_i A \nabla u) \quad \text{in } \mathbb{R}_+^n.
$$

Observe that since $i \neq n$ we still believe we could have $\partial_i u = 0$ on $\partial\mathbb{R}_+^n = \mathbb{R}^{n-1} \times \{0\}$. As before the right-hand side belongs to $(W_0^{1,2}(\mathbb{R}_+^n))^*$

$$
\|\partial_i f - \text{div}(\partial_i A \nabla u)\|_{(W_0^{1,2}(\mathbb{R}_+^n))^*} \lesssim \|f\|_{L^2(\mathbb{R}_+^n)} + \|u\|_{W^{1,2}(\mathbb{R}_+^n)}.
$$

So it sounds believable that by the same strategy as before (testing with $\partial_i u$) we would get

$$
\|\partial_i u\|_{W^{1,2}(\mathbb{R}_+^n)} \lesssim \|f\|_{L^2(\mathbb{R}_+^n)} + \|u\|_{W^{1,2}(\mathbb{R}_+^n)}.
$$

This holds for any $i \in \{1, \ldots, n-1\}$. We call this a *tangential estimate*. But it does not work for $i = n$, since $\partial_n u$ is possibly nonzero on $\partial\mathbb{R}_+^n = \mathbb{R}^{n-1} \times \{0\}$!
So what we get is
\[(6.5) \quad \|\partial_{\alpha\beta}u\|_{L^2(\mathbb{R}^n_+)} \lesssim \|f\|_{L^2(\mathbb{R}^n_+)} + \|u\|_{W^{1,2}(\mathbb{R}^n_+)} \quad \forall \alpha, \beta \in \{1, \ldots, n\} : \quad (\alpha, \beta) \neq (n, n).\]

How do we get information on \(\partial_{nn}u\)? We use that \(\Delta = \partial_{nn} + \sum_{i=1}^n \partial_{xi}x_i!\)

Namely observe that
\[
\partial_n(A_{nn}\partial_n u) = f - \sum_{(\alpha, \beta) \neq (n, n)} \partial_{\alpha}(A_{\alpha\beta}\partial_{\beta} u)
\]

Since \((\alpha, \beta) \neq (n, n)\) we have from the previous estimate (6.5)
\[
\|\partial_{\alpha}(A_{\alpha\beta}\partial_{\beta} u)\|_{L^2(\mathbb{R}^n_+)} \lesssim \|f\|_{L^2(\mathbb{R}^n_+)} + \|u\|_{W^{1,2}(\mathbb{R}^n_+)} \quad \forall \alpha, \beta \in \{1, \ldots, n\}.
\]

and we conclude that
\[
\|\partial_n(A_{nn}\partial_n u)\|_{L^2(\mathbb{R}^n_+)} \lesssim \|f\|_{L^2(\mathbb{R}^n_+)} + \|u\|_{W^{1,2}(\mathbb{R}^n_+)}.
\]

Thus, since \(A_{nn}\partial_n u = \partial_n(A_{nn}\partial_n u) - \partial_n A_{nn}\partial_n u,\)
\[
\|A_{nn}\partial_n u\|_{L^2(\mathbb{R}^n_+)} \lesssim \|\partial_n(A_{nn}\partial_n u)\|_{L^2(\mathbb{R}^n_+)} + \|u\|_{W^{1,2}(\mathbb{R}^n_+)} \lesssim \|f\|_{L^2(\mathbb{R}^n_+)} + \|u\|_{W^{1,2}(\mathbb{R}^n_+)}.
\]

And now we use yet again ellipticity: \(A_{nn} = \langle e_n, Ae_n \rangle \geq \lambda, \) so we have finally shown
\[
\lambda \|\partial_{nn}u\|_{L^2(\mathbb{R}^n_+)} \lesssim \|A_{nn}\partial_{nn} u\|_{L^2(\mathbb{R}^n_+)} \lesssim \|f\|_{L^2(\mathbb{R}^n_+)} + \|u\|_{W^{1,2}(\mathbb{R}^n_+)}.
\]

Now the above argument only delivers an \textit{a priori} argument, since we needed assumed that \(\partial_{xi}u \in W^{1,2}_0(\mathbb{R}^n_+).\) The precise argument goes as follows: Let \(h \in \mathbb{R}^{n-1} \times \{0\}\) and consider \(\delta_h u \in W^{1,2}_0(\mathbb{R}^n_+).\) Then
\[
-\text{div} (A\nabla \delta_h u) = g_h \quad \text{in} \ \mathbb{R}^n_+,
\]
where we can estimate
\[
\|g_h\|_{W^{1,2}_0(\mathbb{R}^n_+)^*} \lesssim |h| \left(\|u\|_{W^{1,2}(\mathbb{R}^n_+)} + \|f\|_{L^2(\mathbb{R}^n_+)}\right)
\]

