References 5
Index 6
Basic formulas and concepts we’ll use a lot 8
Integration by parts, Greens Theorem, Stokes theorem 8
Polar coordinates 8
regular sets 9

Part 1. PDE 1 10

1. Introduction and some basic notation 10
2. Laplace equation 15
  2.1. Sort of a physical motivation 15
  2.2. Definitions 16
  2.3. Fundamental Solution, Newton- and Riesz Potential 17
  2.4. Green Functions 23
  2.5. Mean Value Property for harmonic functions 30
  2.6. Maximum and Comparison Principles 32
  2.7. Harnack Principle 36
  2.8. Perrons method (illustration) 41
  2.9. Weak Solutions, Regularity Theory 49
  2.10. Methods from Calculus of Variations – Energy Methods 55
2.11. Linear Elliptic equations

2.12. Maximum principles for linear elliptic equations

3. Heat equation

3.1. Again, sort of a physical motivation

3.2. Sort of an optimization motivation

3.3. Fundamental solution and Representation

3.4. Mean-value formula

3.5. Maximum principle and Uniqueness

3.6. Harnack’s Principle

3.7. Regularity and Cauchy-estimates

3.8. Variational Methods

4. Wave Equation

4.1. Global Solution via Fourier transform

4.2. Energy methods

5. Black Box – Sobolev Spaces

5.1. Approximation by smooth functions

5.2. Embedding Theorems

5.3. Trace Theorems

5.4. Difference Quotients

6. Existence and basic regularity theory for Laplace Equation

6.1. Existence: Proof of Theorem 6.1

6.2. Uniqueness: Proof of Theorem 6.3

6.3. Interior regularity theory: Proof of Theorem 6.4

6.4. Global/Boundary regularity theory: Proof of Theorem 6.6

6.5. An alterative approach to boundary regularity theory: reflection

6.6. Extension to more general elliptic equations

7. The Role of Harmonic Analysis in PDE – $L^p$-theory
7.1. Short introduction to Calderon-Zygmund Theory
7.2. Calderon-Zygmund operators
7.3. $W^{1,p}$-theory for the Laplace equation
7.4. $W^{1,p}$-theory for a constant coefficient linear elliptic equation
7.5. $W^{1,p}$-theory for a Hölder continuous coefficient linear elliptic equation
8. Schauder theory
9. De Giorgi - Nash - Moser iteration and De Giorgi’s theorem
  9.1. Boundedness
In Analysis
there are no theorems
only proofs
A substantial part of these notes are strongly inspired on [Evans, 2010] and lectures by Heiko von der Mosel (RWTH Aachen). Parts of the parabolic part of these notes have been typed by Julian Scheuer (U Frankfurt).

References


<table>
<thead>
<tr>
<th>Index</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>a priori estimates,</td>
<td>51</td>
</tr>
<tr>
<td>barrier,</td>
<td>47</td>
</tr>
<tr>
<td>bootstrap,</td>
<td>125</td>
</tr>
<tr>
<td>boundary problems,</td>
<td>16</td>
</tr>
<tr>
<td>bump function,</td>
<td>50</td>
</tr>
<tr>
<td>Calderon-Zymgund theory,</td>
<td>99</td>
</tr>
<tr>
<td>Cavalieri’s principle,</td>
<td>8</td>
</tr>
<tr>
<td>chart,</td>
<td>9</td>
</tr>
<tr>
<td>classical solution,</td>
<td>50</td>
</tr>
<tr>
<td>cutoff function,</td>
<td>50</td>
</tr>
<tr>
<td>differentiating the equation,</td>
<td>52</td>
</tr>
<tr>
<td>Dirichlet boundary problem,</td>
<td>16</td>
</tr>
<tr>
<td>Dirichlet principle,</td>
<td>56</td>
</tr>
<tr>
<td>Dirichlet-data,</td>
<td>17</td>
</tr>
<tr>
<td>Dirichlet-problem,</td>
<td>17</td>
</tr>
<tr>
<td>Dirichlet-to-Neumann principle,</td>
<td>39</td>
</tr>
<tr>
<td>distributional,</td>
<td>50</td>
</tr>
<tr>
<td>distributional solutions,</td>
<td>49</td>
</tr>
<tr>
<td>divergence form,</td>
<td>60</td>
</tr>
<tr>
<td>eigenvalues,</td>
<td>65</td>
</tr>
<tr>
<td>Einstein summation convention,</td>
<td>59, 97</td>
</tr>
<tr>
<td>elliptic,</td>
<td>12, 14, 60, 97</td>
</tr>
<tr>
<td>elliptic equation,</td>
<td>97</td>
</tr>
<tr>
<td>energy decay,</td>
<td>83</td>
</tr>
<tr>
<td>Euler-Lagrange-equations,</td>
<td>56</td>
</tr>
<tr>
<td>exterior sphere condition,</td>
<td>47</td>
</tr>
<tr>
<td>flattening the boundary,</td>
<td>109</td>
</tr>
<tr>
<td>fractional Sobolev space,</td>
<td>95</td>
</tr>
<tr>
<td>freezing,</td>
<td>123</td>
</tr>
<tr>
<td>fully nonlinear,</td>
<td>11</td>
</tr>
<tr>
<td>fundamental solution,</td>
<td>17, 19, 73</td>
</tr>
<tr>
<td>Gagliardo-seminorm,</td>
<td>95</td>
</tr>
<tr>
<td>Green’s divergence theorem,</td>
<td>8</td>
</tr>
<tr>
<td>harmonic,</td>
<td>17</td>
</tr>
<tr>
<td>harmonic extension,</td>
<td>37</td>
</tr>
<tr>
<td>heat ball,</td>
<td>74</td>
</tr>
<tr>
<td>heat kernel,</td>
<td>72, 73</td>
</tr>
<tr>
<td>homogeneous,</td>
<td>16</td>
</tr>
<tr>
<td>homogeneous heat equation,</td>
<td>72</td>
</tr>
<tr>
<td>homogeneous Laplace equation,</td>
<td>17</td>
</tr>
<tr>
<td>Hopf Lemma,</td>
<td>66</td>
</tr>
<tr>
<td>hyperbolic,</td>
<td>12, 14</td>
</tr>
<tr>
<td>infinite speed of propagation,</td>
<td>76</td>
</tr>
<tr>
<td>inhomogeneous,</td>
<td>72</td>
</tr>
<tr>
<td>inhomogenous,</td>
<td>16</td>
</tr>
<tr>
<td>Integration by parts,</td>
<td>8</td>
</tr>
<tr>
<td>interior sphere condition,</td>
<td>70</td>
</tr>
<tr>
<td>inverse Fouriertransform,</td>
<td>17</td>
</tr>
<tr>
<td>iteration argument,</td>
<td>33</td>
</tr>
<tr>
<td>Laplace equation,</td>
<td>15</td>
</tr>
<tr>
<td>linear,</td>
<td>11</td>
</tr>
<tr>
<td>linear elliptic equations,</td>
<td>59</td>
</tr>
<tr>
<td>maximum principles,</td>
<td>25</td>
</tr>
<tr>
<td>mean value property,</td>
<td>23</td>
</tr>
<tr>
<td>Neumann,</td>
<td>59</td>
</tr>
<tr>
<td>Neumann boundary problem,</td>
<td>70</td>
</tr>
<tr>
<td>Neumann problem,</td>
<td>59</td>
</tr>
<tr>
<td>Neumann-data,</td>
<td>17</td>
</tr>
<tr>
<td>Neumann-problem,</td>
<td>17</td>
</tr>
<tr>
<td>Newton potential,</td>
<td>18</td>
</tr>
<tr>
<td>nontrivial,</td>
<td>65</td>
</tr>
<tr>
<td>oscillation,</td>
<td>32</td>
</tr>
<tr>
<td>outwards facing unit normal,</td>
<td>17</td>
</tr>
<tr>
<td>parabolic,</td>
<td>12, 14</td>
</tr>
<tr>
<td>parabolic boundary,</td>
<td>75</td>
</tr>
<tr>
<td>Poisson equation,</td>
<td>16</td>
</tr>
<tr>
<td>Poisson formula,</td>
<td>37</td>
</tr>
<tr>
<td>Polar Coordinates,</td>
<td>8</td>
</tr>
<tr>
<td>potential,</td>
<td>18</td>
</tr>
<tr>
<td>principal value,</td>
<td>38</td>
</tr>
<tr>
<td>quasilinear,</td>
<td>11</td>
</tr>
<tr>
<td>radial,</td>
<td>9</td>
</tr>
<tr>
<td>regularity theory,</td>
<td>51</td>
</tr>
<tr>
<td>Riesz potential,</td>
<td>18</td>
</tr>
<tr>
<td>Schauder theory,</td>
<td>99</td>
</tr>
<tr>
<td>semi-group theory,</td>
<td>72</td>
</tr>
<tr>
<td>semilinear,</td>
<td>11</td>
</tr>
<tr>
<td>smooth,</td>
<td>91</td>
</tr>
<tr>
<td>Sobolev space,</td>
<td>88</td>
</tr>
<tr>
<td>Sobolev-Slobodeckij,</td>
<td>95</td>
</tr>
<tr>
<td>stationary,</td>
<td>72</td>
</tr>
<tr>
<td>Stokes’ theorem,</td>
<td>8</td>
</tr>
</tbody>
</table>
strictly convex, 101
strong maximum principle, 25, 75
strong solution, 50
subharmonic, 17, 41
subsolution, 17, 60
superharmonic, 17
supersolution, 17, 60
test-functions, 50
trace sense, 89
trace space, 95
Tychonoff, 79
variation, 71
Viscosity solutions, 49
weak maximum principle, 25, 62, 75
weak solution, 49, 57
Wirtinger’s inequality, 93
Basic formulas and concepts we’ll use a lot

**Integration by parts, Green’s Theorem, Stokes theorem.** If \( \Omega \subset \mathbb{R}^n \) is a (nice) open bounded set with outwards facing unit normal \( \nu = (\nu^1, \ldots, \nu^n) : \partial \Omega \to S^{n-1} \) (\( S^{n-1} \) are simply the vectors \( v \in \mathbb{R}^n \) with \( |v| = 1 \), i.e. the unit sphere) and \( f, g \) are (nice) functions then we have for any \( \alpha \in \{1, \ldots, n\} \)

\[
\int_{\Omega} f \partial_\alpha g = \int_{\partial \Omega} fg \nu^\alpha - \int_{\Omega} \partial_\alpha fg \tag{0.1}
\]

Observe that if \( n = 1 \) and \( \Omega = (a, b) \) then \( \nu(a) = -1 \) and \( \nu(b) = +1 \), and then we have the usual one-dimension integration by parts formula

\[
\int_{(a,b)} f \partial_\alpha g = f(b)g(b) - f(a)g(a) - \int_{(a,b)} \partial_\alpha fg \tag{0.2}
\]

- works also for \( \mathbb{R}^n \) or unbounded set \( \Omega \) – as long as

\[
\lim_{|x| \to \infty} f(x) = \lim_{|x| \to \infty} g(x) = 0
\]

- Green’s formula (divergence theorem) is normally written for vector fields \( G = (G^1, G^2, \ldots, G^n) : \Omega \to \mathbb{R}^n \),

\[
\int_{\Omega} \text{div}(G) = \sum_{\alpha=1}^n \int_{\Omega} \partial_\alpha G = \sum_{\alpha=1}^n \int_{\partial \Omega} \nu^\alpha G^\alpha - \sum_{\alpha=1}^n \int_{\Omega} (\partial_\alpha 1) G^\alpha = \int_{\partial \Omega} G \cdot \nu
\]

**Exercise 0.1.** Use Green’s formula

\[
\int_{\Omega} \text{div}(G) = \int_{\partial \Omega} G \cdot \nu
\]

to prove the integration by parts formula (0.1).

**Exercise 0.2.** Use (0.2) to show (0.1)

(You can use pictures and a simple set \( \Omega \) – I care about the idea, not the most general case)

**Polar coordinates.** Let \( f : B(0, R) \to \mathbb{R}^n \) (nice) then

\[
\int_{B(0,R)} f(x)dx = \int_0^R \int_{\partial B(0,\rho)} f(\theta) d\theta d\rho \tag{0.3}
\]

This is actually Fubini’s theorem (or *Cavalieri’s principle*), and really isn’t that related to polar coordinates (well, there is a sphere...) We call it polar coordinates anyways. By a substitution we can write this as

\[
\int_{B(0,R)} f(x)dx = \int_0^R \rho^{n-1} \int_{\partial B(0,\rho)} f(\rho \theta) d\theta d\rho \tag{0.4}
\]

**Exercise 0.3.** Prove (0.4) using (0.3)
A special case is the case when \( f : B(0, R) \to \mathbb{R} \) is **radial**. \( f \) is called radial if \( f(x) = f(Qx) \) for all rotation matrices \( Q \in O(n) \) (i.e. \( Q \in \mathbb{R}^{n \times n}, Q^t Q = I \)).

**Exercise 0.4.** Show that if \( f \) is radial then there exists \( g : [0, \infty) \to \mathbb{R} \) such that \( f(x) = g(|x|) \).

Thus, one often writes “\( f \) radial” as “\( f(x) = f(|x|) \)” (this is an idiotic notation, but we’ll still use it).

If \( f \) is radial then

\[
(0.5) \quad \int_{B(0,R)} f(x) dx = \int_0^R \rho^{n-1} f(\rho) d\rho |\partial B(0,1)|, 
\]

where \( |\partial B(0,1)| \) denotes the area of \( \partial B(0,1) \), i.e. \( |\partial B(0,1)| = \mathcal{H}^{n-1}(\partial B(0,1)) \).

**Exercise 0.5.** Show \((0.5)\) using \((0.4)\) or \((0.3)\)

**Regular Sets**

We are often going to talk about open sets \( \Omega \) with smooth boundary, \( \partial \Omega \in C^k \) or \( \partial \Omega \in C^\infty \) or similar. When we say \( \Omega \subset \mathbb{R}^n \) with \( \partial \Omega \in C^k \) we mean that \( M := \partial \Omega \) is a \( C^k \)-manifold. That is, for each \( x \in \partial \Omega \) there exists a small ball \( B(x, r) \subset \mathbb{R}^n \) and an associated chart \( \Phi : B(x, r) \to \mathbb{R}^n \), which must be a \( C^k \)-diffeomorphism (\( \Phi \) is invertible and \( \Phi, \Phi^{-1} \) are \( C^k \)-maps in their respective domain) and

\[
\Phi(B(x, r) \cap \partial \Omega) \subset \mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}
\]

and

\[
\Phi(B(x, r) \setminus \partial \Omega) \subset \overline{\mathbb{R}^n_-} = \{(x', x_n) \in \mathbb{R}^n : x_n \le 0\}
\]

and \( \Phi(x) = 0 \). Cf. Figure 0.1.
Part 1. PDE 1

1. INTRODUCTION AND SOME BASIC NOTATION

When studying Partial Differential Equations (PDEs) the first question that arises is: what are partial differential equations.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $u : \Omega \rightarrow \mathbb{R}$ be differentiable. The partial derivatives $\partial_1$ is the directional derivative

$$\partial_1 u(x) \equiv \partial_{x_1} u(x) = \frac{d}{dx_1} u(x) = \frac{d}{dt} \Big|_{t=0} u(x + t e_1),$$

where $e_1 = (1, 0, \ldots, 0)$ is the first unit vector. The partial derivatives $\partial_2, \ldots \partial_n$ are defined likewise.

Sometimes it is convenient to use multiindices: an $n$-multiindex $\gamma$ is a vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ where $\gamma_1, \ldots, \gamma_n \in \{0, 1, 2, \ldots, \}$. The order of a multiindex is $|\gamma|$ defined as

$$|\gamma| = \sum_{i=1}^{n} \gamma_i.$$

For a suitable often differentiable function $u : \Omega \rightarrow \mathbb{R}$ and a multiindex $\gamma$ we denote with $\partial^\gamma u$ the partial derivatives

$$\partial^\gamma u(x) = \partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2} \ldots \partial_{x_n}^{\gamma_n} u(x).$$

For example, for $\gamma = (1, 0, 0, \ldots, 0)$ we have

$$\partial^\gamma u(x) = \partial_{x_1} u,$$

i.e. a partial derivative of first order; and for $\gamma = (1, 2, 0, \ldots, 0)$ we have

$$\partial^\gamma u = \partial_{12} u \equiv \partial_1 \partial_2 u,$$

i.e. a partial derivative of 3rd order.

The collection of all partial derivatives of $k$-th order of $u$ is usually denoted by $D^k u(x) \in \mathbb{R}^{n^k}$ or (the “gradient”) $\nabla^k u$. Usually these are written in matrix form, namely

$$Du(x) = (\partial_1 u(x), \partial_2 u(x), \partial_3 u(x), \ldots, \partial_n u(x))$$

and

$$D^2 u(x) = (\partial_{ij} u)_{i,j=1,\ldots,n} \equiv \begin{pmatrix} \partial_{11} u(x) & \partial_{12} u(x) & \partial_{13} u(x) & \ldots & \partial_{1n} u(x) \\ \partial_{21} u(x) & \partial_{22} u(x) & \partial_{23} u(x) & \ldots & \partial_{2n} u(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \partial_{n1} u(x) & \partial_{n2} u(x) & \partial_{n3} u(x) & \ldots & \partial_{nn} u(x) \end{pmatrix}.$$

**Definition 1.1.** Let $\Omega \subset \mathbb{R}^n$ an open set and $k \in \mathbb{N} \cup \{0\}$. A partial differential equation (PDE) of $k$-th order is an expression of the form

$$F(D^k u(x), D^{k-1} u(x), D^{k-2} u(x), \ldots, Du(x), u(x), x) = 0 \quad x \in \Omega,$$
where \( u : \Omega \to \mathbb{R} \) is the unknown (also the “solution” to the PDE) and \( F \) is a given structure (i.e. map)

\[
F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \ldots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}
\]

- (1.1) is called **linear** if \( F \) is linear in \( u \): meaning if we can find for every \( n \)-multiindex \( \gamma \) with \( |\gamma| \leq k \) a function \( a_\gamma : \Omega \to \mathbb{R} \) (independent of \( u \)) such that

\[
F(D^k u(x), D^{k-1} u(x), \ldots, D u(x), u(x), x) = \sum_{|\gamma| \leq k} a_\gamma(x) \partial^\gamma u(x)
\]

- (1.1) is called **semilinear** if \( F \) is linear with respect to the highest order \( k \), namely if

\[
F(D^k u(x), D^{k-1} u(x), D^{k-2} u(x), \ldots, D u(x), u(x), x) = \sum_{|\gamma|=k} a_\gamma(x) \partial^\gamma u(x) + G(D^{k-1} u(x), D^{k-2} u(x), \ldots, D u(x), u(x), x)
\]

- (1.1) is called **quasilinear** if \( F \) is linear with respect to the highest order \( k \) but the coefficient for the highest order may depend on the lower order derivatives of \( u \).

Namely if we have a representation of the form

\[
F(D^k u(x), D^{k-1} u(x), D^{k-2} u(x), \ldots, D u(x), u(x), x) = \sum_{|\gamma|=k} a_\gamma(D^{k-1} u(x), D^{k-2} u(x), \ldots, D u(x), u(x), x) \partial^\gamma u(x)
\]

- If all the above do not apply then we call \( F \) **fully nonlinear**.

We have a system of partial differential equations of order \( k \), if \( u : \Omega \to \mathbb{R}^m \) is a vector and/or the structure function \( F \) is also a vector

\[
F : \mathbb{R}^{m^k} \times \mathbb{R}^{m^{k-1}} \times \ldots \times \mathbb{R}^m \times \mathbb{R} \times \Omega \to \mathbb{R}^\ell
\]

for \( m, \ell \geq 1 \).

The goal in PDE is usually (besides modeling what PDE describes what situation) to solve PDEs, possibly subject to side-condition (such as prescribed boundary data on \( \partial \Omega \)).

This is rarely possible explicitly (even in the linear case) – which is a huge contrast to ODE. E.g.

\[
u''(x) = 2u(x) \quad x \in \mathbb{R},
\]

then we know that \( u(x) = e^{\sqrt{2}x} A \), and we can compute \( A \) by prescribing some initial value at \( x = 0 \) or similar.

Now if we try that in two dimensions, and consider

\[
\Delta u(x) \equiv \partial_{11} u(x) + \partial_{22} u(x) = 2u(x) \quad x \in B(0,1) \subset \mathbb{R}^2,
\]

it is really difficult to see what \( u \) is (observe that also the amount of initial data – e.g. values at \( \partial B(0,1) \) is not one, but infinitely many!

So in general the best one can hope for is address the following main questions for PDEs are
• Is there a solution to a problem (and if so: in what sense? – we will learn the
distributional/weak sense and strong sense)
• Are solutions unique (under reasonable assumptions like initial data, boundary
data?)?
• What are properties of the solutions (e.g. does the solution depend continuously
on the data of the problem)?

It is important to accept that there are PDEs without (classical) solutions and there is no
general theory of PDEs. There is theory for several types of PDES.

**Example 1.2** (Some basic linear equations).

- **Laplace equation**
  \[ \Delta u := \sum_{i=1}^{n} u_{x_{i}x_{i}} = 0. \]

- **Eigenvalue equation** (aka Helmholtz equation)
  \[ \Delta u = \lambda u. \]

- **Transport equation**
  \[ \partial_{t}u - \sum_{i=1}^{n} b_{i}u_{x_{i}} = 0 \]

- **Heat equation**
  \[ \partial_{t}u - \Delta u = 0 \]

- **Schrödinger equation**
  \[ i\partial_{t}u + \Delta u = 0 \]

- **Wave equation**
  \[ u_{tt} - \Delta u = 0 \]

Second order linear equations are classified into *elliptic, parabolic, hyperbolic* PDE. Roughly
this is understood as follows. Assume that \( u \) depends on \( x \) and \( t \), then

- **elliptic** means the equation is of the form
  \[ u_{xx} + u_{tt} = G(x, y, u, u_{t}, u_{x}) \]

- **parabolic** means
  \[ u_{xx} = G(x, y, u, u_{t}, u_{x}) \]

- **Hyperbolic**
  \[ u_{xx} - u_{tt} = G(x, y, u, u_{t}, u_{x}) \]
  or
  \[ u_{x,t} = G(x, y, u, u_{t}, u_{x}) \]
Let us have a generic second order linear equation
\[ Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = g \]
(for now let us assume that \( A, B, \ldots \) are constant.) We can write the second-order part as
\[ Au_{xx} + Bu_{xy} + Cu_{yy} = \begin{pmatrix} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{pmatrix} : \begin{pmatrix} \partial_{xx}u \\ \partial_{xy}u \\ \partial_{yx}u \\ \partial_{yy}u \end{pmatrix}, \]
where : denotes the matrix scalar product (sometimes: Hilbert-Schmidt product). If \( AC - \frac{1}{4}B^2 > 0 \) the determinant of the coefficient matrix is positive, i.e. either the matrix has two positive eigenvalues \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \) or two negative eigenvalues \( \lambda_1 < 0 \) and \( \lambda_2 < 0 \), and there exists orthogonal matrices \( P \in SO(2) \) such that
\[ P^T \begin{pmatrix} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{pmatrix} P = \text{diag}(\lambda_1, \lambda_2). \]
Then we have
\[ \left( \begin{array}{cc} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{array} \right) : \begin{pmatrix} \partial_{xx}u \\ \partial_{xy}u \\ \partial_{yx}u \\ \partial_{yy}u \end{pmatrix} = \text{diag}(\lambda_1, \lambda_2) : P^T D^2 u P. \]
Now consider \( \tilde{u}(x, y) := u(P(x, y)^t) \), then by the chain rule,
\[ D^2 \tilde{u}(x, y) = P^t(D^2 u)(P(x, y))P, \]
so that if we set \( (\tilde{x}, \tilde{y})^t := P(x, y)^t \) we have
\[ \lambda_1 u_{\tilde{x}\tilde{x}} + \lambda_2 u_{\tilde{y}\tilde{y}} = G(u, u_x, u_y), \]
that is if \( AC - \frac{1}{4}B^2 > 0 \) we can transform our equation into an elliptic equation.

Similarly, if \( AC - \frac{1}{4}B^2 = 0 \), at least one eigenvalue of the matrix in question is negative, one is positive, so we can transform the equation into
\[ \lambda_1 u_{\tilde{x}\tilde{x}} - \lambda_2 u_{\tilde{y}\tilde{y}} = G(u, u_x, u_y), \]
i.e. a hyperbolic equation.

And if \( AC - \frac{1}{4}B^2 < 0 \), one of the eigenvalues is zero, so that we have the structure
\[ \lambda_1 u_{\tilde{x}\tilde{x}} = G(u, u_x, u_y), \]
i.e. we are parabolic.

Whether one is elliptic, parabolic, hyperbolic is not purely an algebraic question – it often determines the ways we can understand properties of the equation in question. Often we think of elliptic equation as equilibrium or stationary equations, parabolic equations as a flow of an energy, and hyperbolic of a wave-like equation – but this is not really always the case, since the Schrödinger equation is parabolic in the previous sense, but it is wave-like. It generally holds: every type of equation warrants its own methods.
One can extend this theory, of course, to higher dimensions. If
\[
\sum_{i,j=1}^{n} A_{ij} \partial_{x_i,x_j} u + \sum_{i=1}^{n} B_i \partial_{x_i} u + Cu = D,
\]
then we may assume that \(A\) is symmetric (any antisymmetric part vanishes because \((\partial_{x_i,x_j} u)_{ij}\) is symmetric) – and thus we can discuss its eigenvalues.

- The equation is **elliptic** if all eigenvalues are nonzero and have the same sign.
- The equation is **parabolic** if exactly one eigenvalue is zero, all others are nonzero and have the same sign.
- The equation is **hyperbolic** if no eigenvalue is zero, and \(n - 1\) eigenvalues have the same sign, and the other one has the opposite sign.

Of course there are more cases (and they may be very challenging to treat). In principle: elliptic means the second order derivatives “move in the same direction”, parabolic means “all but one direction move in the same direction and the remaining direction is of first order only”, and hyperbolic “the second derivatives compete with each other”.

Of course, since in general \(A\) and \(B\) are nonconstant, the type of equation may change and depend on the point \(x\) (e.g. \(tu_{xx} + u_t = 0\)).

**Example 1.3** (Some basic nonlinear equations).

- **Eikonal equation**
  \[ |Du| = 1 \]
- **p-Laplace equation**
  \[ \text{div} \left( |Du|^{p-2} Du \right) \equiv \sum_{i=1}^{n} \partial_i (|Du|^{p-2} \partial_i u) = 0 \]
- **Minimal surface equation**
  \[ \text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0. \]
- **Monge-Ampere**
  \[ \det(D^2 u) = 0. \]
- **Hamilton-Jacobi**
  \[ \partial_t u + H(Du, x) = 0 \]

The notion of what constitutes a solution is important, as a too weak notion allows for too many solutions, and a too strong notion of solution may allow for no solutions at all. We illustrate this for the Eikonal equation:

**Exercise 1.4.** We consider different notions of solutions for the Eikonal equation:
(1) Consider

\[
\begin{aligned}
|u'(x)| &= 1 & x & \in (-1, 1) \\
u(-1) &= u(1) = 0.
\end{aligned}
\]

Show that there is \textbf{no} \( u \in C^0([-1, 1]) \cap C^1((-1, 1)) \) such that (1.2) is satisfied.

(2) Consider instead

\[
\begin{aligned}
|u'(x)| &= 1 & \text{all but finitely many } x & \in (-1, 1) \\
u(-1) &= u(1) = 0.
\end{aligned}
\]

Show that there are \textbf{infinitely} many solutions \( u \in C^0([-1, 1]) \) that are differentiable in all but finitely many points in \((-1, 1)\) such that (1.3) is satisfied.

(3) Show that there is a sequence \( u_k \in C^0([-1, 1]) \) that are differentiable in all but finitely many points in \((-1, 1)\), such that

\[
\sup_{x \in [-1, 1]} |u_k(x) - 0| \xrightarrow{k \to \infty} 0.
\]

(4) Consider instead

\[
\begin{aligned}
|u'(x)| &= 1 & \text{in all but one } x & \in (-1, 1) \\
u(-1) &= u(1) = 0.
\end{aligned}
\]

Show that there are still \textbf{two} solutions \( u \in C^0([-1, 1]) \) that are differentiable in all but at most one points in \((-1, 1)\) such that (1.4) is satisfied.

In this course we will focus on the linear theory (the nonlinear theory is \textit{almost} always based on ideas on the linear theory). Almost each of the linear and nonlinear equations warrants its own course, so we will focus on the basics (namely: mainly elliptic equations).

2. Laplace equation

2.1. \textbf{Sort of a physical motivation}. The following is often used to motivate Laplace’s equation

Assume \( \Omega \) is an open set in \( \mathbb{R}^n \) (usually \( \mathbb{R}^3 \)), and \( u \) describes the density of a fluid or heat that is at an equilibrium state, i.e. no fluid is moving in or out, or not heat is exchanged any more. This means that if we look at any subset \( \Omega' \subset \Omega \) nothing flows out or in that would change the density, that is

\[
\int_{\partial \Omega'} \nabla u \cdot \nu = 0.
\]

By Green’s divergence theorem this is equivalent to saying

\[
\int_{\Omega'} \text{div} (\nabla u) = 0.
\]
Figure 2.1. Solve $\Delta u = 0$ on the annulus (inner radius $r = 2$ and outer radius $R = 4$) with boundary condition $g(\theta) = 0$ if $|\theta| = 2$ and $g(\theta) = 4\sin(5\sigma)$ for $|\theta| = 4$ – where $\sigma \in [0, 2\pi)$ is the angle such that $(\sin(\sigma), \cos(\sigma)) = \theta/|\theta|$. Source: Fourthirtytwo/Wikipedia CC-SA 3

Since this happens for all $\Omega'$ we obtain that

$$\text{div}(\nabla u) = 0$$

So we call $\text{div}(\nabla u) =: \Delta u$ and observe that $\Delta u = \sum_{i=1}^n \partial_{x_i x_i} u = \text{tr}(D^2 u)$.

Often one thinks of Laplace equation $\Delta u = 0$ in $\Omega$ as a heat distribution. Take $\Omega$ a solid, apply some heat at its boundary: at $\theta \in \partial \Omega$ we apply $g(\theta)$ heat. Wait until the heat had time to fully propagate. Then the solution $u : \Omega \to \mathbb{R}$ to the Dirichlet boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial \Omega \end{cases}$$

describes the temperature $u(x)$ at the point $x \in \Omega$. Cf. Figure 2.1.

2.2. Definitions. Let $\Omega \subset \mathbb{R}^n$ be an open set (this will always be the case from now on).

- We consider the \textit{homogeneous} Laplace equation

$$\Delta u = 0 \quad \text{in } \Omega \tag{2.1}$$

where we recall that $\Delta u = \text{tr}(D^2 u) = \sum_{i=1}^n \partial_{x_i x_i} u$.

- The \textit{inhomogenous} equation (sometimes: \textit{Poisson equation}) is, for a given function $f : \Omega \to \mathbb{R}$,

$$\Delta u = f \quad \text{in } \Omega \tag{2.2}$$

Two types of \textit{boundary problems} are common:
• **Dirichlet-problem** or **Dirichlet-data** \( g : \partial \Omega \to \mathbb{R} \)

\[
\begin{align*}
\Delta u &= f \quad \text{in } \Omega \\
u &= g \quad \text{on } \partial \Omega
\end{align*}
\]

• **Neumann-problem** or **Neumann-data** \( g : \partial \Omega \to \mathbb{R} \)

\[
\begin{align*}
\Delta u &= f \quad \text{in } \Omega \\
\partial_{\nu} u &= g \quad \text{on } \partial \Omega
\end{align*}
\]

Here \( \nu : \partial \Omega \to \mathbb{R}^n \) is the *outwards facing unit normal* of \( \partial \Omega \). (Often this is combined with a normalizing assumption like \( \int_{\Omega} u = 0 \), because \( u + c \) is otherwise a solution if \( u \) is a solution – i.e. non-uniqueness occurs).

**Definition 2.1.** A function \( u \in C^2(\Omega) \) is called *harmonic* if \( u \) pointwise solves

\[
\Delta u(x) = 0 \quad \text{in } \Omega
\]

We also say, \( u \) is a solution to the *homogeneous Laplace equation*.

We say that \( u \) is a *subsolution* or *subharmonic* if

\[
\Delta u(x) \geq 0 \quad \text{in } \Omega.
\]

If

\[
\Delta u(x) \leq 0 \quad \text{in } \Omega
\]

we say that \( u \) is a *supersolution* or *superharmonic*.

This notion is very confusing, but it comes from the fact that \(-\Delta u\) is a “positive operator” (i.e. has only positive eigenvalues).

### 2.3. Fundamental Solution, Newton- and Riesz Potential.

There are many trivial solutions (polynomials of order 1) of Laplace equation. But these are not very interesting. There is a special type of solution which is called *fundamental solution* (which, funny enough, is actually not a solution).

It appears when we want to compute the solution to an equation on the whole space

\[
\Delta u(x) = f(x).
\]

For this we make a brief (formal) introduction to Fourier transform:

The Fourier transform takes a map \( f : \mathbb{R}^n \to \mathbb{R} \) and transforms it into \( \mathcal{F}u \equiv \hat{f} : \mathbb{R}^n \to \mathbb{R} \) as follows

\[
\hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\langle \xi, x \rangle} f(x) \, dx.
\]

The *inverse Fourier transform* \( f^\vee \) is defined as

\[
f^\vee(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{+i\langle \xi, x \rangle} f(x) \, dx.
\]
It has the nice property that \((f^\wedge)^\vee = f\).

One of the important properties (which we will check in exercises) is that derivatives become polynomial factors after Fourier transform:

\[(\partial_x g)^\wedge (\xi) = -i\xi \hat{g}(\xi).\]

For the Laplace operator \(\Delta\) this implies

\[(\Delta u)^\wedge(\xi) = -|\xi|^2 \hat{u}(\xi).\]

(Side-remark: In this sense \(-\Delta\) is positive operator).

This means that if we look at the equation \((2.3)\) and apply Fourier transform on both sides we have

\[-|\xi|^2 \hat{u}(\xi) = \hat{f}(\xi),\]

that is

\[\hat{u}(\xi) = -|\xi|^{-2} \hat{f}(\xi),\]

Inverting the Fourier transform we get an explicit formula for \(u\) in terms of the data \(f\).

\[u(x) = -\left(|\xi|^{-2} \hat{f}(\xi)\right)^\vee (x).\]

This is not a very nice formula, so let us simplify it. Another nice property of Fourier transform (and its inverse) is that products become convolutions. Namely

\[(g(\xi) f(\xi))^\vee (x) = \int_{\mathbb{R}^n} g^\vee(x-z) f^\vee(z) \, dz.\]

In our case, for \(g(\xi) = -|\xi|^{-2}\) we get that

\[u(x) = \int_{\mathbb{R}^n} g^\vee(x-z) f(z) \, dz.\]

Now we need to compute \(g^\vee(x-z)\), and for this we restrict our attention to the situation where the dimension is \(n \geq 3\). In that case, just by the definition of the (inverse) Fourier transform we can compute that since \(g\) has homogeneity of order 2 (i.e. \(g(t\xi) = t^{-2} g(\xi)\)), then \(g^\vee\) is homogeneous of order \(2-n\). In particular

\[g^\vee(x) = |x|^{2-n} g^\vee(x/|x|)\] .

Now an argument that radial functions stay radial under Fourier transforms leads us to conclude that

\[g^\vee(x) = c_1 |x|^{2-n}\]

That is, we have arrived that (by formal computations) a solution of \((2.3)\) should satisfy

\[u(x) = c_1 \int_{\mathbb{R}^n} |x-z|^{2-n} f(z) \, dz.\]

The constant \(c_1\) can be computed explicitly, and we will check below that this potential representation of \(u\) really is true. This potential is called the Newton potential (which is a special case of so-called Riesz potentials). The kernel of the Newton potential is called the fundamental solution of the Laplace equation (which, let us stress this again, is not a solution)
Definition 2.2. The fundamental solution \( \Phi(x) \) of the Laplace equation for \( x \neq 0 \) is given as

\[
\Phi(x) = \begin{cases} 
-\frac{1}{2\pi} \log |x| & \text{for } n = 2 \\
-\frac{1}{n(n-2)} \omega_n |x|^{2-n} & \text{for } n \geq 2
\end{cases}
\]

Here \( \omega_n \) is the Lebesgue measure of the unit ball \( \omega_n = |B(0,1)| \).

One can explicitly check that \( \Delta \Phi(x) = 0 \) for \( x \neq 0 \) (indeed, \( \Delta \Phi(x) = \delta_0 \) where \( \delta_0 \) is the Dirac measure at the point zero, cf. remark 2.7).

Exercise 2.3. Show that \( \Phi \in C^\infty(\mathbb{R}^n \setminus \{0\}) \) and compute that \( \Delta \Phi(x) = 0 \) for \( x \neq 0 \).

The following statement justifies (somewhat) the notion of fundamental solution: the fundamental solution \( \Phi(x) \) can be used to construct all solutions to the inhomogeneous Laplace equation:

Theorem 2.4. Let \( u \) be the Newton-potential of \( f \in C^2_c(\mathbb{R}^n) \), that is

\[
u(x) := \int_{\mathbb{R}^n} \Phi(x - y) f(y) dy.\]

Here \( C^2_c(\mathbb{R}^n) \) are all those functions in \( C^2(\mathbb{R}^n) \) such that \( f \) is constantly zero outside of some compact set.

We have

\begin{itemize}
  \item \( u \in C^2(\mathbb{R}^n) \)
  \item \(-\Delta u = f \) in \( \mathbb{R}^n \).
\end{itemize}

Proof. First we show differentiability of \( u \). By a substitution we may write

\[
u(x) := \int_{\mathbb{R}^n} \Phi(x - y) f(y) dy = \int_{\mathbb{R}^n} \Phi(z) f(x - z) dz.
\]

Now if we denote the difference quotient

\[
\Delta^e_h u(x) := \frac{u(x + he_i) - u(x)}{h}
\]

where \( e_i \) is the \( i \)-th unit vector, then we obtain readily

\[
\Delta^e_h u(x) := \int_{\mathbb{R}^n} \Phi(x - y) f(y) dy = \int_{\mathbb{R}^n} \Phi(z) (\Delta^e_h f)(x - z) dz.
\]

One checks that \( \Phi \) is locally integrable (it is not globally integrable!), that is for every bounded set \( \Omega \subset \mathbb{R}^n \),

\[
\int_{\Omega} |\Phi| < \infty.
\]
Indeed, (we show this for $n \geq 3$, the case $n = 2$ is an exercise), if $\Omega \subset \mathbb{R}^n$ is a bounded set, then it is contained in some large ball $B(0, R)$.

\begin{equation}
(2.6) \quad \int_{\Omega} |\Phi| \leq C \int_{B(0,R)} |x|^{2-n} \, dx
\end{equation}

Using Fubini’s theorem,

\begin{align*}
\int_{B(0,R)} |x|^{2-n} \, dx &= \int_0^R \int_{\partial B(0,r)} \theta^{2-n} \, dH^{n-1}(\theta) \, dr \\
&= \int_0^R r^{2-n} \int_{\partial B(0,r)} dH^{n-1}(\theta) \, dr \\
&= c_n \int_0^R r^{2-n} r^{n-1} \, dr \\
&= c_n \int_0^R r^1 \, dr \\
&= c_n \frac{1}{2} R^2 < \infty.
\end{align*}

This establishes (2.5)

On the other hand $(\Delta_h^i f)$ has still compact support for every $h$. In particular, by dominated convergence we can conclude that

\begin{equation*}
\lim_{h \to 0} \Delta_h^i u(x) = \int_{\mathbb{R}^n} \Phi(z) \lim_{h \to 0} (\Delta_h^i f)(x - z) \, dz.
\end{equation*}

that is

\begin{equation*}
\partial_i u(x) = \int_{\mathbb{R}^n} \Phi(z) (\partial_i f)(x - z) \, dz.
\end{equation*}

In the same way

\begin{equation*}
\partial_{ij} u(x) = \int_{\mathbb{R}^n} \Phi(z) (\partial_{ij} f)(x - z) \, dz.
\end{equation*}

Now the right-hand side of this equation is continuous (again using the compact support of $f$). This means that $u \in C^2(\mathbb{R}^n)$.

To obtain that $\Delta u = f$ we first use the above argument to get

\begin{equation*}
\Delta u(x) = \int_{\mathbb{R}^n} \Phi(z) (\Delta f)(x - z) \, dz.
\end{equation*}

Observe that

\begin{equation*}
(\Delta f)(x - z) = \Delta x(f(x - z)) = \Delta_z(f(x - z)).
\end{equation*}

Now we fix a small $\varepsilon > 0$ (that we later send to zero) and split the integral, we have

\begin{equation*}
\Delta u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) \, dy = \int_{\mathbb{B}(\theta, \varepsilon)} \Phi(z) (\Delta f)(x-z) \, dz + \int_{\mathbb{R}^n \setminus \mathbb{B}(\theta, \varepsilon)} \Phi(z) (\Delta f)(x-z) \, dz =: I_\varepsilon + I_\varepsilon.
\end{equation*}
The term $I_\varepsilon$ contains the singularity of $\Phi$, but we observe that

$$I_\varepsilon \xrightarrow{\varepsilon \to 0} 0.$$ 

Indeed, this follows from the absolute continuity of the integral and since $\Phi$ is integrable on $B(0, 1)$:

$$|I_\varepsilon| \leq \sup_{\mathbb{R}^n} |\Delta f| \int_{B(x, \varepsilon)} |\Phi(z)| \xrightarrow{\varepsilon \to 0} 0.$$ 

The term $II_\varepsilon$ does not contain any singularity of $\Phi$ which is smooth on $\mathbb{R}^n \setminus B_\varepsilon(0)$, so we can perform an integration by parts$^1$

$$II_\varepsilon = \int_{\mathbb{R}^n \setminus B(0, \varepsilon)} \Phi(z) (\Delta f)(x-z) \, dz = \int_{\partial B(0, \varepsilon)} \Phi(z) \partial_\nu f(x-z) \, d\mathcal{H}^{n-1}(z) - \int_{\mathbb{R}^n \setminus B(0, \varepsilon)} \nabla \Phi(z) \cdot \nabla f(x-z) \, dz.$$ 

Here $\nu$ is the unit normal to the ball $\partial B(0, \varepsilon)$, i.e. $\nu = \frac{-z}{\varepsilon}$.

By the definition of $\Phi$ one computes that (using (2.5))

$$\left| \int_{\partial B(0, \varepsilon)} \Phi(z) \partial_\nu f(x-z) \, d\mathcal{H}^{n-1}(z) \right| \leq \sup_{\mathbb{R}^n} |\nabla f| \int_{\partial B(0, \varepsilon)} |\Phi(z)| \xrightarrow{\varepsilon \to 0} 0.$$ 

So we perform another integration by parts and have

$$II_\varepsilon = o(1) - \int_{\partial B(0, \varepsilon)} \partial_\nu \Phi(z) f(x-z) \, dz + \int_{\mathbb{R}^n \setminus B(0, \varepsilon)} \frac{\Delta \Phi(z)}{=0} f(x-z) \, dz$$

$$= o(1) - \int_{\partial B(0, \varepsilon)} \partial_\nu \Phi(z) f(x-z) \, dz$$

Here in the last step we used that $\Delta \Phi = 0$ away from the origin, Exercise 2.3.

Now we observe that the unit normal on $\partial B(0, \varepsilon)$ is $\nu(z) = \frac{-z}{\varepsilon}$ and

$$D\Phi(z) = \begin{cases} 
-\frac{1}{2\pi} \frac{z}{|z|} & n = 2, \\
\frac{1}{n(n-2)\omega_n} (2-n) |z|^{1-n} \frac{z}{|z|} & n \geq 3.
\end{cases}$$

Thus, for $|z| = \varepsilon$,

$$\partial_\nu \Phi(z) = \nu \cdot D\Phi(z) = \frac{1}{n\omega_n} \varepsilon^{1-n}$$

$\int_{\Omega} f \partial_i g = \int_{\partial\Omega} f g \nu^i - \int_{\Omega} \partial_i f g,$

where $\nu$ is the normal of $\partial \Omega$ pointing outwards (from the point of view of $\Omega$). $\nu^i$ is the $i$-th component of $\nu$. Fun exercise: Check this rule in 1D, to see the relation what we all learned in Calc 1.
Thus we arrive at
\[
II_\varepsilon = o(1) - \int_{\partial B(0,\varepsilon)} \frac{1}{n\omega_n \varepsilon^{n-1}} f(x - z) \, d\mathcal{H}^{n-1}(z)
\]
\[
= o(1) - \frac{1}{\partial B(0,\varepsilon)} f(x - z) \, d\mathcal{H}^{n-1}(z)
\]
\[
= o(1) - f(x) + \int_{\partial B(0,\varepsilon)} (f(x) - f(x - z)) \, d\mathcal{H}^{n-1}(z)
\]
Here we use the mean value notation
\[
\int_{\partial B(0,\varepsilon)} = \frac{1}{\mathcal{H}^{n-1}(\partial B(0,\varepsilon))} \int_{\partial B(0,\varepsilon)}
\]
Now one shows (exercise!) that for continuous \( f \)
\[
\lim_{\varepsilon \to 0} \int_{\partial B(0,\varepsilon)} (f(x) - f(x - z)) \, d\mathcal{H}^{n-1}(z) = 0.
\]
(Indeed this is essentially Lebesgue’s theorem). That is
\[
II_\varepsilon = o(1) - f(x) \quad \text{as } \varepsilon \to 0
\]
and thus
\[
\Delta u(x) = -f(x) + o(\varepsilon),
\]
and letting \( \varepsilon \to 0 \) we have
\[
\Delta u(x) = -f(x),
\]
as claimed. \( \square \)

**Exercise 2.5.** Show that \( \log |x| \) is locally integrable, i.e. that for any bounded set \( \Omega \subset \mathbb{R}^n \) we have
\[
\int_{\Omega} \log |x| < \infty.
\]

**Exercise 2.6.** Assume \( f \) is continuous. Show that
\[
\lim_{\varepsilon \to 0^+} \int_{\partial B(0,\varepsilon)} |f(x) - f(x - z)| \, d\mathcal{H}^{n-1}(z) = 0.
\]

**Remark 2.7.** One can argue (in a distributional sense, which we learn towards the end of the semester)
\[
-\Delta \Phi = \delta_0,
\]
where \( \delta_0 \) denotes the Dirac measure at 0, namely the measure such that
\[
\int_{\mathbb{R}^n} f(x) \, d\delta_0 = f(0) \quad \text{for all } f \in C^0(\mathbb{R}^n).
Observe that $\delta_0$ is not a function, only a measure. In this sense one can justify that
\[-\Delta u(x) = \Delta \int_{\mathbb{R}^n} \Phi(x-z)f(z) dz \]
\[= \int_{\mathbb{R}^n} \Delta \Phi(x-z)f(z) dz \]
\[= \int_{\mathbb{R}^n} f(z) d\delta_x(z) \]
\[= f(x) \]

2.4. **Green Functions.** Our next goal are Green’s functions. In some way Green functions are a restriction of the fundamental solution to domains $\Omega \subset \mathbb{R}^n$ factoring in also boundary data. Recall that for the fundamental solution $\Phi$ we showed in Theorem 2.4 that for the Newton potential
\[(2.7) \quad u(x) := \int_{\mathbb{R}^n} \Phi(x-y)f(y) dy \]
we have $\Delta u = f$. It is an interesting observation that (for reasonable $f$) we have
\[\lim_{|x| \to \infty} u(x) = 0.\]
That is the Newton potential approach solves an equation of
\[
\begin{cases}
\Delta u = f & \text{in } \mathbb{R}^n \\
u = 0 & \text{on the boundary, i.e. for } |x| \to \infty.
\end{cases}
\]
The Greens function is a way to restrict this construction to domains $\Omega$. Instead of the Fundamental solution $\Phi(x-y)$ we get the Green kernel $G(x,y)$ . Instead of the Newton potential we consider
\[u(x) = \int_{\Omega} G(x,y) f(y) dy\]
and hope that this object solves
\[
\begin{cases}
\Delta u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]
The Greens function $G$ (which depends on $\Omega$) can be computed explicitely only for very specific $\Omega$ (balls, half-spaces) – which is somewhat related to the fact that there is not necessarily a reasonable Fourier transform for generic sets $\Omega$.