Testing the equation with \(\delta_h u\) we then obtain
\[
\sup_{h \in \mathbb{R}^{n-1} \times \{0\}} |h|^{-1}\|\delta_h u\|_{W^{1,2}(\mathbb{R}^n_+)} \lesssim \|u\|_{W^{1,2}(\mathbb{R}^n_+)} + \|f\|_{L^2(\mathbb{R}^n_+)}.
\]

Suitably adapting the argument of Proposition 5.15 we find as desired that
\[
\|\partial_{xi}u\|_{W^{1,2}(\mathbb{R}^n_+)} \lesssim \|u\|_{W^{1,2}(\mathbb{R}^n_+)} + \|f\|_{L^2(\mathbb{R}^n_+)} \quad \forall i = 1, \ldots, n - 1.
\]

Now let \(\varphi \in C_0^\infty(\mathbb{R}^n_+)\) then we have
\[
\int_{\mathbb{R}^n_+} A_{nn} \partial_n u \partial_n \varphi = \int \varphi - \int_{\mathbb{R}^n_+} \sum_{(\alpha, \beta) \neq (n, n)} \partial_{\alpha}(A_{\alpha\beta}\partial_{\beta} u) \varphi
\]

From the previous estimates we conclude that
\[
|\int_{\mathbb{R}^n_+} A_{nn} \partial_n u \partial_n \varphi| \lesssim \|\varphi\|_{L^2(\mathbb{R}^n_+)} \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2}\right).
\]
In particular we have
\[
\left| \int_{\mathbb{R}^n} \partial_n u \partial_n (A_{nn} \varphi) \right| \lesssim \| \varphi \|_{L^2(\mathbb{R}^n_+)} \left( \| u \|_{W^{1,2}(\Omega)} + \| f \|_{L^2(\Omega)} \right).
\]
Take now \( \psi \in C^\infty_c(\mathbb{R}^n_+) \) then \( \varphi := (A_{nn})^{-1} \psi \in C^\infty_c(\mathbb{R}^n_+) \) (using ellipticity), so we have
\[
\left| \int_{\mathbb{R}^n} \partial_n u \partial_n \psi \right| = \left| \int_{\mathbb{R}^n_+} \partial_n u \partial_n \psi \right| \lesssim \| \psi \|_{L^2(\mathbb{R}^n_+)} \left( \| u \|_{W^{1,2}(\Omega)} + \| f \|_{L^2(\Omega)} \right).
\]
But this implies by the definition of distributional derivative (and Riesz representation theorem) that \( \partial_{nn} u \in L^2(\mathbb{R}^n_+) \) awith the estimate
\[
\| \partial_{nn} u \|_{L^2(\mathbb{R}^n_+)} \lesssim \| u \|_{W^{1,2}(\Omega)} + \| f \|_{L^2(\Omega)}.
\]
Thus we have shown
\[
\| u \|_{W^{2,2}(\mathbb{R}^n_+)} \lesssim \max_{i=1,\ldots,n-1} \| \partial_{xi} u \|_{W^{1,2}(\mathbb{R}^n_+)} + \| \partial_{nn} u \|_{L^2(\mathbb{R}^n_+)} \lesssim \| u \|_{W^{1,2}(\Omega)} + \| f \|_{L^2(\Omega)}.
\]
\[\square\]

It is now not difficult (but very cumbersome) to prove the following generalization

**Theorem 6.13** (Global \( W^{k,2} \)-regularity). Assume \( \Omega \) is a bounded open set with smooth boundary \( \partial \Omega \in C^\infty \). Let \( f \in W^{k,2}(\Omega) \) and assume \( u \in W^1_{0,2}(\Omega) \) solves
\[
\begin{cases}
- \text{div} (A \nabla u) = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
in the above sense.

Then \( u \in W^{k+2,2}(\Omega) \), and we have
\[
\| D^{k+2} u \|_{L^2(\Omega)} \leq C(\Omega) \left( \| u \|_{W^{1,2}(\Omega)} + \| f \|_{L^2(\Omega)} \right).
\]

**Exercise 6.14.** Think about how (formally) you could prove now Theorem 6.13

We can also treat more generic boundary data:

**Theorem 6.15** (Global \( W^{k,2} \)-regularity). Assume \( \Omega \) is a bounded open set with smooth boundary \( \partial \Omega \in C^\infty \). Let \( f \in W^{k,2}(\Omega) \) and \( g \in W^{k+2,2}(\Omega) \) and assume \( u \in W^1_{0,2}(\Omega) \) solves
\[
\begin{cases}
- \text{div} (A \nabla u) = f & \text{in } \Omega \\
u = g & \text{on } \partial \Omega,
\end{cases}
\]
in the weak sense (where \( u = g \) on \( \partial \Omega \) simply means \( u - g \in W^1_{0,2}(\Omega) \)).