But one can abstractly show that the Green functions exists for reasonable sets $\Omega$. The idea is as follows: We know that the Newton potential as in (2.7) solves the right equation $\Delta u = f$, but it does not satisfy $u = 0$ on $\partial \Omega$. So let us try to correct the Newton potential and choose the Ansatz
\[u(x) := \int_{\Omega} \Phi(x-y)f(y) dy - \int_{\Omega} H(x,y)f(y) dy\]
By our computations for Theorem 2.4 we have that then for \( x \in \Omega \)
\[
\Delta u(x) := f(x) - \int_{\Omega} \Delta_x H(x, y) f(y) \, dy,
\]
so it would be nice if
\[
\Delta_x H(x, y) = 0 \quad \forall \, x, y \in \Omega.
\]
Moreover we would like that \( u(x) = 0 \) on \( \partial \Omega \), which would be satisfied if
\[
\Phi(x - y) = H(x, y) \quad \forall x \in \partial \Omega, y \in \Omega.
\]
That is, for each fixed \( y \in \Omega \) we should try to find a function \( H(\cdot, y) \) that solves
\[
(2.8) \quad \begin{cases} 
\Delta_x H(\cdot, y) = 0 & \text{in } \Omega, \\
H(\cdot, y) = \Phi(\cdot - y) & \text{on } \partial \Omega.
\end{cases}
\]
Observe that for fixed \( y \in \Omega \) the boundary condition \( \Phi(\cdot - y) \in C^\infty(\partial \Omega) \) is a smooth function, since for \( y \in \Omega \) we clearly have
\[
\inf_{x \in \partial \Omega} |x - y| > 0.
\]
That is, there is a good chance to solve this equation \((2.8)\) (and from Theorem 2.22 we know that there is at most one solution).

**Definition 2.8 (Green function).** For given \( \Omega \), if there exists \( H \) as in \((2.8)\) then we call
\[
G(x, y) := \Phi(x - y) - H(x, y)
\]
the Green function on \( \Omega \).

One can show that \( G \) is symmetric, i.e. that
\[
(2.9) \quad G(x, y) = G(y, x) \quad \forall \, x \neq y \in \Omega.
\]
While the Green function are usually not explicit, some properties and estimates can be shown, and there is an extensive research literature on the subject, e.g. see [Littman et al., 1963]. The Green function is also specially important from the point of view of stochastic processes, see e.g. [Chen, 1999].

We will only investigate the most basic property:

**Theorem 2.9.** Let \( \Omega \subset \subset \mathbb{R}^n \), \( \partial \Omega \in C^1 \), \( f \in C^0(\Omega) \) and \( g \in C^0(\partial \Omega) \). Assume that \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) is a solution to
\[
(2.10) \quad \begin{cases} 
-\Delta u = f & \text{in } \Omega, \\
u = g & \text{on } \partial \Omega.
\end{cases}
\]
Then if \( G \) is the Green function for \( \Omega \) from Definition 2.8 we have for any \( x \in \Omega \),
\[
u(x) = \int_{\Omega} G(x, y) f(y) \, dy - \int_{\partial \Omega} g(\theta) \partial_{\nu(\theta)} G(x, \theta) d\mathcal{H}^{n-1}(\theta).
\]
Proof. Recall the Gauss-Green formula\(^2\) on (smooth enough) domains \(A\),
\[
\int_A u(y) \Delta v(y) - \Delta u(y) v(y) \, dy = \int_{\partial A} u(\theta) \partial_\nu v(\theta) - \partial_\nu u(\theta) v(\theta) \, d\mathcal{H}^{n-1}(\theta).
\] (2.11)
We apply this to formula to \(A = \Omega \setminus B(x, \varepsilon)\) and \(v(y) := G(x, y)\). Observe that by symmetry of \(G\), (2.9),
\[
\Delta_y G(x, y) = \Delta_x G(x, y) = 0 \quad x \neq y,
\]
so, also in view of (2.10), (2.11) becomes
\[
- \int_A G(x, y) f(y) \, dy = \int_{\partial A} u(\theta) \partial_\nu G(x, \theta) - \partial_\nu u(\theta) G(x, \theta) \, d\mathcal{H}^{n-1}(\theta).
\] (2.12)
Now we argue as in the proof of Theorem 2.4. Observe that \(H\) is a smooth function.
We have
\[
\int_{\partial A} u(\theta) \partial_\nu G(x, \theta) d\mathcal{H}^{n-1}(\theta)
\]
\[
= \int_{\partial \Omega} g(\theta) \partial_\nu G(x, \theta) - \int_{\partial B(x, \varepsilon)} u(\theta) \partial_\nu \Phi(x - \theta) \, d\mathcal{H}^{n-1}(\theta) + \int_{\partial B(x, \varepsilon)} u(\theta) \partial_\nu H(x - \theta) \, d\mathcal{H}^{n-1}(\theta)
\]
\[
\varepsilon \to 0 \to \int_{\partial \Omega} g(\theta) \partial_\nu G(x, \theta) - u(x) + 0.
\]
and
\[
\int_{\partial A} \partial_\nu u(\theta) G(x, \theta) d\mathcal{H}^{n-1}(\theta)
\]
\[
= \int_{\partial \Omega} \partial_\nu u(\theta) G(x, \theta) - \int_{\partial B(x, \varepsilon)} \partial_\nu u(\theta) G(x, \theta) \, d\mathcal{H}^{n-1}(\theta)
\]
\[
= 0 - \int_{\partial B(x, \varepsilon)} \partial_\nu u(\theta) G(x, \theta) \, d\mathcal{H}^{n-1}(\theta)
\]
\[
\varepsilon \to 0 \to 0.
\]
This proves the claim. \(\square\)

In special situations one can actually construct explicit Green’s function. Let us firstly consider the Half-space
\[
\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}.
\]
So we need to find a solution to the equation
\[
\begin{cases}
\Delta_x H(\cdot, y) = 0 & \text{in } \mathbb{R}^n_+, \\
H(\cdot, y) = \Phi(\cdot - y) & \text{on } \mathbb{R}^{n-1} \times \{0\} \equiv \partial \mathbb{R}^n_+.
\end{cases}
\]
Since \(H\) at the boundary has to coincide with \(\Phi\) it is likely that \(H\) should be somewhat of the form of \(\Phi\) – only the singularity has to be gotten rid of – the idea is a reflection:
\[
H(x, y) := \Phi(x - y^*)
\]
\(^2\)This is a special case of the integration by parts formula.
where
\[ y^* = (y_1, \ldots, y_n)^* = (y_1, \ldots, y_{n-1}, -y_n). \]

It is a good exercise to check that

1. \( H \) is symmetric, \( H(x, y) = H(y, x) \)
2. \( H \) is smooth in \( \mathbb{R}^n_+ \times \mathbb{R}^n_+ \) (since \( x^* = y \) implies \( x_n = -y_n \), so \( x \) and \( y \) cannot both lie in the upper half-space if this happens)
3. The reflection does not change the PDE, namely \( \Delta x H = 0 \) for \( x, y \in \mathbb{R}^n_+ \).
4. Indeed \( H(x, y) = \Phi(x - y) \) for \( x \in \mathbb{R}^{n-1} \times \{0\} \) and \( y \in \mathbb{R}^n_+ \).

So we set
\[ G(x, y) := \Phi(x - y) - \Phi(x - y^*) = \Phi(x - y) - \Phi(x^* - y) \]

When we now use the representation formula, as in Theorem 2.9, then we need to compute \( \partial_{\nu(y)} G(x, y) \) for \( y \in \mathbb{R}^{n-1} \times \{0\} \). Observe that the outwards unit normal \( \nu(y) = (0, \ldots, 0, -1) \), so we compute
\[ \partial_{\nu(y)} G(x, y) = -\partial_{y_n} \Phi(x - y) + \partial_{y_n} \Phi(x^* - y) = c_n \frac{x_n - y_n}{|x - y|^n} - c_n \frac{x_n + y_n}{|x - y|^n} = \tilde{c}_n \frac{x_n}{|x - y|^n}. \]

If we write the variables in \( \mathbb{R}^n_+ \) as \( x = (x', x_n), x' \in \mathbb{R}^{n-1} \) and \( x_n > 0 \), then as in Theorem 2.9 we indeed obtain, e.g., if
\[ u(x) := c_n \int_{\mathbb{R}^{n-1}} \frac{x_n}{\left(|x' - y'|^2 + |x_n|^2\right)^{n/2}} g(y') dy' \]

then \( u \) satisfies indeed (for “reasonable” \( g \))
\[ \begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ \lim_{x_n \to 0} u(x) = g(x') \\ \lim_{x_n \to \infty} u(x) = 0. \end{cases} \]

The formula for \( u \) is called the Poisson formula on the Half-space \( \mathbb{R}^n_+ \), also the harmonic extension of \( g \) from \( \mathbb{R}^{n-1} \) to \( \mathbb{R}^n_+ \).

**Exercise 2.10.** (1) Show that the constant \( c_n \) in (2.13) is
\[ c_n = \left( \int_{\mathbb{R}^{n-1}} \frac{1}{\left(|x' - y'|^2 + |x_n|^2\right)^{n/2}} dy' \right)^{-1}. \]

**Hint:** Use the maximum principle for \( u \) assuming that \( g \equiv 1 \).
(2) Show that for any \( x_n > 0 \)
\[ c_n = \left( \int_{\mathbb{R}^{n-1}} \frac{x_n}{\left(|x' - y'|^2 + |x_n|^2\right)^{n/2}} dy' \right)^{-1}. \]
Example 2.11 (Dirichlet-to-Neumann formula). Let $g \in C_c^\infty(\mathbb{R}^{n-1})$. Define $u$ via (2.13).

We consider the Neumann-data of $u$:

$$
\begin{align*}
\partial_n u \bigg|_{x_n=0} &= \lim_{x_n \to 0^+} \frac{u(x', x_n) - u(x', 0)}{x_n} \\
&= \lim_{x_n \to 0^+} \frac{u(x', x_n) - g(x')}{x_n} \\
&= c_n \lim_{x_n \to 0^+} \int_{\mathbb{R}^{n-1}} \frac{x_n}{(|x'|^2 + |x_n|^2)^{\frac{3}{2}}} \frac{g(y') - g(x')}{x_n} dy' \\
&= c_n \lim_{x_n \to 0^+} \int_{\mathbb{R}^{n-1}} \frac{1}{(|x'|^2 + |x_n|^2)^{\frac{3}{2}}} (g(y') - g(x')) dy'.
\end{align*}
$$

This looks nice, but it has the problem that the integral does not converge absolutely (only in a principal value sense).

We try this again: Observe by substituting $h' := x' - y'$ we can write

$$
u(x) := c_n \int_{\mathbb{R}^{n-1}} \frac{x_n}{(|h'|^2 + |x_n|^2)^{\frac{3}{2}}} g(x' - h') dh'.
$$

By substituting $h'$ with $-h'$ we also have

$$
u(x) := c_n \int_{\mathbb{R}^{n-1}} \frac{x_n}{(|h'|^2 + |x_n|^2)^{\frac{3}{2}}} g(x' + h') dh'.
$$

So we can write

$$
\begin{align*}
\frac{u(x', x_n) - g(x')}{x_n} &= \frac{1}{2} \frac{2u(x', x_n) - 2g(x')}{x_n} \\
&= \frac{1}{2} c_n \int_{\mathbb{R}^{n-1}} \frac{x_n}{(|h'|^2 + |x_n|^2)^{\frac{3}{2}}} \frac{g(x' + h') + g(x' - h) - 2g(x')}{x_n} dh' \\
&= \frac{1}{2} c_n \int_{\mathbb{R}^{n-1}} \frac{g(x' + h') + g(x' - h) - 2g(x')}{(|h'|^2 + |x_n|^2)^{\frac{3}{2}}} dh' \\
&= \lim_{x_n \to 0^+} \frac{1}{2} c_n \int_{\mathbb{R}^{n-1}} \frac{g(x' + h') + g(x' - h) - 2g(x')}{|h'|^{n-1+1}} dh'.
\end{align*}
$$

In the last step we used that this integral really converges, Exercise 2.12.

This defines an operator

$$
(-\Delta)^\frac{1}{2} g(x') \equiv \sqrt{-\Delta} g(x') := \frac{1}{2} c_n \int_{\mathbb{R}^{n-1}} \frac{g(x' + h') + g(x' - h) - 2g(x')}{|h'|^{n-1+1}} dh'.
$$
which is called the *half-Laplacian*. Indeed using the Fourier transform on $\mathbb{R}^{n-1}$ one can check that

$$\mathcal{F} \left( (-\Delta)^{\frac{1}{2}} g \right)(\xi') = c |\xi'| \mathcal{F} g(\xi') = c \sqrt{|\xi'|^2} \mathcal{F} g(\xi')$$

So this is really the square-root of the Laplacian.

We have proven the *Dirichlet-to-Neumann principle*

If

$$\begin{cases}
\Delta u = 0 & \text{in } \mathbb{R}^n_+ \\
u(x') = g(x') & \text{on } \mathbb{R}^{n-1} \times \{0\}
\end{cases}$$

then

$$\partial_n u \bigg|_{\mathbb{R}^{n-1} \times \{0\}} = c (-\Delta)^{\frac{1}{2}} g \quad \text{on } \mathbb{R}^{n-1} \times \{0\}$$

In 2007, [Caffarelli and Silvestre, 2007], this formula was generalized for $\sigma \in (0, 2)$ to

$$\begin{cases}
\text{div} \left( (x_n)^{1-\sigma} \nabla u \right) = 0 & \text{in } \mathbb{R}^n_+ \\
u(x') = g(x') & \text{on } \mathbb{R}^{n-1} \times \{0\}
\end{cases}$$

then

$$\lim_{x_n \to 0^+} (x_n)^{1-\sigma} \partial_n u \bigg|_{\mathbb{R}^{n-1} \times \{0\}} = c (-\Delta)^{\frac{1}{2}} g \quad \text{on } \mathbb{R}^{n-1} \times \{0\}$$

This paper has more than 1500 citations and is often referred to as the Caffarelli-Silvestre extension formula.

**Exercise 2.12.** Let $g \in C_c^\infty(\mathbb{R}^d)$.

1. For $s \in (0, 1)$ show that for each fixed $x \in \mathbb{R}^d$

   $$y \mapsto \frac{g(y) - g(x)}{|x-y|^{d+s}} \in L^1(\mathbb{R}^d),$$

   i.e.

   $$\int_{\mathbb{R}^d} \frac{|g(y) - g(x)|}{|x-y|^{d+s}} \, dy < \infty.$$

2. For $s \in (0, 2)$ show that for each fixed $x \in \mathbb{R}^d$

   $$y \mapsto \frac{g(x+h) - g(x-h)}{|h|^{d+s}} \in L^1(\mathbb{R}^d),$$

   i.e. that

   $$\int_{\mathbb{R}^d} \frac{|g(x+h) + g(x-h) - 2g(x)|}{|x-y|^{d+s}} \, dy < \infty.$$
2.4.1. *On a ball.* The other situation where we can compute the Green’s function is the ball. For simplicity let us consider $\Omega = B(0,1)$, the unit ball centered at zero. Again the first goal is to find $H(x, y)$ that corrects the fundamental solution. In the case of the half-space $\mathbb{R}^n_+$ we set $H(x, y) = \Phi(x - \tilde{y})$, i.e. we reflected the $y$-variable in a way that did not interfere with the PDE but removed the singularity (and coincided with $\Phi(x - y)$ on the boundary.

So lets do the same for the ball. The canonical operation that reflects points from the ball $B(0, 1)$ outside and vice versa is called the inversion at a sphere, $y^* := \frac{y}{|y|^2} : B(0,1) \rightarrow B(0,1)^c$. (Although it is not explicitly used here, it is good to know: the inversion at the sphere is a conformal transform, i.e. it preserves angles). So a first attempt would be to set

$$H(x, y) := \Phi \left( \left| x - \frac{y}{|y|^2} \right| \right),$$

which takes care of the singularity of $\Phi$ (for $y, x \in B(0,1)$ we have $|x - \frac{y}{|y|^2}| \neq 0$, and does not disturb the PDE for $G(x, y)$. However such a $G(x, y)$ is not equal to $\Phi(x - y)$ for $|x| = 1$. So we need to adapt $G$ to the boundary data. Observe that for $|x| = 1$,

$$|y|^2 \left| x - \frac{y}{|y|^2} \right|^2 = |y|^2 \left( |x|^2 + \frac{1}{|y|^2} - 2 \langle x, \frac{y}{|y|^2} \rangle \right) = \left( |y|^2 |x|^2 + 1 - 2 \langle x, y \rangle \right) = \left( |y|^2 + |x|^2 - 2 \langle x, y \rangle \right) = |x - y|^2.$$

That is why we set

$$G_{B(0,1)}(x, y) := \Phi \left( |y| \left| x - \frac{y}{|y|^2} \right| \right),$$

which satisfies all the requested properties.

From this we obtain (without proof)

**Theorem 2.13 (Poisson’s formula for the ball).** Assume $g \in C^0(\partial B(0, r))$. Define

$$u(x) := c_n \int_{\partial B(0 , r)} \frac{1}{r} \frac{r^2 - |x|^2}{|x - \theta|^n} g(\theta) \, d\mathcal{H}^{n-1}(\theta)$$

Then

1. $u \in C^\infty(B(0, r))$
2. $\Delta u = 0$ in $B(0, r)$
\[(3) \quad \lim_{B(0,r) \ni x \to x_0} u(x) = g(x_0) \quad \text{for any } x_0 \in \partial B(0, r)\]

2.5. Mean Value Property for harmonic functions. An important property (but very
special to the “base Operator $\Delta$”, i.e. not that easily generalizable to more general PDEs)
is the mean value property

**Theorem 2.14** (Harmonic functions satisfy Mean Value Property). Let $u \in C^2(\Omega)$ such
that $\Delta u = 0$, then

\[(2.14) \quad u(x) = \int_{\partial B(x,r)} u(z) d\mathcal{H}^{n-1}(z) = \int_{B(x,r)} u(y) dy\]

holds for all balls $B(x, r) \subset \Omega$.

If $\Delta u \leq 0$ then we have “$\geq$” in (2.14). If $\Delta u \geq 0$ then we have “$\leq$” in (2.14).

**Proof.** Set

$$
\varphi(r) := \int_{\partial B(x,r)} u(y) d\mathcal{H}^{n-1}(y).
$$

Observe that by substitution $z := \frac{y-x}{r}$ we have

$$
\varphi(r) := \int_{\partial B(0,1)} u(x + rz) d\mathcal{H}^{n-1}(z).
$$

Taking the derivative in $r$ we have

$$
\varphi'(r) = \int_{\partial B(0,1)} Du(x + rz) \cdot z d\mathcal{H}^{n-1}(z).
$$

Transforming back we get

$$
\varphi'(r) = \int_{\partial B(x,r)} Du(y) \cdot \frac{y-x}{r} d\mathcal{H}^{n-1}(y).
$$

Observe that $\frac{y-x}{r}$ is the outer unit normal of $\partial B(x, r)$. That is

$$
\varphi'(r) = |\partial B(x,r)|^{-1} \int_{\partial B(x,r)} \partial_{\nu} u(y) d\mathcal{H}^{n-1}(y).
$$

By Stokes or Green’s theorem (aka, integration by parts)

$$
\varphi'(r) = |\partial B(x,r)|^{-1} \int_{B(x,r)} \Delta u(y) dy = 0.
$$

That is,

$$
\varphi'(r) = 0 \quad \forall r \text{ if } B(x, r) \subset \Omega.
$$

which implies that $\varphi$ is constant, and in particular

$$
\varphi(r) = \lim_{\rho \to 0} \varphi(\rho).
$$
But (Exercise 2.6) for continuous functions \( u \),

\[
\lim_{\rho \to 0} \varphi(\rho) = \lim_{\rho \to 0} \int_{\partial B(x,\rho)} u(y) \, d\mathcal{H}^{n-1}(y) = u(x),
\]

we have shown that

\[
(2.15) \quad u(x) = \int_{\partial B(x,r)} u(y) \, d\mathcal{H}^{n-1}(y)
\]

holds whenever \( B(x,r) \subset \Omega \).

Moreover, by Fubini’s theorem

\[
\int_{B(x,r)} u(y) \, dy = \frac{1}{|B(x,r)|} \int_{0}^{r} \int_{\partial B(x,\rho)} u(\theta) \, d\mathcal{H}^{n-1}(\theta) \, d\rho
\]

\[
= \frac{1}{|B(x,r)|} \int_{0}^{r} |\partial B(x,\rho)| \int_{\partial B(x,\rho)} u(\theta) \, d\mathcal{H}^{n-1}(\theta) \, d\rho
\]

\[
= \frac{1}{|B(x,r)|} \int_{0}^{r} |\partial B(x,\rho)| \, u(x) \, d\rho
\]

\[
= u(x) \frac{1}{|B(x,r)|} \int_{0}^{r} \int_{\partial B(x,\rho)} 1 \, d\mathcal{H}^{n-1}(\theta) \, d\rho
\]

\[
= u(x) \frac{|B(x,r)|}{|B(x,r)|} = u(x).
\]

Together with (2.15) we have shown the claim for \( \Delta u = 0 \). The inequality arguments are left as an exercise. 

\[\square\]

The converse holds as well (and there is actually a whole literature on “how many balls” one has to assume the mean value property to get harmonicity, cf. [Llorente, 2015, Kuznetsov, 2019])

**Theorem 2.15** (Mean Value property implies harmonicity). Let \( u \in C^2(\Omega) \). If for all balls \( B(x,r) \subset \Omega \),

\[
(2.16) \quad u(x) = \int_{\partial B(x,r)} u(\theta) \, d\mathcal{H}^{n-1}(\theta)
\]

then

\[
\Delta u = 0 \quad \text{in} \; \Omega
\]

**Proof.** Assume the claim is false.

Then there exists some \( x_0 \in \Omega \) such that \( \Delta u(x_0) \neq 0 \), so (by continuity of \( \Delta u \)) w.l.o.g. \( \Delta u > 0 \) in a small neighbourhood \( B(x_0,R) \) of \( x_0 \).
On the other hand, setting as above

\[ \varphi(r) := \int_{\partial B(x_0, r)} u(\theta) \] (2.16) \equiv u(x_0) \]

we have \( \varphi'(r) = 0 \) for all \( r > 0 \) such that \( B(x_0, r) \subset \Omega \). But as computed before, for \( r < R \),

\[ \varphi'(r) = C(r) \int_{B(x_0, r)} \Delta u \, dy > 0. \]

This \((0 = \varphi'(r) > 0)\) is a contradiction, so the claim is established. \qed

2.6. Maximum and Comparison Principles. The mean value property as above is very rigid in the sense that it holds only for very special operators such as the Laplacian. A much more general property (which for the Laplacian \( \Delta \) is a direct consequence of the mean value property) are maximum principles, which should be seen as a “forced convexity/concavity property” for sub-/supersolutions of a large class of PDEs (2nd order elliptic).

In one-dimension a subsolution of Laplace’s equation satisfies

\[ u'' \geq 0 \]

that is, subsolutions are exactly the convex \( C^2 \)-functions.

On the hand, if \( u : \Omega \to \mathbb{R} \) is a smooth convex function, then \( D^2 u(x) \geq 0 \) (in the sense of matrices), so \( \Delta u = \text{tr} D^2 u = \sum(\text{eigenvalues of } D^2 u) \geq 0. \)

On the other hand, the converse does not hold: if we take \( u(x, y) = 2x^2 - y^2 \) then \( u \) is not convex, but \( \Delta u \geq 0. \)

Still, subsolutions have some properties of convex functions (and supersolutions have some properties of concave functions): comparison principles:

Convexity means that on any interval \((a, b)\) the maximum of \( u \) is obtained at \( a \) or at \( b \) – and if the maximum is obtained in a point \( c \in (a, b) \) then \( u \) is constant. The curious fact is that these properties still hold in arbitrary dimension for solutions of the Laplace equation (and later a large class of elliptic 2nd order equations), they are the so-called weak maximum principle and strong maximum principle.

There is also a “physical” way to explain maximum principles: For example, assume that a solid \( \Omega \) is heated from the sides with a heat source \( g : \partial \Omega \to \mathbb{R} \) and assume there is some heat source from the middle, but it only subtracts heat, \(-\Delta u \leq 0\), then what is the maximal heat at any point in the interior (letting the system become stationary)? well the maximum heat in the inside is the heat at the boundary (\textit{weak maximum principle}). And if the heat at any point in the interior is exactly the maximum value of the heat, since the system is stationary, if it is colder at any other point then the heat would have distributed to that point – meaning any other point must have the same heat (\textit{strong maximum principle}).
**Corollary 2.16** (Strong Maximum-principle). Let $u \in C^2(\Omega)$ be subharmonic, i.e. $\Delta u \geq 0$ in $\Omega$. If there exists $x_0 \in \Omega$ at which $u$ attains a global maximum then $u$ is constant in the connected component of $\Omega$ containing $x_0$.

*Proof.* By taking a possibly smaller $\Omega$ we can assume w.l.o.g. $\Omega$ is connected and $u$ still attains its global maximum in $x_0 \in \Omega$.

Let

$$A := \{ y \in \Omega : u(y) = u(x_0) \}.$$ 

We will show that $A = \Omega$ (and thus $u$ is constant) by showing that the following three properties hold

- $A$ is nonempty
- $A$ is relatively closed (in $\Omega$).
- $A$ is open

Then $A$ is an open and closed set in $\Omega$, and since $A$ is not the empty set it is all of $\Omega$.

Clearly $A$ is nonempty since $x_0 \in A$.

Also $A$ is relatively closed by continuity of $u$: If $\Omega \ni y_k \xrightarrow{k \to \infty} y_0 \in \Omega$ then

$$u(y_0) = \lim_{k \to \infty} u(y_k) = u(x_0)$$

and thus $y_0 \in A$.

To show that $A$ is open let $y_0 \in A$. Since $\Omega$ is open we can find a small ball $B(y_0, \rho) \subset \Omega$.

Observe that $x_0$ is a global maximum of $u$ in $B(y_0, \rho)$.

The mean value property, Theorem 2.14, and then the fact that $u(x_0) \geq u(y)$ for all $y$ in $B(y_0, \rho)$, imply

$$u(x_0) = u(y_0) \leq \int_{B(y_0, \rho)} u(y) \, dy \leq \int_{B(y_0, \rho)} u(x_0) \, dy = u(x_0).$$

Since left-hand side and right-hand side coincide the inequality is actually an equality.

That is, we have

$$u(x_0) = \int_{B(y_0, \rho)} u(y) \, dy,$$

in other words

$$\int_{B(y_0, \rho)} u(y) - u(x_0) \, dy = 0.$$

Since $u(y) - u(x_0)$ by assumption $\leq 0$ the above integral becomes

$$-\int_{B(y_0, \rho)} |u(y) - u(x_0)| \, dy = 0.$$
that is
\[ u(y) \equiv u(x_0) \quad \text{in } B(y_0, \rho), \]
that is \( B(y_0, \rho) \subset A \). That is, \( A \) is open. \( \Box \)

**Remark 2.17.** The statement of Corollary 2.16 is false if one replaces global with local maximum (even though local maxima are locally global maxima). A counterexample is for example

\[ u(x) := \begin{cases} 
0 & x \leq 0 \\
x^3 & x > 0 
\end{cases} \]

Then \( u \in C^2(\mathbb{R}) \) and
\[ \Delta u = u'' \geq 0 \quad \text{in } (-1, 1) \]

Clearly \( u \) attains several local maxima, namely in \((-1, 0)\) we have \( u \equiv 0 \), but also clearly \( u \) is not constant. The argument above in the proof of Corollary 2.16 fails, since the point 0 is not a local maximum, and thus the set
\[ A := \{ x \in (-1, 1) : u(x) = 0 \} \]

is not open.

For the next statement, and henceforth, we use the notation \( A \subset \subset B \) (“\( A \) is compactly contained in \( B \)”) which means that \( A \) is bonded and its closure \( \overline{A} \subset B \). I.e. for two open sets \( A, B \) the condition \( A \subset \subset B \) means in particular that \( \partial A \) has positive distance from \( \partial B \).

**Corollary 2.18** (Weak maximum principle). Let \( \Omega \subset \subset \mathbb{R}^n \) and \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) be subharmonic, i.e. \( -\Delta u \leq 0 \) in \( \Omega \). Then
\[ \sup_{\Omega} u = \sup_{\partial \Omega} u, \]
i.e. “the maximal value is attained at the boundary”\(^3\).

**Remark 2.19.** This statement also holds on unbounded sets \( \Omega \), one just has to define the meaning of \( \sup_{\partial \Omega} \) in a suitable sense (i.e. “\( \sup_{\partial \mathbb{R}^n} \)” should be interpreted as \( \lim \sup_{|x| \to \infty} \)).

**Proof of Corollary 2.18.** Clearly by continuity
\[ \sup_{\Omega} u \geq \sup_{\partial \Omega} u. \]

To prove the converse let us argue by contradiction and assume that
\[ \sup_{\Omega} u > \sup_{\partial \Omega} u. \quad (2.17) \]

Since \( u \) is continuous and \( \Omega \) bounded this must mean that there exists a local maximum point \( x_0 \in \Omega \) such that
\[ u(x_0) = \sup_{\Omega} u > \sup_{\partial \Omega} u. \quad (2.18) \]

\[^3\text{again: think of convex functions which do have this property} \]

...
But in view of Corollary 2.16 (strong maximum principle) \( u \) is then constant on the connected component of \( \Omega \) containing \( x_0 \). But this implies that on the boundary of this connected component the value of \( u \) is still \( u(x_0) \), which implies

\[
\sup_{\partial \Omega} u \geq u(x_0).
\]

But this contradicts the assumption (2.18). \( \square \)

**Remark 2.20.** A particular consequence of the strong maximum principle is the following. If for \( \Omega \subset \subset \mathbb{R}^n \) we have \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) satisfying

\[
\begin{aligned}
\Delta u &\geq 0 \quad \text{in } \Omega \\
u &\equiv g \quad \text{on } \partial \Omega
\end{aligned}
\]

for some \( g \in C^0(\partial \Omega) \). Then the following (equivalent) statements are true:

- If \( g \leq 0 \) but \( g \not\equiv 0 \) on \( \partial \Omega \) then we have that \( u < 0 \) in all of \( \Omega \).
- If \( g \leq 0 \) then either \( u \equiv 0 \) or \( u < 0 \) everywhere in \( \Omega \).

Such a behaviour is special to the PDEs of order two. Even for

\[
\Delta^2 u = \Delta(\Delta u) = 0 \quad \text{in } \Omega
\]

the above statement may not hold (see e.g. [Gazzola et al., 2010]).

**Corollary 2.21** (Strong Comparison Principle). Let \( \Omega \subset \subset \mathbb{R}^n \) open and connected. Assume that \( u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) satisfy

\[
\Delta u_1 \geq \Delta u_2 \quad \text{in } \Omega.
\]

If \( u_1 \leq u_2 \) on \( \partial \Omega \), then exactly one of the following statements is true

(1) either \( u_1 \equiv u_2 \)

(2) or \( u_1(x) < u_2(x) \) for all \( x \in \Omega \).

**Proof.** Let \( w := u_1 - u_2 \), then we have

\[
\begin{aligned}
\Delta w &\geq 0 \quad \text{in } \Omega \\
w &\leq 0 \quad \text{in } \partial \Omega
\end{aligned}
\]

The claim now follows from Remark 2.20. \( \square \)

The maximum principle is a great tool to get uniqueness for linear equations!

**Theorem 2.22** (Uniqueness for the Dirichlet problem). Let \( \Omega \subset \subset \mathbb{R}^n \), \( f \in C^0(\Omega) \) and \( g \in C^0(\partial \Omega) \) be given. Then there is at most(!) one solution \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) of

\[
\begin{aligned}
\Delta u = f \quad &\text{in } \Omega \\
 u = g \quad &\text{on } \partial \Omega
\end{aligned}
\]
Proof. Assume there are two solutions, \( u, v \) solving this equation. If we set \( w := u - v \) then \( w \) is a solution to the equation

\[
\begin{cases}
\Delta w = 0 & \text{in } \Omega \\
w = 0 & \text{on } \partial \Omega
\end{cases}
\]

In view of Corollary 2.18 we then have

\[
\sup_{\Omega} w \leq \sup_{\partial \Omega} w = 0.
\]

That is, \( w \leq 0 \) in \( \Omega \). But observe that \( -w \) solves the same equation, which implies that

\[
\sup_{\Omega} (-w) \leq \sup_{\partial \Omega} (-w) = 0,
\]

that is \( -w \leq 0 \) in \( \Omega \). But this readily implies that \( w \equiv 0 \) in \( \Omega \), that is \( v \equiv w \). \( \square \)

So comparison principles are a fantastic tool for obtaining uniqueness for PDEs. Let us also note that via the so-called Perron’s method (which relies heavily on maximum principles) we also can obtain existence, Section 2.8. But first we need another comparison principle: Harnack inequality.

2.7. Harnack Principle. Above we learned, e.g. in Corollary 2.16 of the strong maximum principle. Another type of maximum principle is the Harnack inequality.

Theorem 2.23. Let \( \Omega \subset \mathbb{R}^n \) open. For any open, connected, and bounded \( U \subset \subset \Omega \) there exists a constant \( C = C(U, \Omega) \) such that for any solution \( u \in C^2(\Omega) \) with \( u \geq 0 \) and such that

\[
\Delta u = 0 \quad \text{in } \Omega
\]

we have

\[
\sup_{U} u \leq C \inf_{U} u
\]

Proof. The proof is based on the mean value formula, Theorem 2.14, namely for any \( x \in U \) and any \( r < \text{dist} (U, \partial \Omega) \) we have

\[
u(x) = \int_{B(x, r)} u(z) \, dz
\]

Let now \( R := \frac{1}{4} \text{dist} (U, \partial \Omega) \). For any \( x_0 \in U \) and any \( x \in B(x_0, R) \) we have (here we use \( u \geq 0 \) and that \( B(x, R) \subset B(y, 2R) \) for \( x, y \in B(x_0, R) \))

\[
u(x) = \int_{B(x, R)} u(z) \, dz \leq 2^n \int_{B(y, 2R)} u(z) \, dz = 2^n u(y).
\]

Again, this holds for any \( x, y \in B(x_0, R) \). Taking the supremum for \( x \in B(x_0, R) \) and the infimum on \( y \in B(x_0, R) \) we get

\[
\sup_{B(x_0, R)} u \leq 2^n \inf_{B(x_0, R)} u.
\]

(2.19)
That is we have the Harnack principle on any Ball $B(x_0, R)$. Since $U$ is bounded, open and compactly contained in $\Omega$ we can now cover all of $U$ by finitely many balls $(B_\ell)_{\ell=1}^N$ which lie inside $\Omega$ centered at points in $U$ and of radius $R$.

Take any $i_1, i_2 \in \{1, \ldots, N\}$ and assume that $B_{i_1} \cap B_{i_2} \neq \emptyset$. Since then $\inf_{B_{i_1}} u \leq \sup_{B_{i_2}} u$ Harnack’s principle on the ball $B_{i_1}$ and the ball $B_{i_2}$ implies

$$\sup_{B_{i_1}} u \leq 2^{2n} \inf_{B_{i_2}} u$$

whenever $B_{i_1} \cap B_{i_2} \neq \emptyset$.

Repeating the same argument, assume now that $i_1, i_2, i_3 \in \{1, \ldots, N\}$ such that $B_{i_1} \cap B_{i_2} \neq \emptyset$ and $B_{i_2} \cap B_{i_3} \neq \emptyset$. Then

$$\sup_{B_{i_1}} u \leq 2^{2n} \inf_{B_{i_2}} u \leq 2^{2n} \sup_{B_{i_3}} u \leq 2^{4n} \inf_{B_{i_3}} u$$

whenever $B_{i_1} \cap B_{i_2} \neq \emptyset$ and $B_{i_2} \cap B_{i_3} \neq \emptyset$.

By induction we readily conclude the following fact: Whenever we have $i, j \in \{1, \ldots, N\}$ such that there are $i_1, \ldots, i_K \in \{1, \ldots, N\}$ with $i_1 = i$ and $i_K = j$ and $B_{i_\ell} \cap B_{i_{\ell+1}} \neq \emptyset$ for all $\ell$ then we have

$$\sup_{B_i} u \leq 2^{2nK} \inf_{B_j} u.$$ 

Cf. Figure 2.2. Since $U$ is connected and it is covered by $N$ balls we conclude that

$$\sup_{U} u \leq \sup_{i \in \{1, \ldots, N\}} \sup_{B_i} u \leq 2^{2nN} \inf_{j \in \mathbb{N}} \inf_{B_j} u \leq 2^{2nN} \inf_{U} u.$$ 

Observe that $N$ heavily depends on $U \subset\subset \Omega$ – and the closer the boundary of $U$ is to $\Omega$, the larger $N$ tends to be. Thus we have shown that

$$\sup_{U} u \leq C(U, \Omega) \inf_{U} u.$$ 

□

We observe from the proof above that we can proof Harnack inequality on a ball with a uniform constant.

**Corollary 2.24.** For any dimension $n \in \mathbb{N}$ there exists a constant $C = C(n)$ such that the following holds:

Let $B(x_0, R)$ be a ball. If $u \in C^2(B(x_0, R))$ with $u \geq 0$ in $B(x_0, R)$ satisfies

$$\Delta u = 0 \quad \text{in} \quad B(x_0, R)$$

then

$$\sup_{B(x_0, R/2)} u \leq C \inf_{B(x_0, R/2)} u.$$ 

**Exercise 2.25.** Let $\Omega \subset \mathbb{R}^n$ be any open set. Assume there is $u \in C^0(\Omega)$ such that

$$u \geq 0 \quad \text{in} \quad \Omega$$
Figure 2.2. From Harnack’s inequality on balls we can conclude Harnack’s inequality on any set $U \subset \subset \Omega$: Harnack’s principle repeatedly applied on balls implies $\sup_{B_{15}} u \leq 2^{2^{15}} \inf_{B_{15}} u$ (as long as each ball is small enough, so that e.g. twice the ball is in $\Omega$). Any set $U \subset \subset \Omega$ can be covered by finitely many such small balls. So we have $\sup_U u \leq C(U, \Omega) \inf_U u$.

and for some $\lambda \in (0, 1)$ and $\Lambda > 1$ we know that

$$u(x) \leq \Lambda \int_{B(x, r)} u$$

and

$$u(x) \geq \lambda \int_{B(x, r)} u$$

holds for all $x \in \Omega$ with $B(x, r) \subset \subset \Omega$.

Show that there exists a constant $C > 0$ only depending on $n, \lambda, \Lambda$ such that

$$\sup_{B(y, \rho)} u \leq C \inf_{B(y, \rho)} u$$

holds for all balls $B(y, 2\rho) \subset \Omega$.

An important consequence of Harnack inequality is that it implies Hölder continuity. This is of course more relevant if we do not a priori that $u \in C^2$ – but we still illustrate this, because this principles applies to many equations.
Example 2.26 (Harnack implies Hölder estimates). Assume
\[ \Delta u = 0 \quad \text{in } \Omega \]
For \( r > 0 \) and any \( x \Omega \) such that \( B(x, 2r) \subset \Omega \).
\[ M(x_0, r) := \sup_{B(x_0, r)} u, \quad m(x_0, r) := \inf_{B(x_0, r)} u. \]
(We assume both values are finite)
Then
\[ \Delta (M(x_0, r) - u) = 0 \]
and \( M(x_0, r) - u \geq 0 \) in \( B(x_0, r) \) so we have from Harnack’s inequality Corollary 2.24 for a uniform constant \( C \),
\[ \sup_{B(x_0, r/2)} (M(x_0, r) - u) \leq C \inf_{B(x_0, r/2)} (M(x_0, r) - u), \]
and thus
\[ M(x_0, r) - m(x_0, r/2) \leq C (M(x_0, r) - M(x_0, r/2)). \]
Similarly,
\[ \sup_{B(x_0, r/2)} (u - m(x_0, r)) \leq C \inf_{B(x_0, r/2)} (u - m(x_0, r)), \]
and thus
\[ M(x_0, r/2) - m(x_0, r) \leq C (m(x_0, r/2) - m(x_0, r)). \]
We add those two equations
\[ M(x_0, r/2) - m(x_0, r) + M(x_0, r) - m(x_0, r/2) \leq C (m(x_0, r/2) - m(x_0, r) + M(x_0, r) - M(x_0, r/2)). \]
and thus
\[ M(x_0, r/2) - m(x_0, r/2) \leq M(x_0, r/2) - m(x_0, r/2) \leq M(x_0, r) - m(x_0, r) + M(x_0, r) - M(x_0, r/2). \]
\[ \leq C (M(x_0, r) - m(x_0, r) - (M(x_0, r/2) - m(x_0, r/2))). \]
And thus we have
\[ M(x_0, r/2) - m(x_0, r/2) \leq C (M(x_0, r) - m(x_0, r) - (M(x_0, r/2) - m(x_0, r/2))). \]
which by absorbing becomes
\[ (C + 1) (M(x_0, r/2) - m(x_0, r/2)) \leq C (M(x_0, r) - m(x_0, r)). \]
That is
\[ (M(x_0, r/2) - m(x_0, r/2)) \leq \frac{C}{C + 1} (M(x_0, r) - m(x_0, r)). \]
Set
\[ \gamma := \frac{C}{C + 1} < 1. \]
If we then set the oscillation
\[ \text{osc}_{B(x_0, r)} u := M(x_0, r) - m(x_0, r), \]
we have shown
\[ \text{osc}_{B(x_0, r/2)} u \leq \gamma \text{ osc}_{B(x_0, r)} u. \]

We can *iterate* this: for any \( k \in \mathbb{N} \) we have
\[ \text{osc}_{B(x_0, r/2^k)} u \leq \gamma^k \text{ osc}_{B(x_0, r)} u. \]

Now let \( \rho < r \), then there exists exactly one \( k \in \mathbb{N} \) such that \( \rho \in [r/2^{k-1}, r/2^k) \). And we have (the oscillation is monotone, Exercise 2.28)
\[ \text{osc}_{B(x_0, \rho)} u \leq \text{osc}_{B(x_0, 2^k r)} u \leq \gamma^k \text{ osc}_{B(x_0, r)} u. \]

Now observe that \( \gamma = 2^{-\sigma} \) for some \( \sigma > 0 \). So,
\[ \gamma^k = (2^k)^{-\sigma} \lesssim_{\sigma} \left(\frac{r}{\rho}\right)^{-\sigma} = \frac{\rho^\sigma}{r^\sigma}. \]

Thus we have shown, for any \( \rho < r \)
\[ \text{osc}_{B(x_0, \rho)} u \leq \frac{\rho^\sigma}{r^\sigma} \text{ osc}_{B(x_0, r)} u. \]

If \( B(x_0, 2r) \subset \Omega \) we in particular have
\[ \sup_{x_1 \in B(x_0, r)} \text{osc}_{B(x_0, r)} u \leq \frac{\rho^\sigma}{r^\sigma} \text{ osc}_{B(x_0, 2r)} u. \]

This implies Hölder continuity, Exercise 2.27.

**Exercise 2.27.** Show that if for any \( \rho \in (0, r) \) we have
\[ \sup_{x_1 \in B(x_0, r)} \text{osc}_{B(x_0, \rho)} u \leq C \rho^\sigma, \]
then \( u \) is Hölder continuous, namely
\[ \sup_{x, y \in B(x_0, r)} \frac{|u(x) - u(y)|}{|x - y|^\sigma} < \infty. \]

**Exercise 2.28.** Show that if \( u \) is a bounded function then if we set
\[ \text{osc}_A u := \sup_A u - \inf_A u. \]

Show that if \( A \subset B \) then
\[ \text{osc}_A u \leq \text{osc}_B u. \]
2.8. Perron’s method (illustration). Comparison principles (weak, strong maximum principle, and Harnack) are not only great for estimates – they can also be used to show existence (for equations that have these comparison principles – which many have not. To illustrate this we jump a little bit ahead, and recall that we can already solve the Laplace equation in a ball $B(x, R)$ (via the Green’s function method, Theorem 2.13). Namely, we shall accept that if $f \in C^0(\partial B_R(y))$ then for a certain constant $c_n > 0$ if we set

$$w(x) := \frac{R^2 - |x - y|^2}{c_n R} \int_{\partial B_R(y)} \frac{f(z)}{|z - x|^n} dz, \quad x \in B_R(y)$$

then $w \in C^0(\overline{B_R(y)}) \cap C^2(B_R(y))$ and

$$\begin{cases}
\Delta w = 0 & \text{in } B(y, r) \\
w = f & \text{on } \partial B(y, R).
\end{cases}$$

For general open and bounded sets set with smooth boundary $\partial \Omega$, it is not so easy to get an explicity formula. But one can use Perron’s method and local replacements to show existence of solutions of

$$\begin{cases}
\Delta u = 0 & \text{in } \Omega \\
u = g & \text{on } \partial \Omega
\end{cases}$$

where $g \in C^0(\partial \Omega)$.

First we extend the notion of solution and subsolution to upper- and lowercontinuous functions.

**Definition 2.29.** Let $\Omega \subset \mathbb{R}^n$ open.

1. A function $f : \Omega \to (-\infty, \infty)$ is called subharmonic in $\Omega$ if it is continuous and for any $x \in \Omega$, $r > 0$ $B_r(x) \subset \Omega$ we have

$$f(x) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} f(y) dy.$$

2. A function $f : \Omega \to [-\infty, \infty)$ is called harmonic in $\Omega$ if it is continuous and for any $x \in \Omega$, $r > 0$ $B_r(x) \subset \Omega$

$$f(x) = \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} f(y) dy.$$

Similar to Theorem 2.15 one can show that if $u \in C^2$ then subharmonicity as defined above coincides with subharmonicity in the sense of $-\Delta u \leq 0$. One can show that any harmonic function as defined above must be $C^2$ and thus Theorem 2.15 says that indeed our notion of harmonicity coincides with the earlier one.

We now need a first important ingredient: Perron’s method works locally, so somehow one has to pass from the notion of local subsolutions to global subsolutions.
Lemma 2.30. Let $f : \Omega \to \mathbb{R}$ be continuous and assume that for any $x \in \Omega$ there exists $r = r(x)$ such that for any $r \in (0, r(x))$ we have

$$f(x) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} f(y) dy.$$ 

Then $f$ is subharmonic.

Proof. Denote $\rho(x)$ the maximal value such that

$$\rho(x) := \sup\{\rho > 0 : f(x) \leq \frac{1}{|\partial B_{\rho(x)}|} \int_{\partial B_{\rho(x)}(x)} f(y) dy \text{ for all } r \in (0, \rho)\}.$$ 

We observe that

$$f(x) \leq \frac{1}{|\partial B_{\rho(x)}|} \int_{\partial B_{\rho(x)}(x)} f(y) dy,$$

which follows from the continuity (for each fixed $x$ and $r > 0$)

$$r \mapsto \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} f(y) dy.$$

Observe also that

$$\lim_{r \to 0^+} \frac{1}{|\partial B_r|} \int_{\partial B(x,r)} f(y) dy = f(x).$$

Now we show that $\rho$ is lower semicontinuous, i.e.

$$\liminf_{\Omega \ni y \to x} \rho(y) \geq \rho(x).$$

Indeed, assume that there exists a sequence $y_k \to x$ and some $\varepsilon > 0$ such that $\rho(y_k) \leq \rho(x) - \varepsilon$ then there must be some $r_k \leq \rho(x) - \varepsilon$ such that

$$f(y_k) \geq \frac{1}{|\partial B_{r_k}|} \int_{\partial B_{r_k}(x)} f(z) dz + \varepsilon.$$ 

Clearly $r_k > 0$. Up to taking a subsequence we can assume that $r_k \xrightarrow{k \to \infty} \tilde{r} \in [0, \rho(x) - \varepsilon]$. Then we have (by continuity of $f$)

$$f(x) \geq \frac{1}{|\partial B_{\tilde{r}}|} \int_{\partial B_{\tilde{r}}(x)} f(z) dz + \varepsilon.$$ 

This is a contradiction, since $\tilde{r} < \rho(x)$. The contradiction also holds if $\tilde{r} = 0$, then the integral on the right-hand side would be replaced with $f(x)$.

So we indeed have

$$\liminf_{\Omega \ni y \to x} \rho(y) \geq \rho(x).$$

In particular on any compact subset $K \subset \Omega$, $\rho$ attains its global minimum in some $x_0 \in K$, and $\rho(x_0) > 0$. Call this minimum $\rho_{\text{min}}$.

We need to show that $\rho(x) = \text{dist}(x, \partial \Omega)$ for all $x \in K$ (since $K \subset \subset \Omega$ is arbitrary this implies the claim.)
Assume that \( x \in K \) and \( \rho(x) < \text{dist}(x, \partial \Omega) \). Take \( \delta \in (0, \rho_{\text{min}}) \) such that \( R < \rho(x) + \delta < \text{dist}(x, \partial \Omega) \).

Let \( h \) be the solution to
\[
\begin{cases}
\Delta h = 0 & \text{in } B(x, R) \\
h = f & \text{on } \partial B(x, R).
\end{cases}
\]
We know that \( h \) exists, since we are in a ball and have the explicit Poisson formula. We then have that \( h \) satisfies the mean value equality, and thus
\[
h(x) = |\partial B(R)| \int_{\partial B(x, R)} h = |\partial B(R)| \int_{\partial B(x, R)} f.
\]
If only we could show that \( h(x) \geq f(x) \) we’d have that
\[
f(x) \leq |\partial B(R)| \int_{\partial B(x, R)} f \quad \forall R < \rho(x) + \delta,
\]
which contradicts the definition of \( \rho(x) \).