Then \( u \in W^{k+2,2}(\Omega) \), and we have
\[
\| D^{k+2} u \|_{L^2(\Omega)} \leq C(\Omega) \left( \| u \|_{W^{1,2}(\Omega)} + \| f \|_{L^2(\Omega)} + \| g \|_{W^{k+2,2}(\Omega)} \right).
\]
Exercise 6.16. Think about how (formally) you could prove now Theorem 6.15.

Hint: consider \( v := u - g \).

6.5. An alternative approach to boundary regularity theory: reflection. There is another (in general quite delicate) argument for boundary regularity that we want to very briefly discuss.

Assume that we have the equation
\[
\begin{cases}
\Delta u = f & \text{in } \mathbb{R}^n_+ \\
u = 0 & \text{in } \mathbb{R}^{n-1} \times \{0\}.
\end{cases}
\]

If we want to find regularity at the boundary, we could use the following reflection argument.

For \( x = (x', x_n) \in \mathbb{R}^n_+ \) set
\[
\tilde{u}(x', x_n) := u(x', |x_n|).
\]

Now observe that this is a Lipschitz operation and for continuous functions \( u \) we have that \( \tilde{u} \) is also continuous (thanks to the boundary data being zero). So if \( u \in W^{1,2}_0 \) then it seems believable (a proof is needed however) that \( \tilde{u} \in W^{1,2}(\mathbb{R}^n_+) \). The formal computation goes like this: Clearly
\[
\partial_n \tilde{u}(x', x_n) = \begin{cases} 
\partial_n u(x', x_n) & x_n > 0 \\
-\partial_n u(x', -x_n) & x_n < 0
\end{cases}
\]

Observe (think about the Heaviside function) that this does not imply that \( \partial_n u \) exists in distributional sense! But (formally) we now can show for any \( \varphi \in C_c^\infty(\mathbb{R}^n) \)
\[
\int_{\mathbb{R}^n} \tilde{u} \partial_n \varphi = \int_{\mathbb{R}^n_+} u(x', x_n) \partial_n \varphi(x) - \int_{\mathbb{R}^n_-} u(x', -x_n) \partial_n \varphi(x)
\]

substitution
\[
= \int_{\mathbb{R}^n_+} u(x', x_n) \partial_n \varphi - \int_{\mathbb{R}^n_+} u(x', +x_n) (\partial_n \varphi) (x', -x_n)
\]

\[
+ \int_{\mathbb{R}^n_+} u(x', x_n) \partial_n \varphi + \int_{\mathbb{R}^n_+} u(x', x_n) \partial_n (\varphi(x', -x_n))
\]

\[
= -\int_{\mathbb{R}^n_+} \partial_n u(x', x_n) \varphi(x', x_n) - \int_{\mathbb{R}^n_+} \partial_n u(x', x_n) \varphi(x', -x_n)
\]

\[
- \int_{\mathbb{R}^{n-1} \times \{0\}} u(x', 0) \varphi(x', 0) dx' - \int_{\mathbb{R}^{n-1} \times \{0\}} u(x', 0) \varphi(x', 0) dx'
\]

substitution
\[
= -\int_{\mathbb{R}^n_+} \partial_n u(x', x_n) \varphi - \int_{\mathbb{R}^n_-} (\partial_n u) (x', -x_n) \varphi(x', +x_n)
\]

def
\[
= -\int_{\mathbb{R}^n_+} \partial_n u \varphi - \int_{\mathbb{R}^n_-} \partial_n \tilde{u} \varphi
\]

\[
= -\int_{\mathbb{R}^n} \partial_n \tilde{u} \varphi
\]
This shows that $\tilde{u} \in W^{1,2}(\mathbb{R}^n)$ (some details need to be ironed out, but that’s the idea).

Now we compute in a similar way the PDE:
\[
\partial_{x_i} \tilde{u}(x) = (\partial_{x_i} u)(x', |x_n|) \quad i = 1, \ldots, n-1
\]
and
\[
\partial_{x_n} \tilde{u}(x) = \begin{cases} 
(\partial_{x_n} u)(x', x_n) & x_n > 0 \\
- (\partial_{x_n} u)(x', -x_n) & x_n < 0 = (\partial_{nn} u)(x', |x_n|)
\end{cases}
\]

Thus we have
\[
\Delta \tilde{u} = f(x', |x_n|) \quad \text{in } \mathbb{R}^n.
\]

Now we can obtain boundary regularity by interior regularity theory.

This argument is beautiful, but its downsides are that the reflection needs to be adapted to the PDE at hand – which can be extremely difficult (or impossible).

6.6. **Extension to more general elliptic equations.** With the above arguments one can also treat more general linear PDE (and indeed the arguments for nonlinear elliptic pde are mostly based on “using linear theory for nonlinear pde”, and thus follow the general spirit of the above argument).

\[
\begin{cases}
- \text{div} (A \nabla u) + b \cdot \nabla u + cu = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

*Email address: armin@pitt.edu*