How do we show \( h(x) \geq f(x) \)? This is the maximum principle.

Consider \( f - h \). We then have for any \( y \in B(x, R) \) and any \( r \leq \min\{\rho(y), B(x, R)\} \)
\[
(f - h)(y) \leq |\partial B(r)|^{-1} \int_{\partial B(r)} (f - h)(z) dz.
\]
This rules out that there is any local maximum of \( f - h \) anywhere in \( B(r) \), and thus there is no local maximum of \( f - h \) in \( B(x, R) \). In particular we have that
\[
f(x) - h(x) \leq \sup_{y \in B(x, R)} (f - h)(y) \leq \sup_{\partial B(x, R)} f - h = 0.
\]
Thus \( f(x) \leq h(x) \), thus we have shown
\[
f(x) \leq |\partial B(R)| \int_{\partial B(x, R)} f \quad \forall R < \rho(x) + \delta,
\]
a contradiction to \( \rho(x) \). Thus \( \rho(x) = \text{dist}(x, \partial \Omega) \) and we can conclude.

\( \square \)

(Very roughly) the idea of Perron’s method is as follows.

**Perron: Step 1**

Consider the collection of all subsolutions (which is a nonempty set)
\[
S_g := \{ v \in C^0(\overline{\Omega}) : \quad v \leq g \quad \text{on } \partial \Omega, \quad v \text{ is subharmonic in } \Omega \}
\]
We need to show \( S_g \) is nonempty. This is easy. Take \( v := \min_{x \in \partial \Omega} g(x) \). Then \( v \) is constant, so \( -\Delta v = 0 \) (in particular \( v \) is subharmonic). And clearly \( v \leq g \) on \( \partial \Omega \).

**Perron: Step 2**
Here comes the trick: let $u$ be simply the largest subsolution, for $x \in \overline{\Omega}$

$$u(x) := \sup_{v \in S_g} v(x).$$

The idea is that since $u$ is the largest subsolution, then even locally there cannot be a larger one. However if locally $u$ was not harmonic, then we can use a harmonic replacement technique on a ball to get a contradiction.

First we need to ensure that $u$ is well-defined. Here we use the maximum principle, Corollary 2.18 and Corollary 2.16. Observe that these arguments were based on the continuity of a subsolution $v$ and the mean value formula so they still apply to our situation, and we have

$$v(x) \leq \sup_{\partial \Omega} g \quad \forall v \in S_g, \quad \forall x \in \overline{\Omega}.$$ 

This implies that for each $x \in \overline{\Omega}$ the family $\{v(x) : v \in S_g\}$ has an upper bound, so the supremum is well-defined. That is $u$ is well-defined.

Next we observe that (formally) $u$ is still subharmonic. Let $x \in \Omega$ and consider any ball $B_r(x) \subset \Omega$.

$$u(x) = \sup_{v \in S_g} v(x) \leq \sup_{v \in S_g} \frac{1}{|B_r|} \int_{B_r(x)} v(y)dy \leq \frac{1}{|B_r|} \int_{B_r(x)} \sup_{v \in S_g} v(y)dy \leq \frac{1}{|B_r|} \int_{B_r(x)} u(y)dy.$$ 

Alas the integral of $u$ may not exist (for all we know $u$ could be non-measurable!). That won’t happen, indeed we have

**Lemma 2.31.** $u$ is lower semicontinuous, that is

$$u(x) \leq \liminf_{y \to x} u(y)$$

Think of $u(t) := \sup_{r>0} t^r$ for $t \in [0,1]$ to see that $u$ may not be continuous!

**Proof.** Fix any $x \in \overline{\Omega}$ and let $\varepsilon > 0$. Then there must be some $\bar{v} \in S_g$ such that

$$u(x) \leq \bar{v}(x) + \varepsilon.$$ 

Since $\bar{v}$ is continuous, there exists $\delta > 0$ such that for any

$$|ar{v}(x) - \bar{v}(y)| \leq \varepsilon \quad \forall y \in B(x, \delta) \cap \overline{\Omega}.$$ 

Consequently, for any $y \in \overline{\Omega}$,

$$u(x) - u(y) \leq \bar{v}(x) - \bar{v}(y) + \varepsilon \leq 2\varepsilon \quad \forall y \in B(x, \delta) \cap \overline{\Omega}.$$ 

Observe that we cannot do the same argument in the other direction, since $x$ is fixed and $y$ is variable. In any case, now we have

$$u(x) \leq u(y) + 2\varepsilon \forall y \in B(x, \delta) \cap \overline{\Omega}$$

which implies

$$u(x) \leq \liminf_{y \to x} u(y) + \varepsilon.$$
The above lemma makes $u$ measurable, and since it is bounded

$$\min_{\partial \Omega} g \leq u(x) \leq \sup_{\partial \Omega} g.$$  

$u$ is integrable. But still it does not say that $u$ is a subsolution (because we haven’t shown that $u$ is continuous).

Fix now $\bar{x} \in \Omega$. Then there must be a sequence of subharmonic $\tilde{v}_n \in S_g$ such that $\lim_{n \to \infty} \tilde{v}_n(\bar{x}) = u(\bar{x})$. Set

$$v_n(z) := \max\{\tilde{v}_1(x), \tilde{v}_2(x), \ldots, \tilde{v}_n(x), \min g\}.$$  

As a (finite) maximum of continuous functions $v_n \in C^0(\Omega)$. As we did for $u$ above, we can also easily check that $v_n$ is still a subharmonic function. Moreover we have monotonicity

$$v_n(x) \leq v_{n+1}(x) \quad \forall x \in \Omega,$$

all while still ensuring $\lim_{n \to \infty} \tilde{v}_n(\bar{x}) = u(\bar{x})$.

Take now a ball $B(\pi, R) \subset \Omega$ ($\pi$ is in the interior of $\Omega$!). We now replace now $v_n$ inside of $B(\pi, R)$ with its harmonic replacment, i.e. we set

$$w_n(x) := \begin{cases} \frac{R^2 - |x - \pi|^2}{c_n R^2} \int_{\partial B_R(\pi)} \frac{v_n(z)}{|z - x|^n} \, dz & x \in B_R(\bar{x}) \\ v_n(x) & x \in \Omega \setminus B_R(\bar{x}) \end{cases}.$$  

Then we have $w_n \in C^0(\Omega)$. Since $v_n$ was monotonically increasing, so is $w_n$.

$$w_n(x) \leq w_{n+1}(x) \quad \forall x \in \Omega.$$

**Lemma 2.32.** We have the following properties

1. $w_n(x) \geq v_n(x)$ and
2. $w_n \in S_g$.

**Proof.** (1) We have $w_n \equiv v_n$ in $\Omega \setminus B(\pi, R)$. Since $w_n$ is harmonic in $B(\pi, R)$ we have $(v - w)$ is subharmonic in $B(\pi, R)$, and since $v - w = 0$ on $\partial B(\pi, R)$ the maximum principle implies $v - w \leq 0$ in $B(\pi, R)$, i.e.

$$v(x) \leq w(x) \quad \forall x \in B(\pi, R).$$

(2) Since $w_n \equiv v_n$ in $\Omega \setminus B(\pi, R)$ we have that $w_n(x) \leq g(x)$ for all $x \in \partial \Omega$. We have

$$v_n(x) \leq \frac{1}{|B(r)|} \int_{B(x, r)} v_n(y) \, dy \overset{(1)}{\leq} \frac{1}{|B(r)|} \int_{B(x, r)} w_n(y) \, dy$$

So for all $x \in \mathbb{R}^n \setminus B(\bar{x}, R)$, $v_n = w_n$ is subharmonic.
Let now $x \in B(\bar{x}, R)$ (which is open). Since $w_n$ is harmonic for all $r < \text{dist}(x, \partial B(\bar{x}, R))$ we have

$$w_n(x) \leq \frac{1}{|B(r)|} \int_{B(x,r)} w_n(y).$$

We conclude that $w_n$ is subharmonic in $\Omega$ by Lemma 2.30.  

□

Since $w_n \in S_g$ we conclude that

$$v_n(x) \leq w_n(x) \leq u(x) \quad \forall x \in \overline{\Omega}$$

and thus in particular

$$\lim_{n \to \infty} w_n(\bar{x}) = u(\bar{x}).$$

Lemma 2.33. For $x \in \overline{B(\bar{x}, R/2)}$ set

$$w(x) := \lim_{n \to \infty} w_n(\bar{x}).$$

(This exists since $w_n$ is bounded by $u$ and monotonicity). Then $w$ is harmonic in $B(\bar{x}, R/2)$ and $w \leq u$ in $B(\bar{x}, R/2)$.

Proof. For each $n \in \mathbb{N}$ we know that $w_n$ is harmonic in $B(\bar{x}, R)$ (by definition).

So $w_n - w_m$ for $n, m \in \mathbb{N}$ is harmonic in $B(\bar{x}, R)$. We want to apply Harnack’s inequality, Theorem 2.23, so let us assume $n \geq m$, then we have $w_n - w_m \geq 0$, and thus

$$\sup_{x \in B(\bar{x}, R/2)} (w_n(x) - w_m(x)) \leq C \inf_{y \in B(\bar{x}, R/2)} (w_n(y) - w_m(y)) \quad \forall n \geq m,$$

i.e.

$$\sup_{x \in B(\bar{x}, R/2)} |w_n(x) - w_m(x)| \leq C \inf_{y \in B(\bar{x}, R/2)} |w_n(y) - w_m(y)| \quad \forall n \geq m.$$

In particular,

$$\sup_{x \in B(\bar{x}, R/2)} |w_n(x) - w_m(x)| \leq C |w_n(\bar{x}) - w_m(\bar{x})| \frac{n,m \to \infty}{n,m \to \infty} 0.$$

That is, $w_n$ is a Cauchy sequence with respect to uniform convergence in $\overline{B(\bar{x}, R/2)}$, and since $w_n$ is continuous we conclude that there must be some $w \in C^0(\overline{B(\bar{x}, R/2)})$ such that $w$ is the uniform limit of $w_n$ in $B(\bar{x}, R/2)$.

Since $w_n$ is harmonic in $B(\bar{x}, R/2)$ (in the sense of Definition 2.29(2)), so is $w$ (by the uniform convergence).

Since $w_n(x) \leq u(x)$ for all $x \in \overline{\Omega}$ (because $w_n \in S_g$), we conclude that $w(x) = \lim_{n \to \infty} w_n(x) \leq u$ for all $x \in \overline{B(\bar{x}, R/2)}$.  

□

Lemma 2.34. Take $w$ from Lemma 2.32. Then $w = u$ in $B(\bar{x}, R/2)$.  

Proof. We already know \( w \leq u \) from Lemma 2.32.

So assume that there is \( \tilde{y} \in B(\bar{x}, R/2) \) such that \( w(\tilde{y}) > u(\tilde{y}) \).

Since \( w(\tilde{y}) = \lim_{n \to \infty} w_n(\tilde{y}) \) there must be some \( n \) such that \( w_n(\tilde{y}) > u(\tilde{y}) \).

But this is a contradiction since \( w_n \in S_g \), and thus \( u(\tilde{y}) = \sup_{v \in S_g} v(\tilde{y}) \geq w_n(\tilde{y}) > u(\tilde{y}) \).

We can conclude. \( \square \)

Corollary 2.35. Let \( u(x) := \sup_{v \in S_g} v(x) \). Then \( u \in C^0(\Omega) \) and \( \Delta u = 0 \) in \( \Omega \)

Proof. For every \( \bar{x} \in \Omega \) there exists a small neighborhood \( B(\bar{x}, R/2) \) where \( u \) equals a harmonic function, Lemma 2.34. So \( u \) must be harmonic and continuous around any point \( x \in \Omega \). \( \square \)

It remains to show that \( u = g \) on \( \partial \Omega \).

Lemma 2.36. Assume that \( \partial \Omega \in C^\infty \) and \( g \) is continuous in \( \partial \Omega \). Let \( u(x) := \sup_{v \in S_g} v(x) \), \( u \in C^0(\Omega) \) (not yet up to the boundary!) be the harmonic function from before.

Then \( u \in C^0(\overline{\Omega}) \) and for any \( \theta \in \partial \Omega \) we have

\[
\lim_{x \to \theta} u(x) = g(\theta).
\]

Proof. Since \( u(x) := \sup_{v \in S_g} v(x) \) and \( v \in S_g \) must satisfy \( v \leq g \) on \( \partial \Omega \) we conclude that To see the other direction, we build what is called a barrier. A barrier at \( \theta \) is a continuous function \( b \in C^0(\overline{\Omega}) \) which is superharmonic (i.e. \( -b \) is subharmonic) in \( \Omega \) and \( b(x) \geq 0 \) for all \( x \in \overline{\Omega} \) and \( b(x) = 0 \) if and only if \( x = \theta \).

Fix \( \theta \in \partial \Omega \). Since \( \partial \Omega \) is smooth, there exists (nontrivial exercise!) a ball \( B(\bar{z}, R) \subset \mathbb{R}^n \setminus \Omega \) such that \( \overline{B(\bar{z}, R)} \cap \overline{\Omega} = \{ \theta \} \) (this is called the exterior sphere condition of \( \partial \Omega \)).

Here is our barrier function

\[
b(x) := \begin{cases}
R^{2-n} - |x - \bar{z}|^{2-n} & \text{if dimension } n \geq 3 \\
- \log(R) + \log(|x - \bar{z}|) & \text{if } n = 2.
\end{cases}
\]

Then \( b \in C^\infty(\mathbb{R}^n \setminus \{ \bar{z} \}) \) and since it involves the fundamental solution we know that \( \Delta b(x) = 0 \) for all \( x \in \mathbb{R}^n \setminus \{ \bar{z} \} \). Since \( \bar{z} \not\in \overline{\Omega} \) we conclude that \( \Delta b = 0 \) in \( \Omega \).

For \( x \in \mathbb{R}^n \setminus \overline{B(\bar{z}, R)} \) we have \( b(x) > 0 \) (observe that \( 2 - n \) is a negative power!) and we have \( b(\theta) = 0 \). Since \( \overline{B(\bar{z}, R)} \cap \overline{\Omega} = \{ \theta \} \) this satisfies the barrier definition in \( \Omega \).
Fix $\varepsilon > 0$. Since $g$ is continuous on $\partial \Omega$ there exists $\delta > 0$ such that

$$|g(x) - g(\theta)| < \varepsilon \quad \forall x \in \partial \Omega, |x - \theta| < \delta.$$  

Set

$$\lambda := \inf_{z \in \partial \Omega \setminus B(\theta, \delta)} \beta(z) > 0.$$  

and

$$\Lambda := 2 \sup_{\partial \Omega} |g| < \infty.$$  

Then $\bar{v}$ is still harmonic in $\Omega$ (in particular it is subharmonic). Moreover for $x \in \partial \Omega$, if $|x - \theta| < \delta$ then

$$\bar{v}(x) - g(x) = g(\theta) - g(x) - \varepsilon - b(x) \frac{\Lambda}{\lambda} \leq |g(\theta) - g(x)| - \varepsilon \leq 0.$$  

If on the other hand $|x - \theta| \geq \delta$ then

$$\bar{v}(x) - g(x) = g(\theta) - \varepsilon - g(x) - b(x) \frac{\Lambda}{\lambda} \leq \frac{\bar{v}(x) - g(x)}{\lambda} \leq g(\theta) - g(x) - \Lambda$$

$$\leq 2 \sup_{\partial \Omega} |g| - \Lambda \leq 0.$$  

So we have $\bar{v} \leq g$ on $\partial \Omega$, and thus $\bar{v} \in S_g$. Since $u = \sup_{v \in S_g} v$ we find that

$$u(\theta) \geq \bar{v}(\theta) = g(\theta) - \varepsilon.$$  

Since this holds for all $\theta \in \partial \Omega$ we have shown

$$u(x) \geq g(x) - \varepsilon \quad \text{for all } x \in \partial \Omega.$$  

This again holds for any $\varepsilon > 0$ so that we have

$$u(x) \geq g(x) \quad \text{for all } x \in \partial \Omega.$$  

We conclude that $u(x) = g(x)$ for all $x \in \partial \Omega$ and we can conclude. $\square$

We finally can conclude

**Corollary 2.37.** Let $u(x) := \sup_{v \in S_g} v(x)$. Then $u \in C^0(\overline{\Omega})$ and $\Delta u = 0$ in $\Omega$ and $u = g$ on $\partial \Omega$.

Let us summarize some features of Perron’s method.

- Perron’s method shows existence of solutions via obtaining “a largest subsolution” (a “smallest supersolution” would work similarly).
- it relies on the ability to locally improve a subsolution to obtain a global solution (but observe that we worked hard to show that a local subsolution everywhere is a global subsolution).
• Perron’s method relies \textit{extremely} on comparison principles. Take an equation without comparison principle (e.g. 4th order, or systems of equations), and there is essentially no hope of running this idea.

• Perron’s method likes to work with some form of continuity, not differentiability; in particular we need to define a notion of “weak” subsolution that makes sense for continuous functions. This works for second order equations with comparison principles often via theories like \textit{Viscosity solutions}.

2.9. \textbf{Weak Solutions, Regularity Theory.} Now we look at our first encounter with \textit{distributional solutions}. Let $u \in L^1_{\text{loc}}(\Omega)$, that is $u$ is a measurable function on $\Omega$ which is integrable on every compactly contained set $K \subset \Omega$, i.e.

$$\int_K |u| < \infty.$$  

$u$ certainly has no reason to be differentiable, it might not even be continuous. How on earth are we going to define

$$\Delta u = 0 \quad \text{in } \Omega?$$

The idea is that if $u \in C^2(\Omega)$ then

(2.20)  

$$\Delta u = 0 \quad \text{in } \Omega$$

is equivalent to saying that

(2.21)  

$$\int_\Omega u \Delta \varphi = 0 \quad \text{for all } \varphi \in C^\infty_c(\Omega).$$

(Recall that $C^\infty_c(\Omega)$ are those smooth functions that have compact support $\text{supp } \varphi \subset \subset \Omega$).

Indeed, for $\varphi \in C^\infty_c(\Omega)$ and $u \in C^2(\Omega)$ we have by integration by parts

$$\int_\Omega u \Delta \varphi = \int_\Omega \Delta u \varphi.$$  

So for $u \in C^2(\Omega)$ we clearly have that (2.21) is equivalent to

(2.22)  

$$\int_\Omega \Delta u \varphi = 0 \quad \text{for all } \varphi \in C^\infty_c(\Omega).$$

Now if (2.20) holds then clearly (2.22) holds.

On the other hand assume that (2.22) holds, but (2.20) is false. That is assume there is $x_0 \in \Omega$ such that (w.l.o.g.)

$$\Delta u(x_0) > 0.$$  

Since $u \in C^2(\Omega)$ we have $\Delta u \in C^0(\Omega)$ and thus there exists a ball $B(x_0, r) \subset \subset \Omega$ such that

(2.23)  

$$\Delta u > 0 \quad \text{on } B(x_0, r)$$
Now let $\varphi \in C^\infty_c(\Omega)$ a bump function (or cutoff function), namely a function $\varphi$ such that $\varphi \geq 1$ in $B(x_0, r/2)$ and $\varphi \equiv 0$ in $\Omega \setminus B(x_0, r)$, and $\varphi \geq 0$ everywhere. These bump functions really exist: they can be build by essentially scaled and glued versions of 

$$
\eta(x) := \begin{cases} 
  e^{-\frac{1}{1-|x|^2}} & \text{for } |x| < 1 \\
  0 & \text{for } |x| > 1
\end{cases}
$$

See Figure 2.9.

For this bump function $\varphi$ we have from (2.23)

$$
\int_\Omega \varphi \Delta u > 0
$$

which contradicts (2.22). This proves the equivalence of (2.21) and (2.20) for $C^2$-functions $u$.

However, we notice that while (2.20) only makes sense for functions $u$ that are twice differentiable, the statement (2.21) makes sense for all functions $u \in L^1_{\text{loc}}(\Omega)$. This warrants the following definition:

**Definition 2.38** (Weak solutions of the Laplace equation). For a function $u \in L^1_{\text{loc}}(\Omega)$ we say that (2.20) is satisfied in the weak sense (or in the distributional sense) if

$$
\int_\Omega u \Delta \varphi = 0 \quad \text{for all } \varphi \in C^\infty_c(\Omega).
$$

holds. The functions $\varphi$ used to “test” the equation are for this very reason called test-functions.

To distinguish the notion of solution we used before, we say that if $\Delta u = 0$ in a differentiable function sense then $u$ is a strong solution or classical solution.

Above, we already have shown the following statement

**Proposition 2.39.** Let $u \in C^2(\Omega)$. Then the following two statements are equivalent:

1. $u$ is a weak solution to the Laplace equation $\Delta u = 0$ in $\Omega$
2. $u$ is a classical solution of $\Delta u = 0$ in $\Omega$. 

Weyl proved that this equivalence holds for $u \in L^1_{\text{loc}}$ (i.e. with no a priori differentiability at all) – this is our first result of regularity theory: showing that weak solutions which are a priori only integrable are actually differentiable. Observe: the reason this works here is that we have a homogeneous equation $\Delta u = 0$, and that $\Delta$ is a constant-coefficient linear elliptic operator (and one can spend much more time for proving similar results for more general linear elliptic operators). Having said that, in some sense, the regularity theory for elliptic equations is always somewhat based on the following Theorem, Theorem 2.40 (albeit in a hidden way).

**Theorem 2.40 (Weyl’s Lemma).** Let $u \in L^1_{\text{loc}}(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open. If $u$ is a weak solution of Laplace equation, i.e.

$$
\int_{\Omega} u \Delta \varphi = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega).
$$

then $u \in C^\infty(\Omega)$ and $\Delta u$ in the classical sense.

Observe that this theorem (rightfully) does not say anything about $u$ on $\partial \Omega$, this is a purely interior result!

The proof of Theorem 2.40 exhibits the structure that many proofs in PDE have. First on obtains some *a priori estimates* (namely under the assumption that everything is smooth we find good estimates). Then we show that these estimates hold also for rough solutions by an approximation argument.

The a priori estimates for the Laplace equations are called the Cauchy estimates. These are truly amazing: They say that if we solve the Laplace equation we can estimate all derivatives, in pretty much any norm simply by the $L^1$-norm of the function.

**Lemma 2.41 (Cauchy estimates).** Let $u \in C^\infty(\Omega)$ be harmonic, $\Delta u = 0$ in $\Omega$. Then we have for any ball $B(x_0, r) \subset \Omega$ and for any multiindex $\gamma$ of order $|\gamma| = k$,

$$
|\partial^\gamma u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0, r))}.
$$

In particular we have for any $\Omega_2 \subset \subset \Omega$ that

$$
\sup_{\Omega_2} |D^k u| \leq C(\text{dist}(\Omega_2, \Omega), k)\|u\|_{L^1(\Omega)}
$$

*Proof of the Cauchy estimates, Lemma 2.41.* For $k = 0$ we argue with the mean value property for harmonic functions, Theorem 2.15. We have for any $\rho$ such that $B(x_0, \rho) \subset \Omega$ and any $x \in B(x_0, \rho/2)$,

$$
|u(x)| = \left| \int_{B(x, \rho/2)} u(z) \, dz \right| \leq \frac{C}{\rho^n} \int_{B(x, \rho/2)} |u(z)| \, dz \leq \frac{C}{\rho^n} \int_{B(x_0, \rho)} |u(z)| \, dz.
$$
That is, we have obtained that for if \( \Delta u = 0 \) on \( B(x_0, \rho) \) then
\[
(2.24) \quad \sup_{B(x_0, \rho/2)} |u| \leq \frac{C}{\rho^n} \|u\|_{L^1(B(x_0, \rho))}.
\]
This proves in particular the case \( k = 0 \) (taking \( \rho =: r \)).

For the case \( k = 1 \) we use a technique called “differentiating the equation” (and in more general situations where this is used in a discretized version we will study later is due to Nirenberg, cf. Section 5.4). Observe that \( \Delta u = 0 \) in \( \Omega \) implies
\[
\Delta \partial_i u = \partial_i \Delta u = 0 \quad \text{in} \ \Omega.
\]
So if we set \( v := \partial_i u \) we have that \( \Delta v = 0 \) in \( \Omega \). For \( x \in B(x_0, \rho/4) \), again from the mean value property for harmonic functions, Theorem 2.15, we get with an additional integration by parts
\[
|\partial_i u(x)| = \left| \int_{B(x, \rho/4)} \partial_i u(z) \, dz \right| = \frac{C}{\rho^n} \left| \int_{\partial B(x, \rho/4)} u(\theta) \nu^i d\mathcal{H}^{n-1}(\theta) \right|
\leq \frac{C}{\rho^n} \rho^{n-1} \sup_{B(x, \rho/4)} |u|
\leq \frac{C}{\rho^n} \rho^{n-1} \sup_{B(x_0, \rho/2)} |u|
\]
Now in view of the estimates in the step \( k = 0 \), namely (2.24), we arrive at
\[
\sup_{B(x_0, \rho/4)} |\nabla u(x)| \leq \frac{C}{\rho^{n+1}} \|u\|_{L^1(B(x_0, \rho))}.
\]
Differentiating the equation again, we find by induction that (the constant changes in each appearance!)
\[
|\nabla^k u(x_0)| \leq \sup_{B(x_0, A^{-k}\rho)} |\nabla^k u(x)| \leq \frac{C}{\rho^{n+1}} \|\nabla^{k-1} u\|_{L^1(B(x_0, A^{1-k}\rho))} \leq \ldots \leq \frac{C}{\rho^{n+k}} \|u\|_{L^1(B(x_0, \rho))}.
\]
If we want to show the estimate on \( \Omega_2 \subset \subset \Omega \) we now pick \( \rho < \text{dist}(\Omega_2, \partial\Omega) \) and obtain the claim.

\[\square\]

**Proof of Weyl’s Lemma: Theorem 2.40.** We use a mollification argument, i.e. we approximate \( u \) with smooth functions \( u_\varepsilon \) that also solve (in the classical sense) the Laplace equation.

Let \( \eta \in C_c^\infty(B(0, 1)) \) be another bump function, this time with the condition \( \eta(x) = \eta(-x) \), i.e. \( \eta \) is even, \( \eta \geq 0 \) everywhere, and normalized such that
\[
\int_{\mathbb{R}^n} \eta = 1.
\]
We rescale \( \eta \) by a factor \( \varepsilon > 0 \) and set
\[
\eta_\varepsilon(x) := \varepsilon^{-n} \eta(x/\varepsilon).
\]
Then the convolution\(^4\) is defined as
\[
u_\varepsilon(x) := \eta_\varepsilon * u(x) := \int_{\mathbb{R}^n} \eta_\varepsilon(y - x) u(y) \, dy
\]
Clearly this is not well-defined for all \( x \), if \( u \in L^1_{\text{loc}}(\Omega) \) only. But it is defined for all \( x \in \Omega \) such that \( \text{dist}(x, \partial \Omega) > \varepsilon \), since \( \text{supp} \eta_\varepsilon(\cdot - x) \subset B(x, \varepsilon) \).

But observe that derivatives on \( u_\varepsilon \) hit only the kernel \( \eta_\varepsilon \) (which is smooth) (there is a dominated convergence to be used to show that, and for this we need \( L^1_{\text{loc}}! \))
\[
\partial^\gamma u_\varepsilon(x) := \eta_\varepsilon * u(x) := \int_{\mathbb{R}^n} \partial^\gamma \eta_\varepsilon(y - x) u(y) \, dy
\]
That is \( u_\varepsilon \in C^\infty(\Omega_{-\varepsilon}) \) where
\[
\Omega_{-\varepsilon} = \{ x \in \Omega, \ \text{dist}(x, \partial \Omega) > \varepsilon \}.
\]

The fun part (which we used above already) is that convolutions behave well with differential operators, namely we will show now that \( \Delta u_\varepsilon = 0 \) in \( \Omega_{-\varepsilon} \).

For this let \( \psi \in C^\infty_c(\Omega_{-\varepsilon}) \) a testfunction, then we have
\[
\int_{\Omega_{-\varepsilon}} u_\varepsilon(x) \Delta \psi(x) \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(y) \eta_\varepsilon(x-y) \Delta \psi(x) \, dy \, dx = \int_{\mathbb{R}^n} u(y) \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) \Delta \psi(x) \, dx \, dy
\]
Now, by integration by parts (for any fixed \( y \in \mathbb{R}^n \))
\[
\int_{\mathbb{R}^n} \eta(x-y) \Delta \psi(x) \, dx = \int_{\mathbb{R}^n} \Delta_x \eta(x-y) \psi(x) \, dx = \int_{\mathbb{R}^n} \Delta_y \eta(x-y) \psi(x) \, dx = \Delta_y \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) \psi(x) \, dx
\]
So if we set
\[
\varphi(y) := \eta_\varepsilon * \psi(y) \equiv \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) \psi(x) \, dx
\]
then we have by the support condition on \( \psi \) that \( \varphi \in C^\infty_c(\Omega) \), and thus
\[
\int_{\Omega_{-\varepsilon}} u_\varepsilon(x) \Delta \psi(x) \, dx = \int_{\mathbb{R}^n} u(y) \Delta \varphi(y) \, dy (2.21) = 0.
\]
This argument works for any \( \psi \in C^\infty_c(\Omega_{-\varepsilon}) \), that is \( u_\varepsilon \) is weakly harmonic in \( \Omega_{-\varepsilon} \). But since \( u_\varepsilon \in C^\infty(\Omega_{-\varepsilon}) \) this implies in view of Proposition 2.39 that in the strong sense
\[
\Delta u_\varepsilon = 0 \quad \text{in} \ \Omega_{-\varepsilon}.
\]
So now \( u_\varepsilon \) is a smooth solution to Laplace’s equation, so we use the a priori estimates of Lemma 2.41.

**Fix** \( \Omega_2 \subset \subset \Omega \). Between \( \Omega_2 \) and \( \Omega \) we can squeeze two more set \( \Omega_3 \), and \( \Omega_4 \),
\[
\Omega_2 \subset \subset \Omega_3 \subset \subset \Omega_4 \subset \subset \Omega.
\]
\(^4\)we have seen this operation for the Fourier Transform argument above after (2.3), there we used a nonsmooth kernel \(| \cdot |^{2-n} \) for the convolution.
For any $\varepsilon$ small enough, namely

$$\varepsilon < \text{dist}(\Omega_3, \partial \Omega_4) \quad \text{and} \quad \varepsilon < \text{dist}(\Omega_3, \partial \Omega_4)$$

we have that $\Delta u^\varepsilon = 0$ in $\Omega_3$, so by the Cauchy estimates, Lemma 2.41, we have for any $k \in \mathbb{N}$

$$\sup_{\Omega_2} |\nabla^k u^\varepsilon| \leq C(k, \Omega_2, \Omega_3) \|u^\varepsilon\|_{L^1(\Omega_3)}.$$

Now we estimate, by Fubini,

$$\|u^\varepsilon\|_{L^1(\Omega_3)} \leq \int_{\Omega_3} \int_{\mathbb{R}^n} |\eta_\varepsilon(x - y)| |u(y)| \, dy \, dx = \int_{\mathbb{R}^n} |u(y)| \int_{\Omega_3} |\eta_\varepsilon(x - y)| \, dx \, dy$$

Since $\varepsilon$ is small enough we have that

$$\text{supp} \left( \int_{\Omega_3} |\eta_\varepsilon(x - \cdot)| \, dx \right) \subset \Omega_4.$$

So we get

$$\|u^\varepsilon\|_{L^1(\Omega_3)} \leq \|u\|_{L^1(\Omega_4)} \sup_{y \in \mathbb{R}^n} \int_{\Omega_3} |\eta_\varepsilon(x - y)| \, dx \leq \|u\|_{L^1(\Omega_4)} \int_{\mathbb{R}^n} |\eta_\varepsilon(z)| \, dz.$$

Now we use the definition of $\eta_\varepsilon$ to compute via substitution\(^5\)

$$\int_{\mathbb{R}^n} |\eta_\varepsilon(z)| \, dz = \varepsilon^{-n} \int_{\mathbb{R}^n} |\eta(z/\varepsilon)| \, dz = \varepsilon^{-n} \int_{\mathbb{R}^n} |\eta(z)| \, dz = \int_{\mathbb{R}^n} |\eta(\tilde{z})| \, d\tilde{z} = 1.$$

The last equality is due to the normalization of $\eta$, $\int \eta = 1$.

That is, we have shown that for any $k \in \mathbb{N} \cup \{0\}$

$$\sup_{\Omega_2} |\nabla^k u^\varepsilon| \leq C(k, \Omega_2, \Omega_3) \|u\|_{L^1(\Omega_3)},$$

and the right-hand side is finite since $u \in L^1_{loc}(\Omega)$ and $\Omega_4 \subset \subset \Omega$.

This estimate holds for any $\varepsilon > 0$, so $u^\varepsilon$ and all its derivative are uniformly equicontinuous (in $\varepsilon$). By Arzela-Ascoli (and a diagonal argument in $k$) we find a converging subsequence $\varepsilon \to 0$ and a function $u_0 \in C^\infty(\Omega_2)$ such that for any $k \in \mathbb{N} \cup \{0\}$

$$|\nabla^k u^\varepsilon(x) - \nabla^k u_0(x)| \xrightarrow{\varepsilon \to 0} 0 \quad \text{locally uniformly in } \Omega_2.$$

We claim that $u = u_0$ in almost every point (since $u$ is an $L^1_{loc}$-function it is actually a the class of maps equal up to almost every point, $u_0$ is a continuous representative of the class $u$). Indeed, by the normalization $\int \eta = 1$ which implies $\int \eta_\varepsilon = 1$ we have

$$|u^\varepsilon(x) - u(x)| = \left| \int \eta_\varepsilon(y - x) (u(y) - u(x)) \, dy \right| \leq C(\eta) \int_{B(x, \varepsilon)} |u(y) - u(x)| \, dy.$$

So, by the Lebesgue differentiation theorem, we have for almost every $x \in \Omega_2$,

$$\lim_{\varepsilon \to 0} |u^\varepsilon(x) - u(x)| = 0,$$

\(^5\)observe for $\tilde{z} = z/\varepsilon$ we have in $n$ space dimensions $d\tilde{z} = \varepsilon^{-n} dz$
that is
\[ u_0 = u \quad \text{a.e. in } \Omega_2. \]
Thus \( u \in C^\infty(\Omega_2) \), and \( \Delta u = 0 \) in classical sense in \( \Omega_2 \).

Since this holds for any \( \Omega_2 \subset \Omega \) we have shown
\[ u \in C^\infty(\Omega), \] and \( \Delta u = 0 \) in classical sense in \( \Omega \).

\[ \square \]

**Corollary 2.42** (Liouville). Let \( u \in C^2(\mathbb{R}^n) \) and \( \Delta u = 0 \) in all of \( \mathbb{R}^n \). If \( u \) is a bounded function then \( u \equiv \text{const} \).

**Proof.** Fix \( x_0 \in \mathbb{R}^n \). In view of Lemma 2.41 we have for such a function \( u \), for any radius \( r > 0 \),
\[ |Du(x_0)| \leq \frac{C}{r^{n+1}} \|u\|_{L^1(B(x_0,r))} \]
If \( u \) is bounded,
\[ \|u\|_{L^1(B(x_0,r))} \leq Cr^n \sup_{\mathbb{R}^n} |u| < \infty \]
and thus
\[ |Du(x_0)| \leq Cr^{-1} \sup_{\mathbb{R}^n} |u|. \]
This holds for any \( r > 0 \), so if we let \( r \to \infty \), we get
\[ |Du(x_0)| = 0, \]
which holds for any \( x_0 \in \mathbb{R}^n \). That is, \( Du \equiv 0 \), and by the fundamental theorem of calculus this means \( u \) is a constant.

\[ \square \]

2.10. **Methods from Calculus of Variations – Energy Methods.** As we have seen, comparison principles is a strong tool for uniqueness (and also existence). These arguments also work in some situations of nonlinear pdes, where the theory of distributional solutions does not work, but the theory of Viscosity solutions can be applied, see [Koike, 2004].

On the other hand, the comparison methods are (currently) restricted to first or second-order equations, and to scalar equations. For systems or higher-order PDEs they seem not to be that helpful.

In this section we have a short look on energy methods, which is a basic tool of distributional theory. They do not rely on any comparison principle, and they are often used for higher-order differential equations and systems. On the other hand for some fully nonlinear equations ("non-variational" equations, equations “not in divergence form”) they cannot be well applied.

The ideas should be reminiscent of the arguments we employed for the weak solutions in Theorem 2.40.
Assume that we have
\begin{align}
\Delta u = f \quad &\text{in } \Omega \\
u = 0 \quad &\text{on } \partial \Omega
\end{align}

We have seen before Theorem 2.40 that this equation is related to the integral equation
\[ \int_{\Omega} Du \cdot D\varphi + f \varphi = 0 \quad \forall \varphi \in C^\infty_c(\Omega). \]

The interesting point is that this expression is a Frechet-Derivative of a function acting on the map \( u \) in direction \( \varphi \).

Indeed one can characterize solutions as minimizers of an energy functional. This is sometimes called the \textit{Dirichlet principle}.

**Theorem 2.43** (Energy Minimizers are solutions and vice versa). Assume \( f \in C^0(\overline{\Omega}) \).

Denote the class of permissible functions
\[ X := \{ u \in C^2(\overline{\Omega}), \quad u = 0 \quad \text{on } \partial \Omega \} \]
and define the energy
\[ \mathcal{E}(u) := \int_{\Omega} \frac{1}{2} |Du|^2 + fu. \]

Let \( u \in X \) be a minimizer of \( \mathcal{E} \) in \( X \), i.e.
\[ \mathcal{E}(u) \leq \mathcal{E}(v) \quad \forall v \in X. \]

Then \( u \) solves (2.25).

Conversely, if \( u \in X \) solves (2.25), then \( u \) is a minimizer of \( \mathcal{E} \) in the set \( X \).

**Proof.** We compute what is called the \textit{Euler-Lagrange-equations} of \( \mathcal{E} \): Let \( \varphi \in C^\infty_c(\Omega) \), then certainly \( u + t\varphi \in X \) for all \( t \in \mathbb{R} \). That is the minimizing property says that the function
\[ E(t) := \mathcal{E}(u + t\varphi) \]
has a minimum in \( t = 0 \). By Fermats theorem (one checks easily that \( E \) is differentiable in \( t \))
\[ \left. \frac{d}{dt} \right|_{t=0} E(t) \equiv E'(0) = 0. \]

Now observe that
\[ \left. \frac{d}{dt} \right|_{t=0} |D(u + t\varphi)|^2 = 2\langle Du, D\varphi \rangle \]
and
\[ \left. \frac{d}{dt} \right|_{t=0} f (u + t\varphi) = f \varphi. \]

Thus, we arrive at
\[ 0 = \left. \frac{d}{dt} \right|_{t=0} E(t) = \int_{\Omega} Du \cdot D\varphi + f \varphi = 0. \]
That is, \( u \) is a weak solution of (2.25). But \( u \in C^2(\Omega) \), so we argue similar to the proof of Proposition 2.39:

By an integration by parts (for \( \varphi \in C^\infty_c(\Omega) \) there are no boundary terms), we thus have

\[
0 = \int_\Omega Du \cdot D\varphi + f\varphi = 0 = -\int_\Omega (\Delta u - f)\varphi.
\]

Since \( \Delta u - f \) is continuous, and the last estimate holds for any smooth \( \varphi \in C^\infty_c(\Omega) \) we get that (as for Proposition 2.39, or otherwise by the fundamental lemma of calculus of variations, Lemma 2.44,

\[
\Delta u - f = 0.
\]

That is the first claim is proven: minimizers are solutions.

For the converse assume \( u \) solves (2.25). Let \( w \) be any other map in \( X \). Then we have

\[
\int_\Omega (\Delta u - f)(u - w) = 0.
\]

Observe that \( u \) and \( w \) have the same boundary value 0 on \( \partial \Omega \). Thus, when we perform the following integration by parts we do not find boundary terms,

\[
0 = -\int_\Omega \nabla u \cdot \nabla (u - w) + f(u - w) = 0.
\]

Now we compute (using Young’s inequality or Cauchy-Schwarz \( 2ab \leq a^2 + b^2 \))

\[
\int_\Omega |\nabla u|^2 + fu \overset{(2.26)}{=} \int_\Omega \nabla u \cdot \nabla w + fw \\
\leq \int_\Omega \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |\nabla w|^2 + fw \\
= \frac{1}{2} \int_\Omega |\nabla u|^2 + \mathcal{E}(w)
\]

Subtracting \( \frac{1}{2} \int_\Omega |\nabla u|^2 \) from both sides in the estimate above we obtain

\[
\mathcal{E}(u) \leq \mathcal{E}(w).
\]

That is, we have shown: if \( u \) solves the equation, then \( u \) is a minimizer. \( \square \)

Above we have used the following statement for continuous functions. It is worth recording that this works also for locally integrable functions.

**Lemma 2.44** (Fundamental Lemma of the Calculus of Variations). Let \( \Omega \subset \mathbb{R}^n \) be any open set and assume \( f \in L^1_{\text{loc}}(\Omega) \), i.e. for any \( \Omega' \subset \subset \Omega \) we have

\[
\int_{\Omega'} |f| < \infty.
\]

(1) If

\[
\int_{\Omega} f(x) \varphi(x) \geq 0 \quad \text{for all } \varphi \in C^\infty_c(\Omega) \text{ that are nonnegative, } \varphi \geq 0,
\]
then
\[ f \geq 0 \text{ almost everywhere in } \Omega. \]

(2) If
\[ \int_{\Omega} f(x) \varphi(x) = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega) \text{ that are nonnegative}, \varphi \geq 0, \]
then
\[ f \equiv 0 \text{ almost everywhere in } \Omega. \]

The proof is left as an exercise, it is a combination of convolution arguments as in Theorem 2.40 and the argument used for Proposition 2.39.

**Theorem 2.45 (Uniqueness).** Assume \( f \in C^0(\overline{\Omega}) \cap L^1(\Omega) \)

Denote the class of permissible functions
\[ X := \{ u \in C^2(\overline{\Omega}), \ u = 0 \text{ on } \partial\Omega \} \]

Then there is at most one solution \( u \in X \) to (2.25)

**Proof.** Assume \( u, w \in X \) are two solutions, then
\[ \Delta(u - w) = 0. \]

Multiplying by \( u - w \) and integrating by parts (observe that there are no boundary terms since \( u = w \) on \( \partial\Omega \), we obtain
\[ \int_{\Omega} |\nabla(u - w)|^2 = 0. \]

But this implies \( \nabla(u - w) \equiv 0 \), so \( u - w \equiv \text{const} \). Since \( u = w \) on the boundary that constant is zero, and \( u \equiv w \).

**Exercise 2.46.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set with smooth boundary. Assume \( f \in C^0(\overline{\Omega}) \) and \( A \in C^2(\overline{\Omega}, \mathbb{R}^{n \times n}) \), \( A \) symmetric, and all eigenvalues strictly positive in \( \overline{\Omega} \), and let \( c \in C^0(\overline{\Omega}) \).

Denote the class of permissible functions
\[ X := \{ u \in C^2(\overline{\Omega}), \ u = 0 \text{ on } \partial\Omega \} \]

and define the energy
\[ E(u) := \int_{\Omega} \frac{1}{2} \langle ADu, Du \rangle_{\mathbb{R}^n} + \frac{1}{2} \int c|u|^2 + fu. \]

Let \( u \in X \) be a minimizer of \( E \) in \( X \), i.e.
\[ E(u) \leq E(v) \quad \forall v \in X. \]

Then \( u \) solves
\[ \begin{cases} \text{div} (A\nabla u) - cu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \]
Conversely, if $u \in X$ solves (2.27), then $u$ is a minimizer of $\mathcal{E}$ in the set $Y$.

These methods can be extended, e.g. for higher order differential equations (where no maximum principle holds), e.g. the Neumann boundary problem. Let $\nu : \partial \Omega \to \mathbb{R}^n$ be the outwards facing unit normal. The Neumann problem is the equation

\begin{equation}
\begin{cases}
\Delta u = f & \text{in } \Omega \\
\partial_{\nu} u = 0 & \text{on } \partial \Omega
\end{cases}
\end{equation}

Exercise 2.47. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Assume $f \in C^0(\overline{\Omega})$.

Denote the class of permissible functions

$$Y := \{ u \in C^2(\overline{\Omega}) \}$$

and define the energy

$$\mathcal{E}(u) := \int_{\Omega} \frac{1}{2} |Du|^2 + fu.$$ 

Let $u \in Y$ be a minimizer of $\mathcal{E}$ in $Y$, i.e.

$$\mathcal{E}(u) \leq \mathcal{E}(v) \quad \forall v \in Y.$$ 

Then $u$ solves (2.28).

Conversely, if $u \in Y$ solves (2.25), then $u$ is a minimizer of $\mathcal{E}$ in the set $Y$.

Exercise 2.48 (Uniqueness modulo constants). Let $\Omega \subset \mathbb{R}^n$ be a connected, bounded, open set with smooth boundary. Assume $f \in C^0(\overline{\Omega})$. Assume $f \in C^0(\overline{\Omega})$.

Denote the class of permissible functions

$$Y := \{ u \in C^2(\overline{\Omega}) \}$$

Then any two solutions $u, v \in Y$ to (2.28) must satisfy $u - v \equiv \text{constant}$

2.11. Linear Elliptic equations. From now on we often use the Einstein summation convention, often described as “summing over repeated indices”. We write

$$a_{ij} \partial_{ij} u \iff \sum_{i,j} a_{ij} \partial_{ij} u.$$ 

$$b_i \partial_{i} u \iff \sum_{i} b_i \partial_{i} u.$$ 

bul

$$b_i \partial_{j} u \not\iff \sum_{i,j} b_i \partial_{j} u.$$ 

In particular

$$\Delta u \iff \partial_{ii} u.$$
Second order elliptic equations are a class of equations that in some sense are \textit{governed} by the Laplacian operator.

\textbf{Definition 2.49} (Linear elliptic equations). \hspace{0.5em} (1) ("non-divergence form") linear second order operators are defined to be operators of the form 
\[ L := a_{ij} \partial_{ij} + b_i \partial_i + c \]
for coefficients \( a_{ij}, b_i, c : \Omega \to \mathbb{R} \). They act as follows on functions \( u \in C^2(\Omega) \)
\[ Lu(x) := a_{ij}(x) \partial_{ij}u(x) + b_i(x) \partial_iu(x) + c(x) u(x). \]

\( L \) is called a constant coefficient operator, if the coefficients \( a_{ij}, b_i \) and \( c \) are all constant.

(2) ("\textit{divergence form}") linear second order operators are defined to be operators of the form 
\[ L := \partial_i (a_{ij} \partial_j) + b_i \partial_i + c \]
for coefficients \( a_{ij}, b_i, c : \Omega \to \mathbb{R} \). They act as follows on functions \( u \in C^2(\Omega) \)
\[ Lu(x) := \partial_i (a_{ij}(x) \partial_ju(x)) + b_i(x) \partial_iu(x) + c(x) u(x). \]

(3) Clearly, divergence on non-divergence form are very similar if \( a_{ij} \) is smooth enough, but they are different if \( a \) is not smooth (or, has happens often in applications: \( a \) depends on \( u \)).

(4) (divergence-form or non-divergence form) operators \( L \) are called \textit{elliptic} (also often called uniformly elliptic and bounded) if there exists an \textit{ellipticity constants} \( \Lambda > 0 \) such that
\[ \xi^T A \xi \equiv \xi^i a_{ij} \xi^j \geq \frac{1}{\Lambda} \]
and
\[ \sup_{\Omega} |a_{ij}|, |b_i|, |c| < \infty. \]

For simplicity, although this is not strictly necessary we will below always assume \( A \) is symmetric.

\textbf{Example 2.50.} \hspace{0.5em} • The operator \( \Delta \) is clearly elliptic in the above sense, with
\[ a_{ij} = \delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases} \]

• Operators like \( \text{div} (|\nabla u|^{p-2}\nabla u) \) are not (uniformly elliptic), since \( |\nabla u| = 0 \) cannot be excluded. These operators are called degenerate elliptic.

\textbf{Definition 2.51.} \( u \in C^2(\Omega) \) is called a \textit{subsolution} of \(-Lu = f\) for an elliptic operator \( L \), if
\[ -Lu \leq 0 \quad \text{in } \Omega \]
and a \textit{supersolution} if
\[ -Lu \geq 0 \quad \text{in } \Omega. \]

\( u \in C^2(\Omega) \) is called a solution if it is both sub- and supersolution.
In the following we will restrict ourselves to elliptic non-divergence operators!

2.12. **Maximum principles for linear elliptic equations.** The first result is a generalization of the weak maximum principle for $\Delta$, Corollary 2.18.

**Theorem 2.52** (Weak maximum principle for $c = 0$). Let $\Omega \subset \subset \mathbb{R}^n$, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be an $L$-subsolution, i.e.

$$-Lu \leq 0 \text{ in } \Omega$$

If $L$ is (non-divergence form) linear elliptic operator with $c \equiv 0$, then

$$\sup_{\Omega} u = \sup_{\partial \Omega} u.$$

If instead of (2.29) we have

$$-Lu \geq 0 \text{ in } \Omega$$

then

$$\inf_{\Omega} u = \inf_{\partial \Omega} u.$$

**Proof.** First we assume instead of (2.29)

$$-Lu > 0 \text{ in } \Omega$$

Clearly, by continuity of $u$ in $\overline{\Omega}$,

$$\sup_{\Omega} u \geq \sup_{\partial \Omega} u$$

If we had

$$\sup_{\Omega} u > \sup_{\partial \Omega} u,$$

then we would find the global (and thus a local) maximum $x_0 \in \Omega$, at which we have $Du(x_0) = 0$ and $D^2u(x_0) \leq 0$. But this implies (recall $c \equiv 0$)

$$Lu(x_0) = a_{ij}(x_0) \partial_{ij} u(x_0) + b_i(x_0) \partial_i u(x_0)$$

Since $a_{ij}(x_0)$ is elliptic, and $\partial_{ij} u(x_0) \geq 0$ we have

$$a_{ij}(x_0) \partial_{ij} u(x_0) \geq 0.$$

(This is a general Linear Algebra fact, if $A, B$ are symmetric, nonnegative matrices, then their Hilbert-Schmidt Scalar product $A : B := a_{ij}b_{ij} \geq 0$, Exercise 2.54.) That is, we have

$$Lu(x_0) \geq 0$$

which is a contradiction to (2.30).

We conclude that under the assumption (2.30) we have

$$\sup_{\Omega} u = \sup_{\partial \Omega} u.$$
In order to weaken the assumption to (2.30) we consider, for some \( \gamma > 0 \), \( v_\gamma(x) := e^{\gamma x_1} \), where \( x_1 \) is the first component of \( x = (x_1, \ldots, x_n) \). Observe that
\[
Lv_\gamma(x) = \left(a_{11}(x)\gamma^2 + b_1(x)\gamma\right) e^{\gamma x_1}
\]
Since \( L \) is elliptic we have \( a_{11} \geq \frac{1}{\Lambda} \) and \( b_1 \geq -\Lambda \), so
\[
Lv_\gamma(x) = a_{11}(x)\gamma^2 + b_1(x)\gamma \geq e^{\gamma x_1} \left(\frac{1}{\Lambda} \gamma - \Lambda\right).
\]
If we choose \( \gamma = 3\Lambda \) we thus find
\[
Lv_\gamma(x) > 0 \quad \text{in} \, \Omega.
\]
Consequently, under the assumption (2.29) we have for any \( \varepsilon > 0 \), for \( w_\varepsilon := u + \varepsilon v_\gamma \),
\[
Lw_\varepsilon(x) > 0 \quad \text{in} \, \Omega.
\]
and thus by the first step
\[
\sup_\Omega w_\varepsilon = \sup_{\partial \Omega} w_\varepsilon
\]
Since \( w_\varepsilon = u + \varepsilon v_\gamma \) and \( v_\gamma \) is continuous (and \( \Omega \) is bounded) we have
\[
\left| \sup_\Omega u - \sup_{\partial \Omega} u \right| \leq C(\Omega)\varepsilon.
\]
Letting \( \varepsilon \to 0 \) we obtain the claim.

The inf claim follows by taking \(-u\) instead of \( u\).

**Exercise 2.53.** Let \( A \in \mathbb{R}^{n \times n} \) be a symmetric matrices, i.e. \( A^t = A \). Show that the following two conditions are equivalent

1. \( A \geq 0 \) in the sense of matrices, i.e.
   \[
   \xi^t A \xi \geq 0
   \]
2. all eigenvalues of \( A \) are nonnegative.

**Exercise 2.54.** Let \( A, B \in \mathbb{R}^{n \times n} \) be two symmetric matrices, i.e. \( A^t = A \), \( B^t = B \). Assume that \( A, B \geq 0 \) in the sense of matrices, i.e.
\[
\xi^t A \xi \geq 0, \quad \xi^t B \xi \geq 0 \quad \forall \xi \in \mathbb{R}^n
\]
Show that
\[
A : B \equiv \sum_{i=1}^n \sum_{j=1}^n A_{ij} B_{ij} \geq 0.
\]
Also in the case \( c \neq 0 \) a type of weak maximum principle holds (essentially mimicking the above argument):
Theorem 2.55 (Weak maximum principle for $c \leq 0$). Let $\Omega \subset \subset \mathbb{R}^n$, and consider

$$L := a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x).$$

where $c \leq 0$ in $\Omega$.

Assume $u \in C^2(\Omega) \cap C^0(\Omega)$.

(1) If $u$ solves

$$-Lu \leq 0 \quad \text{in } \Omega$$

Then

$$\sup_{\Omega} u \leq \sup_{\partial \Omega} u_+,$$

where $u_+$ denotes the positive part of $u$, namely

$$u_+ = \max\{0, u\}.$$

(2) If on the other hand $u$ solves

$$-Lu \geq 0 \quad \text{in } \Omega$$

we have

$$\inf_{\Omega} u \geq \inf_{\partial \Omega} (-u_-),$$

where $u_-$ denotes the positive part of $u$, namely

$$u_+ = \max\{0, u\}, \quad u_- = -\min\{0, u\}.$$

(3) In particular, if $Lu = 0$ then

$$\sup_{\Omega} |u| \leq \sup_{\partial \Omega} |u|$$

Proof. Let us assume $-Lu \leq 0$. First we observe that if

$$\sup_{\Omega} u \leq 0$$

then there is nothing to show, since we have $u_+ \geq 0$ by definition and thus

$$\sup_{\Omega} u \leq 0 \leq \sup_{\partial \Omega} u_+.$$

So w.l.o.g. we may assume that $\sup_{\Omega} u > 0$. Set

$$\Omega_+ := \{x \in \Omega : u(x) > 0\} \neq \emptyset.$$

Since $u$ is continuous $\Omega_+ = u^{-1}((0, \infty))$ is a nonempty, open set.

Define the elliptic operator $L_0$ by

$$L_0 u := Lu - cu = a_{ij}\partial_{ij}u + b_i\partial_iu.$$

Since $-Lu \leq 0$ we have $-L_0u \leq cu \leq 0$ in $\Omega_+$. — since by assumption $c \leq 0$. So, using the weak maximum principle for $c \equiv 0$, Theorem 2.52,

$$\sup_{\Omega} u \leq \sup_{\Omega_+} u_+ \overset{\text{2.52}}{=} \sup_{\partial \Omega_+} u_+ \leq \sup_{\partial \Omega} u_+.$$
In the last step we used that $\partial\Omega_+ \subset \overline{\Omega}$ can be split into two parts: the part $\partial\Omega_+ \subset \Omega$ (on this part we have $u = u_+ = 0$), and the part $\partial\Omega_+ \subset \partial\Omega$ where $u_+ \geq 0$.

This settles the claim for $-Lu \leq 0$.

If we assume $-Lu \geq 0$ then $-u$ satisfies $-L(-u) \geq 0$, and we obtain the claim from the previous case

$$-\inf_{\Omega} u = \sup_{\Omega} (-u) \leq \sup_{\partial\Omega^+} (-u) = \sup_{\partial\Omega^-} u = -\inf_{\partial\Omega^-} (-u)$$

so

$$\inf_{\Omega} u \geq \inf_{\partial\Omega^-} (-u).$$

For the last case assume that $-Lu = 0$. By the arguments before we have then (observe that $|u| = u_+ + u_-$).

$$\sup_{\Omega} u \leq \sup_{\partial\Omega^+} u \leq \sup_{\partial\Omega^-} |u|.$$  

and

$$\inf_{\Omega} u \geq \inf_{\partial\Omega^-} (-u),$$

which can be rewritten as

$$-\inf_{\Omega} u \leq -\inf_{\partial\Omega^-} (-u) = \sup_{\partial\Omega^-} u \leq \sup_{\partial\Omega^-} |u|.$$  

Now at least one of the following cases holds:

$$\sup_{\Omega} |u| = \sup_{\Omega} u, \text{ or } \sup_{\Omega} |u| = -\inf_{\Omega} u$$

but in both cases the estimates above imply

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega^-} |u|.$$  

\[\square\]

**Exercise 2.56** (Counterexample for $c \geq 0$). Consider

$$Lu = \Delta u + 5u$$

for $\Omega = (-1, 1) \times (-1, 1)$. Take

$$u = (1 - x^2) + (1 - y^2) + 1$$

Show that

1. $-Lu \leq 0$ in $\Omega$
2. $\sup_{\Omega} u \geq u(0) = 3$
3. $\sup_{\partial\Omega} u = 1$
4. Why is this no contradiction to Theorem 2.55?
Corollary 2.57 (Eigenvalues of $\Delta$). $\Delta$ with Dirichlet-boundary has no nonnegative eigenvalues. Namely there is no nontrivial solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ for $\lambda \geq 0$ to
\[
\begin{cases}
\Delta u = \lambda u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
(Here, nontrivial means $u \not\equiv 0$).

Proof. The above equation is for $L := \Delta - \lambda$ equivalent to
\[
\begin{cases}
-Lu = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
Since $\lambda \geq 0$, Theorem 2.55 is applicable, so for any solution to the above equation we’d have
\[
\sup_{\Omega} |u| \leq \sup_{\partial \Omega} |u| = 0.
\]
Thus $u \equiv 0$, i.e. $u$ is the trivial solution. \qed

As it was the case for the $\Delta$-operator, Theorem 2.22, the weak maximum principle implies uniqueness results.

Corollary 2.58 (Uniqueness for the Dirichlet problem). Let $L$ be as above a non-divergence form linear elliptic operator, $\Omega \subset \subset \mathbb{R}^n$ with smooth boundary, $c \leq 0$, $f \in C^0(\Omega)$, $g \in C^0(\partial \Omega)$. Then there exists at most one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of the Dirichlet boundary problem
\[
\begin{cases}
Lu = f & \text{in } \Omega \\
u = g & \text{on } \partial \Omega
\end{cases}
\]

Exercise 2.59. Prove Corollary 2.58.

Corollary 2.60 (Comparison principle). Let $L$ be a linear elliptic differential operator (non-divergence form), and assume that $c \leq 0$ in $\Omega \subset \subset \mathbb{R}^n$. Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy $-Lu \leq -Lv$ in $\Omega$. Then $u \leq v$ on $\partial \Omega$ implies $u \leq v$ in $\Omega$.

Exercise 2.61. Prove Corollary 2.60.

Corollary 2.62 (Continuous dependence on data). Let $L$ be a linear elliptic differential operator (non-divergence form), and assume that $c \leq 0$ in $\Omega \subset \subset \mathbb{R}^n$.

Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy
\[
\begin{cases}
-Lu = f & \text{in } \Omega \\
u = g & \text{on } \partial \Omega
\end{cases}
\]
where $f \in C^0(\overline{\Omega})$ and $g \in C^0(\partial \Omega)$. 
Then for some constant $C = C(a,b,c,\Omega)$ we have

$$\sup_{\Omega} |u| \leq C \left( \sup_{\partial \Omega} |g| + \sup_{\Omega} |f| \right).$$

**Exercise 2.63.** Prove Corollary 2.62.

**Hint:** Set $v_\lambda := u + \lambda e^{\mu |x-x_0|^2} \sup_{\Omega} |f|$ where $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$. Choose $\mu \gg 1$. Then choose $\lambda$ so that $Lv_\lambda \leq 0$ and use the weak maximum principle. Then choose $\lambda$ so that $Lv_\lambda \geq 0$, and again use the weak maximum principle.

Our next goal is the the strong maximum principle, for this we use the following result by Hopf:

**Lemma 2.64** (Hopf Boundary point Lemma). Let $B \subset \mathbb{R}^n$ be a ball, and let $L$ be as above. Let $u \in C^2(B) \cap C^0(\overline{B})$ and assume that for $x_0 \in \partial B$ we have

- $u(x) < u(x_0)$ for all $x \in B$
- $-Lu \leq 0$ in $B$
- One of the following
  - (1) $c \equiv 0$
  - (2) $c \leq 0$ and $u(x_0) \geq 0$
  - (3) $u(x_0) = 0$

Then for $\nu$ the outwards facing normal of $B$ at $x_0$ (i.e. if $B = B(y_0, \rho)$ then for $\nu = \frac{y_0 - x_0}{\rho}$

$$\partial_\nu u(x_0) > 0,$$

if that derivative exists.
An illustration of the setup of Lemma 2.64 is in Figure 2.4. Observe that \( \partial_u u(x_0) \geq 0 \) is clear, the Hopf-Lemma says this must be a strict inequality!

**Proof.** W.l.o.g. we may assume

\begin{equation}
B = B(0,R), \quad c \leq 0, \quad u(x_0) = 0, \quad u < 0 \quad \text{in} \quad B(0,R) : \tag{2.31}
\end{equation}

Indeed, the condition \( B = B(0,R) \) can be assumed simply by shifting. As for the other conditions set (recall that \( c_+ = \max\{c, 0\} \))

\[ \tilde{L} := L - c_+ . \]

and

\[ \tilde{u} := u - u(x_0) . \]

Then in \( B \),

\[ -\tilde{L}\tilde{u} = -(L - c_+)(u - u(x_0)) = -Lu + c_+ u + cu(x_0) - c_+ u(x_0) \leq c_+(u - u(x_0)) + cu(x_0) \]

If \( c \equiv 0 \) then we readily have \( -\tilde{L}\tilde{u} \leq 0 \).

If \( c \leq 0 \) we have \( c_+ \equiv 0 \), and again obtain \( -\tilde{L}\tilde{u} \leq 0 \).

If \( u(x_0) = 0 \) then \( c_+ u \leq 0 \), since \( u \leq u(x_0) = 0 \) by assumption.

Since \( c - c_+ \leq 0 \) we observe that \( \tilde{L} \) is an operator that satisfies the missing conditions in (2.31). Thus, indeed, (2.31) can be assumed w.l.o.g.

So assume (2.31) from now on.

Set for some \( \alpha > 0 \)

\[ v_\alpha(x) := e^{-\alpha|x|^2} - e^{-\alpha R^2} . \]

Clearly \( 0 \leq v_\alpha \leq 1 \) in \( B = B(0,R) \). Moreover

\[ v_\alpha \equiv 0 \quad \text{on} \quad \partial B(0,R) . \]

For \( \rho \in (0,R) \) denote by \( A(\rho,R) \) the annulus \( B(0,R) \setminus B(0,\rho) \). We will show next

\begin{equation}
\text{For any } \rho \in (0,R) \text{ there exists } \alpha > 0 \text{ such that } -Lv_\alpha < 0 \quad \text{in} \quad A(\rho,R) : \tag{2.32}
\end{equation}

For this we first compute

\begin{equation}
\partial_i v_\alpha(x) = -2\alpha x_i e^{-\alpha|x|^2} . \tag{2.33}
\end{equation}

Next we compute

\[ \partial_{ij} v_\alpha(x) = \left( -2\alpha \delta_{ij} + 4\alpha^2 x_i x_j \right) e^{-\alpha|x|^2} \]

so (using the ellipticity conditions, \( a_{ij} x_i x_j \geq \lambda |x|^2 \), and \( |a|, |b|, |c| \leq \Lambda \),

\[ -Lv(x) = -a_{ij} \partial_{ij} v - b_i \partial_i v - cv \]

\[ = -a_{ij} \left( -2\alpha \delta_{ij} + 4\alpha^2 x_i x_j \right) e^{-\alpha|x|^2} - b_i \left( -2\alpha x_i e^{-\alpha|x|^2} \right) - ce^{-\alpha|x|^2} + \underbrace{ce^{-\alpha R^2}}_{\leq 0} \]

\[ \leq \left( 2\alpha \Lambda - 4\alpha^2 \lambda |x|^2 + 2\alpha \Lambda |x| + \Lambda \right) e^{-\alpha|x|^2} . \]
That is, for \( x \in A(\rho, R) \),
\[
-Lv(x) \leq \left( -4\alpha^2 \rho^2 + 2\alpha \Lambda + 2\alpha \Lambda R + \Lambda \right) e^{-\alpha |x|^2} \leq 0 \text{ for } \alpha \gg 1
\]
If we take \( \alpha \) large, the (negative) \( \alpha^2 \)-term dominates, that is for \( \alpha \gg 1 \) (depending on \( \rho > 0, \Lambda, \lambda \) and \( R \)) we have (2.32).

Next, we consider the equation for \( u + \varepsilon v \), which in view of (2.32) becomes
\[
-L(u + \varepsilon v) < 0 \text{ in } A(\rho, R).
\]

The weak maximum principle, Theorem 2.55, implies
\[
(2.34) \quad \sup_{A(\rho, R)} u + \varepsilon v \leq \sup_{\partial A(\rho, R)} (u + \varepsilon v)_+.
\]

The boundary \( \partial A(\rho, R) \) is the union of \( \partial B(0, R) \) and \( \partial B(0, \rho) \).

On \( \partial B(0, R) \) we know \( v \equiv 0 \) and since \( u \) is continuous and \( u < 0 \) in \( B(0, R) \) we have \( u \leq 0 \) on \( \partial B(0, R) \). That is \( (u + \varepsilon v)_+ = 0 \) on \( \partial B(0, R) \).

On \( \partial B(0, \rho) \), since \( u < 0 \) on \( B(0, R) \) we have \( \sup_{\partial B(0, \rho)} u < 0 \), and consequently, since \( v \leq 1 \) we have for all \( 0 < \varepsilon < \varepsilon_0 := -\sup_{\partial B(0, \rho)} u \)
\[
u \quad u + \varepsilon v < 0 \text{ on } \partial B(0, \rho)
\]
That is (2.34) implies
\[
(2.35) \quad u + \varepsilon v \leq 0 \text{ in } A(\rho, R).
\]

Now fix \( \rho \in (0, R) \), choose \( \varepsilon, \alpha \) so that the above is true.

Denote \( \nu := \frac{x_0}{|x_0|} \) the outwards unit normal to \( \partial B \) at \( x_0 \in \partial B \). Observe that for all small \( 0 < t \ll 1 \) (depending on \( \rho \)) we have \( x_0 - tv \in A(\rho, R) \).

Recall that by assumption \( u(x_0) = 0 \), then (2.35) implies for any small \( t > 0 \),
\[
u \quad u(x_0 - tv) + \varepsilon v(x_0 - tv) \leq 0 = u(x_0) + \varepsilon v(x_0).
\]
This leads to (again: for all \( 0 < t \ll 1 \))
\[
\frac{u(x_0 - tv) - u(x_0)}{t} \leq -\varepsilon \frac{v(x_0 - tv) - v(x_0)}{t}
\]
Letting \( t \to 0^+ \) on both sides we obtain
\[
(2.36) \quad -\partial_{\nu} u(x_0) \leq \varepsilon \partial_{\nu} v(x_0).
\]

Observe that (2.33) implies
\[
\partial_{\nu} v(x_0) = \partial_i v(x_0) \frac{(x_0)_i}{R} = -2\alpha \frac{|x_0|^2}{R} e^{-\alpha R^2} < 0
\]
That is (2.36) implies
\[
-\partial_{\nu} u(x_0) < 0
\]
which implies the claim. □

The Hopf Lemma, Lemma 2.64 implies the strong maximum principle.

**Corollary 2.65** (Strong maximum principle). Let $\Omega \subset \mathbb{R}^n$ be an open and connected set, (but $\Omega$ may be unbounded). Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$-Lu \leq 0 \quad \text{in } \Omega.$$

Then we have the following: If there exists $x_0 \in \Omega$ such that

$$u(x_0) = \sup_{\Omega} u$$

then $u \equiv u(x_0)$ in $\Omega$.

**Proof.** Assume the claim is false. Via the modification as in the proof of Lemma 2.64, we may assume w.l.o.g. $u \leq 0$ in $\Omega$ and $u(x_0) = 0$ for some $x_0 \in \Omega$, but $u \not\equiv 0$.

Let

$$\Omega_- := \{x \in \Omega : u(x) < 0\}.$$

Observe that $\Omega_- \text{ is open } (u \text{ is continuous)}$ and $\Omega_- \neq \emptyset$ (because $u \leq 0$ and $u \not\equiv 0$).

Since $x_0 \in \Omega$ and $u(x_0) = 0$, the boundary of $\Omega_-$ cannot be contained in $\partial\Omega$, i.e. we have

$$\partial\Omega_- \cap \Omega \neq \emptyset.$$

Indeed, this follows from connectedness: Let $\gamma \subset \Omega$ be a continuous path from $x_0$ to a point in $\Omega_-$. Then there has to be a point on $\gamma$ where $\gamma$ leaves $\Omega_-$. This point lies in $\partial\Omega_-$ and in $\Omega$.

This means we can find a point $x_1 \in \Omega_-$ which is close to $\partial\Omega_-$ but not close to $\partial\Omega$, i.e.

$$x_1 \in \Omega_- \quad \rho := \text{dist} (x_1, \partial\Omega_-) < 10 \text{dist} (x_1, \partial\Omega).$$

By definition of the distance

$$B(x_1, \rho) \subset \Omega_-, \quad \overline{B(x_1, \rho)} \setminus \Omega_- \neq \emptyset.$$

Let $x_2 \in \partial B(x_1, \rho) \setminus \Omega_-$. Since by construction $x_2 \in \partial\Omega_- \cap \Omega$ we have $u(x_2) = 0$ by continuity. Moreover $u < 0$ in $B(x_1, \rho) \subset \Omega_-.$

Since everything takes place well within $\Omega$, the conditions of the Hopf Lemma, Lemma 2.64, are satisfied and thus for $\nu$ the outwards facing normal at $x_2$ to $\partial B(x_1, \rho)$

$$\partial_\nu u(x_2) > 0.$$
But on the other hand $x_2 \in \Omega$ is a local maximum for $u$, so $Du(x_2) = 0$, which is a contradiction. The claim is then proven. □

A consequence of the Hopf Lemma, Lemma 2.64, and the strong maximum principle, Corollary 2.65, is the uniqueness for the Neumann problem.

**Corollary 2.66** (Uniqueness for Neumann-boundary problem). Let $\Omega \subset \subset \mathbb{R}^n$ be open and connected. Moreover we assume a boundary regularity of $\partial \Omega$, the interior sphere condition\(^6\):

Assume that for any $x_0 \in \partial \Omega$ there exists a ball $B \subset \Omega$ such that $x_0 \in B$.

Then the following holds for any elliptic operator as above with $c \equiv 0$: For any given $f \in C^0(\Omega)$ and any $g \in C^0(\partial \Omega)$ there is at most one solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of the Neumann boundary problem

$$
\begin{cases}
-Lu = f & \text{in } \Omega \\
\partial_\nu u = g & \text{on } \partial \Omega,
\end{cases}
$$

up to constant functions. That means, the difference of two solutions $u, v$ is constant, $u - v \equiv c$.

**Proof.** The difference of two solutions $u, v$, $w := u - v$ satisfies\(^7\)

$$
\begin{cases}
-Lw \leq 0 & \text{in } \Omega \\
\partial_\nu w = 0 & \text{on } \partial \Omega,
\end{cases}
$$

Firstly, assume that there exists $x_0 \in \Omega$ such that $\sup_\Omega w = w(x_0)$. Then, by the strong maximum principle, Corollary 2.65, we have $w \equiv w(x_0)$ and the claim is proven. If this is not the case, then there must be $x_0 \in \partial \Omega$ with $w(x_0) > w(x)$ for all $x \in \Omega$. If we take a ball from the interior sphere condition of $\partial \Omega$ at $x_0$ then on this ball $B$ we can apply Hopf Lemma, Lemma 2.64, which leads to $\partial_\nu w(x_0) > 0$, which is ruled out by the Neumann boundary assumption $\partial_\nu w = 0$. □

3. **Heat equation**

3.1. **Again, sort of a physical motivation.** This is somewhat similar to Section 2.1.

The Laplacian $\Delta u(x)$ describes the difference between the average value of a function around a point $x$ and the value at the point $x$ (cf. the mean value formula)

$$
\Delta u \approx \int_{\partial B(x,r)} u - u(x).
$$

If we think of $u$ as a temperature, then $\Delta u(x) > 0$ means that the material surrounding $x$ is hotter than $u$, and $\Delta u(x) < 0$ means the surroundings are colder than $u$. Heat will flow

\(^6\)This condition does not allow for outwards facing cusps. One can show that every set $\Omega$ whose boundary $\partial \Omega$ is a sufficiently smooth manifold satisfies the interior sphere condition

\(^7\)actually we have $=$ in the equation below, but the argument works for $\leq$ as well
from the hotter areas to the lower areas, and the speed of this propagation is proportional
to the difference in temperature (second law of thermodynamics). That is,
\[ \partial_t u = c \Delta u \]
could describe the change in heat distribution over time (where \( c \) is a material property
like conductivity). So if we solve
\[
\begin{cases}
\partial_t u - \Delta u = 0 & \text{in } \Omega \times (0, T) \\
u(0, \cdot) = u_0 & \text{on } \Omega \\
u(x, t) = g(x, t) & \text{on } \partial \Omega \times (0, T)
\end{cases}
\]
then \( u(x, t) \) describes the heat of the body at time \( t \) at the point \( x \) in the body \( \Omega \), of a
system that started with the heat distribution \( u_0 \) and heat source at \( \partial \Omega \) which is \( g(x, t) \).
The equation is thus called the heat equation, or it is said that \( u \) solves the heat flow.
We can believe that as time passes, there will be less and less change in the energy, so at
\( T = \infty \) maybe we have that \( \partial_t u = 0 \). That is at \( T = \infty \) the solution \( u(\infty, x) \) solves
\[ -\Delta u = 0 \]
that is stationary solutions (could be, this is not always true) appear as \( \lim_{t \to \infty} \) of flows.

3.2. Sort of an optimization motivation. We have discussed in Section 2.10 that we
can solve the equation
\[
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
by minimizing the energy
\[ E(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 - uf \]
among functions with \( u = 0 \) on \( \partial \Omega \) (to make this precise we need Sobolev spaces).
So, in some sense \( \nabla E \) (which we usually write as the variation \( \delta E \) corresponds to \( \Delta u + f \).
(\( \delta E = 0 \) means that we have found a minimizer of this convex functional.

What is the relation to
\[
\begin{cases}
\partial_t u - \Delta u = f & \text{in } \Omega \times (0, T) \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
Well, this is
\[ \partial_t u = -\delta E(u). \]
If \( u \) was a finite dimensional vector, then
\[ \partial_t u = -\nabla E(u) \]
would be that \( u \) follows the steepest gradient descent.
3.3. **Fundamental solution and Representation.** We consider

\[
\begin{align*}
\partial_t u - \Delta u &= f \quad \text{in } \mathbb{R}^{n+1}_+ \\
u(0, \cdot) &= g \quad \text{on } \mathbb{R}^n.
\end{align*}
\]

(3.1)

If \( f = 0 \), then (3.1) is called *homogeneous heat equation*. For \( f \neq 0 \) it is called *inhomogeneous*.

Trivial solutions of the homogeneous equation constant maps \( u(x, t) \equiv c \), or (not completely trivial) time-independent harmonic functions \( u(x, t) := v(x) \) with \( \Delta v = 0 \) (these are called *stationary* solutions).

For elliptic equations we had the notion of a fundamental solution, Section 2.3; There exists a similar concept for the heat equation, the *heat kernel*, which we will (formally) derive now.

If we fix \( x \in \mathbb{R}^n \) and look at (3.1) as an equation in time \( t \) then it looks like an ODE, and naively the solution should be (Duhamel principle!)

\[
u(x, t) = e^{t\Delta} u(x, 0) + \int_0^t e^{(t-s)\Delta} f(x, s) \, ds.
\]

Of course, \( e^{t\Delta} \) does not make any sense for now (it can be defined via *semi-group theory*).

To make (still formally, but more precise) sense of the “ODE argument”, we use the Fourier-transform (with respect to the variables \( x \in \mathbb{R}^n \)):

Let \( u \) be a solution of \( \partial_t u = \Delta u \). Taking the Fourier transform (in \( x \)) on both sides we find

\[
\frac{d}{dt} \hat{u}(\xi, t) = \hat{\partial_t u}(\xi, t) = \hat{\Delta u}(\xi, t)
= -|\xi|^2 \hat{u}(\xi, t).
\]

(There should be a constant \( c \) in front of \(-|\xi|^2\), but we ignore that for now)

Let \( \xi \) be fixed and let

\[
v(t) = \hat{u}(\xi, t).
\]

Then the above reads as

\[
\frac{d}{dt} v(t) = -|\xi|^2 \hat{v}(t).
\]

There is one solution to this ODE (starting from a given value \( v(0) \)):

\[
v(t) = e^{-t|\xi|^2} v(0).
\]

Observe that in particular \( v(\infty) = 0 \), \( \partial_t v(\infty) = 0 \), etc. (i.e. we have strong “decay at infinity”).

Ansatz: \( v(0) = 1 \), resp. \( u(0) = \delta_0 \). This means

\[
\hat{u}(\xi, t) = e^{-t|\xi|^2}.
\]
In this case we have
\[ u(x, t) = \frac{1}{(4\pi t)^\frac{n}{2}} e^{-\frac{|x|^2}{4t}} , \]
which seems to be a special solution.

**Definition 3.1.**
\[
\Phi(x, t) = \begin{cases} 
\frac{1}{(4\pi t)^\frac{n}{2}} e^{-\frac{|x|^2}{4t}}, & t > 0, x \in \mathbb{R}^n \\
0, & t < 0, x \in \mathbb{R}^n.
\end{cases}
\]
is called \textit{fundamental solution} or \textit{heat kernel}.

One has
\[
\partial_t \Phi - \Delta \Phi = 0, \quad \text{for} \quad t > 0
\]
and
\[
\lim_{t \to 0} \Phi(x_0, t) = \begin{cases} 
0, & x_0 \neq 0 \\
\infty, & x_0 = 0.
\end{cases}
\]

**Lemma 3.2.**
\[
\forall t > 0 : \int_{\mathbb{R}^n} \Phi(x, t) \, dx = 1.
\]

**Proof.**
\[
\int_{\mathbb{R}^n} \Phi(x, t) \, dx = \hat{\Phi}(0, t) = 1.
\]

Analogously to the fundamental solution for the Laplace equation, the heat kernel \( \Phi \) generates solutions to the heat equation. Indeed, if we set
\[
u(x, t) := \Phi(\cdot, t) * g(x)
= \int_{\mathbb{R}^n} \Phi(x - y, t)g(y) \, dy
\]
Then
\[
\hat{\nu}(\xi, t) = (\Phi(\cdot, t) * \hat{g})(\xi) = \hat{\Phi}(\xi, t)\hat{g}(\xi).
\]
That is,
\[
\hat{\nu}(\xi, 0) = \hat{g}(\xi), \quad (\frac{d}{dt} + |\xi|^2)\hat{\nu}(\xi, t) = 0.
\]
Revert the Fourier-transformation to obtain
\[
\begin{cases}
(\partial_t - \Delta)u = 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+ \\
u(x, 0) = g(x) \quad x \in \mathbb{R}^n.
\end{cases}
\]
Motivated by this calculation we set
\[
u(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t)g(y) \, dy.
\]
**Theorem 3.3** (Potential representation). Let $g \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Let $u$ as in (3.3). Then $u$ is defined in $\mathbb{R}^n$ and there holds:

(i) $u \in C^\infty(\mathbb{R}^{n+1})$,

(ii) $\partial_t u - \Delta u = 0$ in $\mathbb{R}^{n+1}$ and

(iii) $\forall x_0 \in \mathbb{R}^n: \lim_{(x,t) \to (x_0,0)} u(x,t) = g(x_0)$.

Next we search a potential representation for

$$ (\partial_t - \Delta) u = f \quad \text{in } \mathbb{R}^{n+1} $$

$$ u(\cdot, 0) = 0 \quad \text{on } \mathbb{R}^n. $$

From the argument in the beginning, using the inverse Fourier transform, and Duhamel principle,

$$ u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s) f(y,s) \, dyds. $$

(3.3)

**Theorem 3.4.** Let $f \in C^2(\mathbb{R}^n \times [0, \infty))$ with compact support and let $u$ as in (3.3). Then

(i) $u \in C^2(\mathbb{R}^n \times (0, \infty))$,

(ii) $(\partial_t - \Delta) u = f \quad \text{in } \mathbb{R}^n \times (0, \infty)$

(iii) $\forall x_0 \in \mathbb{R}^n: \lim_{(x,t) \to (x_0,0)} u(x,t) = 0$.

**3.4. Mean-value formula.** (cf. [Evans, 2010, Chapter 2.3])

Use the fundamental solution to construct a parabolic ball, or heat ball

$$ E(x,t; r) \subset \mathbb{R}^{n+1}. $$

**Definition 3.5** (Heat ball). Let $(x,t) \in \mathbb{R}^{n+1}$. Set

$$ E(x,t; r) = \left\{ (y,s) \in \mathbb{R}^{n+1}: s \leq t, \Phi(x - y, t - s) \geq \frac{1}{r^n} \right\}. $$

Cf. Figure 3.1.

**Theorem 3.6** (mean value). Let $X \subset \mathbb{R}^{n+1}$ be open and $u \in C^2(X)$ solve $(\partial_t - \Delta) u = 0$ in $X$. Then there holds

$$ u(x,t) = \frac{1}{4r^n} \int_{E(x,t; r)} u(y,s) \frac{|x - y|^2}{(t - s)^2} \, dyds $$

for all $E(x,t; r) \subset X$. 

(3.6)
3.5. Maximum principle and Uniqueness.

**Definition 3.7.** Let \( \Omega \subset \mathbb{R}^n \) be an open set and denote with \( \Omega_T := \Omega \times (0, T] \) for some time \( T > 0 \). It is important to note that the top \( \Omega \times \{T\} \) belongs to \( \Omega_T \). The *parabolic boundary* \( \Gamma_T \) of \( \Omega_T \) is the boundary of \( \Omega_T \) without the top,
\[
\Gamma_T = \overline{\Omega_T} \setminus \Omega_T = \partial \Omega \times [0, T) \cup \Omega \times \{0\}.
\]

See Figure 3.2.

**Theorem 3.8.** Let \( U \) be bounded and \( u \in C^2(U_T) \cap C^0(\overline{U_T}) \) be a solution of \( \partial_t u = \Delta u \) in \( U_T \). Then there holds

1. the weak maximum principle:
\[
\max_{\overline{U_T}} u = \max_{\Gamma_T} u
\]

2. and the strong maximum principle: If \( U \) is connected and if there is \( (x_0, t_0) \in U_T \) (i.e. \( t_0 \in (0, T] \), \( x \in U \)) with
\[
u(x_0, t_0) = \max_{\overline{U_T}} u,
\]
then \( u \) is constant on all prior times, i.e.
\[
u(x, t) = u(x_0, t_0) \quad \forall (x, t) \in U_{t_0}.
\]

**Exercise 3.9.** Show that the strong maximum principle Theorem 3.8(2) implies the weak maximum principle Theorem 3.8(1).
Proof of Theorem 3.8 (2). Suppose there is \((x_0, t_0) \in U_T\) with
\[
u(x_0, t_0) = M = \max_{U_T} u.\tag{3.10}
\]
Since \(t_0 > 0\), there exists a small heat ball \(E(x_0, t_0, r_0) \subset U_T\) and we have by Theorem 3.6
\[
M = u(x_0, t_0) = \frac{1}{4r_0^n} \int_{E(x_0, t_0, r_0)} u(y, s) \frac{|y - x|^2}{(t - s)^2} \, ds \, dy \leq M. \tag{3.11}
\]
Hence \(u \equiv M\) in \(E(x_0, t_0; r_0)\).

Now we need to show \(u = M\) in all of \(U_{t_0}\). It suffices to show \(u \equiv M\) in any \(U_{t_1}\) for any \(t_1 < t_0\), by continuity \(u \equiv M\) in all of \(U_{t_0}\). So let \((x_1, t_1) \in U_{t_0}\), \(t_1 < t_0\). Then there exists a continuous path \(\gamma: [0, 1] \to U\) connecting \(x_0\) and \(x_1\). In the spacetime set
\[
\Gamma(r) = (\gamma(r), rt_1 + (1-r)t_0). \tag{3.12}
\]
Let
\[
\rho = \max\{r \in [0, 1]: u(\Gamma(r)) = M\}. \tag{3.13}
\]
Show that \(\rho = 1\). Suppose \(\rho < 1\). Then we use the proof above to find a heat ball
\[
E = E(\Gamma(\rho), r'), \tag{3.14}
\]
where \(u = M\). Since \(\Gamma\) crosses \(E\) (time parameter is decreasing along \(\Gamma\)), we obtain a contradiction to the maximality of \(\rho\). \(\square\)

**Exercise 3.10.** Use Theorem 3.8 to show the following infinite speed of propagation:

Assume \(u \in C^2(\Omega_T)\) satisfies
\[
\begin{cases}
\partial_t u - \Delta u = 0 & \text{in } \Omega_T \\
u = 0 & \text{on } \partial \Omega \times [0, T] \\
u = g & \text{in } \Omega \times \{0\}
\end{cases}
\]

(1) Show the following: if there exists any \(x_0 \in \Omega\) such that \(g(x) > 0\) then \(u(x, t) > 0\) in every point in \((x, t) \in \Omega_T\).

(2) Think about how this is a non-relativistic behaviour: any at an arbitrary point influences the whole universe instantaneously.

For general \(X \subset \mathbb{R}^{n+1}\) open we have a similar maximum principle:

**Exercise 3.11.** In Theorem 3.15 we learned of the strong maximum principle in parabolic Cylinders. Use this to obtain the strong maximum principle in general open sets \(X\):

Let \(X \subset \mathbb{R}^{n+1}\) be a bounded, open set. Assume that \(u \in C^\infty(X)\) and
\[
\partial_t u - \Delta u \text{ in } X.
\]
Assume moreover that for some \((x_0, t_0) \in X\) we have
\[
M := u(x_0, t_0) = \sup_{(x, t) \in X} u(x, t).
\]

(1) Describe (in words) in which set \(C\) the function is necessarily constant
\[
C := \{(x, t) \in X : u(x, t) = M\}.
\]

(2) Assume the set \(X\) (grey) and the point \((x_0, t_0)\) are given in the picture. Draw (in orange) the set \(C\) from the question above.

**Theorem 3.12** (Uniqueness on bounded domains). Let \(U \Subset \mathbb{R}^n\) bounded and \(g \in C^0(\Gamma_T)\), \(f \in C^0(U_T)\). Then there is at most one solution \(C^2(U_T) \cap C^0(\overline{U_T})\) to
\[
\begin{align*}
\partial_t u - \Delta u &= f & \text{in } U_T \\
u &= g & \text{on } \Gamma_T.
\end{align*}
\]

**Exercise 3.13.** Prove Theorem 3.12.

**Theorem 3.14.** Let \(u \in C^2(\mathbb{R}^n \times (0, T]) \cap C^0(\mathbb{R}^n \times [0, T])\) be a solution of
\[
(\partial_t - \Delta)u = 0 \quad \text{in } \mathbb{R}^n \times (0, T)
\]
\[
u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}
\]
with the growth condition
\[
u(x, t) \leq Ae^{a|x|^2} \quad \forall (x, t) \in \mathbb{R}^n \times [0, T]
\]
for some \( a, A > 0 \). Then there holds
\begin{equation}
\sup_{\mathbb{R}^n \times [0, T]} u \leq \sup_{\mathbb{R}^n} g.
\end{equation}

**Proof.** It suffices to show this estimate for small times, by splitting up the time interval into many small time steps. For this reason we assume first:
\begin{equation}
4aT < 1.
\end{equation}
For \( \varepsilon > 0 \) and \( \mu \) chosen below, let
\begin{equation}
v(x, t) = u(x, t) - \frac{\mu}{(T + \varepsilon - t)^{\frac{n}{2}}} e^{\frac{|x|^2}{4(T + \varepsilon - t)}}
\end{equation}
for some \( \mu > 0 \). Then \( v_t - \Delta v = 0 \) in \( \mathbb{R}^n \times [0, T] \) (observe that \( t \) appears in the negative above). Theorem 3.8 implies
\begin{equation}
\forall U \subset \subset \mathbb{R}^n: \max_{\overline{U} \cap T} v \leq \max_{\partial U \times [0, T]} v(x, t).
\end{equation}
We have
\begin{equation}
v(x, 0) = g(x) - \frac{\mu}{(T + \varepsilon)^{\frac{n}{2}}} e^{\frac{|x|^2}{4(T + \varepsilon)}} \leq \sup_{\mathbb{R}^n} g.
\end{equation}
Let \( U = B_R(0) \), then
\begin{equation}
\max_{B_R(0) \times [0, T]} v \leq \max \left( \sup_{\mathbb{R}^n} g, \max_{|x|=R, t \in [0, T]} v(x, t) \right).
\end{equation}
For \( |x| = R \) and \( t \in (0, T) \)
\begin{align*}
v(x, t) &= u(x, t) - \frac{\mu}{(T + \varepsilon - t)^{\frac{n}{2}}} e^{\frac{|x|^2}{4(T + \varepsilon - t)}} \\
&\leq A e^{aR^2} - \frac{\mu}{(T + \varepsilon - t)^{\frac{n}{2}}} e^{\frac{R^2}{4(T + \varepsilon - t)}} \\
&\leq A e^{aR^2} - \frac{\mu}{(T + \varepsilon)^{\frac{n}{2}}} e^{\frac{R^2}{4(T + \varepsilon)}}
\end{align*}
Since \( 4aT < 1 \), there exist \( \varepsilon > 0, \gamma > 0 \), such that
\begin{equation}
a + \gamma = \frac{1}{4(T + \varepsilon)}
\end{equation}
and hence
\begin{equation}
v(x, t) \leq A e^{aR^2} - \frac{\mu}{(T + \varepsilon)^{\frac{n}{2}}} e^{aR^2 + \gamma R^2}.
\end{equation}
In particular, the right term dominates for \( R \gg 0 \): in particular for all large \( R > 0 \) we have \( v(x, t) \leq g(0) \). So for large \( R \) and \( |x| = R \) we have for all \( t \in (0, T) \),
\begin{equation}
v(x, t) \leq g(0) \leq \sup_{\mathbb{R}^n} g
\end{equation}
and so
\begin{equation}
\max_{(x,t) \in B_R(0) \times [0,T]} v(x,t) \leq \sup_{\mathbb{R}^n} g \quad \forall R >> 1.
\end{equation}
Letting $R \to \infty$ we find that
\begin{equation}
\sup_{\mathbb{R}^n \times [0,T]} v(x,t) \leq \sup_{\mathbb{R}^n} g,
\end{equation}
i.e.
\begin{equation}
\sup_{\mathbb{R}^n \times [0,T]} \left( u(x,t) - \frac{\mu}{(T + \varepsilon - t)^{\frac{n}{2}}} e^{\frac{|x|^2}{4(T + \varepsilon - t)}} \right) \leq \sup_{\mathbb{R}^n} g
\end{equation}
This holds for any any $\mu > 0$.

Letting $\mu \to 0$ for fixed $x$ gives the claim under the assumption that $4aT < 1$.

If $4aT \geq 1$, we can slice the time interval $(0, T]$ into parts $(0, T_1] \cup (T_1, T_2] \cup \ldots \cup (T_K, T]$ with $4a(T_{i+1} - T_i) < 1$ for all $i$. Using the estimate in each of these time intervals we conclude. \hfill \Box

\textbf{Theorem 3.15.} Let $g \in C^0(\mathbb{R}^n)$, $f \in C^0(\mathbb{R}^n \times [0,T])$. Then there is at most one solution $u \in C^2(\mathbb{R}^n \times (0, T)) \cap C^0(\mathbb{R}^n \times [0,T])$ of
\begin{equation}
(\partial_t - \Delta) u = f \quad \text{in} \quad \mathbb{R}^n \times (0,T)
\end{equation}
\begin{equation}
u(x,0) = 0 \quad \text{for} \quad x \in \mathbb{R}^n.
\end{equation}
with
\begin{equation}
|u(x,t)| \leq Ae^{a|x|^2} \quad \forall (x,t) \in \mathbb{R}^n \times (0,T).
\end{equation}

\textbf{Exercise 3.16.} Prove Theorem 3.15

Without the assumption (3.31), Theorem 3.15 may fail. These solutions are sometimes called \textit{non-physical solutions}, since they grow too fast.

\textbf{Exercise 3.17.} \textit{(cf. [John, 1991])} Define the following Tychonoﬀ-function,
\begin{equation}
u(x,t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.
\end{equation}

Here $g^{(k)}$ denotes the $k$-th derivative of $g$, given as
\begin{equation}g(t) := \begin{cases}
e^{(-t-\alpha)} & t > 0 \\
0 & t \leq 0.
\end{cases}
\end{equation}

(1) Show that $u \in C^2(\mathbb{R}^n) \cap C^0(\mathbb{R} \times [0,\infty))$.

(2) Show moreover that
\begin{equation}\begin{cases}(\partial_t - \Delta) u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,T), \\
u(x,0) = 0 \quad \text{für} \quad x \in \mathbb{R}^n.
\end{cases}
\end{equation}
(3) Find a different solution \( v \not\equiv u \) of (3.32).
(4) Why (without proof) does this not contradict 3.15?

3.6. Harnack’s Principle. In the parabolic setting an “immediate” Harnack principle is not true in general, to compare sup and inf of a function one needs to wait for an (arbitrary short) amount of time.

**Theorem 3.18** (Parabolic Harnack inequality). Assume \( u \in C^2(\mathbb{R}^n \times (0, T]) \cap L^\infty(\mathbb{R}^n \times [0, T]) \) and solves
\[
\partial_t u - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0, T)
\]
and
\[
u \geq 0 \quad \text{in } \mathbb{R}^n \times (0, T)
\]
Then for any compactum \( K \subset \mathbb{R}^n \) and any \( 0 < t_1 < t_2 < T \) there exists a constant \( C \), so that
\[
\sup_{x \in K} u(x, t_1) \leq C \inf_{y \in K} u(y, t_2)
\]

**Proof.** By the representation formula, Section 3.3, and uniqueness of the Cauchy problem
\[
u(x_2, t_2) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t_2)^{\frac{n}{2}}} e^{-\frac{|x_2-y|^2}{4t_2}} u_0(y) \, dy.
\]
Now, for \( t_1 < t_2 \) whenever \( |x_1|, |x_2| \leq \Lambda < \infty \), there exists a constant \( C = C(|t_2 - t_1|, \Lambda) \) so that
\[
-\frac{|x_2 - y|^2}{4t_2} \geq -\frac{|x_1 - y|^2}{4t_1} - C \quad \forall y \in \mathbb{R}^n
\]
See Exercise 3.19.

Consequently,
\[
u(x_2, t_2) \geq \left( \frac{t_1}{t_2} \right)^n e^{-C} \int_{\mathbb{R}^n} \frac{1}{(t_1)^{\frac{n}{2}}} e^{-\frac{|x_1-y|^2}{4t_1}} u_0(y) \, dy = \left( \frac{t_1}{t_2} \right)^n e^{-C} \nu(x_1, t_1).
\]

**Exercise 3.19.** Show the following estimate, which we used for Harnack-principle, Theorem 3.18:

If \( K \subset \mathbb{R}^n \) is compact and \( 0 < t_1 < t_2 < \infty \), then there exists a constant \( C > 0 \) depending on \( K \) and \( (t_2 - t_1) \), such that
\[
\frac{|x_1 - y|^2}{t_2} \leq \frac{|x_2 - y|^2}{t_1} + C \quad \forall x_1, x_2 \in K, \ y \in \mathbb{R}^n.
\]

**Exercise 3.20** (Counterexample Harnack). \( 1 \) Let \( u_0 : \mathbb{R}^n \to [0, \infty) \) a smooth function with compact support such that \( u_0(0) = 1 \). Set
\[
u(x, t) := \int_{\mathbb{R}^n} \Phi(x - y, t) \ u_0(y) \quad t > 0
\]
Show that
\[ \inf_{x \in \mathbb{R}^n} u(x, t) = 0 \quad \text{for all } t > 0. \]
However
\[ \sup_{x \in \mathbb{R}^n} u(x, t) > 0 \quad \text{for all } t > 0. \]
Why does this not contradict Harnack’s principles, Theorem 3.18?

(2) Let \( \xi \in \mathbb{R}^n \) be given and \( u \) defined as
\[ u_{\xi}(x, t) := (t + 1)^{-\frac{3}{2}} e^{-\frac{|x + \xi|^2}{4(t+1)}}. \]
Show that \( u \) is a solution of \( (\partial_t - \Delta)u = 0 \) in \( \mathbb{R}^n \times (0, \infty) \).
Moreover show for each fixed \( t > 0 \) there is no constant \( C = C(t) > 0 \) such that
\[ \sup_{x \in [-1, 1]} u_{\xi}(x, t) \leq C \inf_{y \in [-1, 1]} u_{\xi}(y, t) \quad \forall \xi \in \mathbb{R}^n. \]
Why does this not contradict Harnack’s principles, Theorem 3.18?

Hint: Choose \( x = -\frac{\xi}{|\xi|} \) and \( y = 0 \). What happens if \( |\xi| \to \infty \)?

3.7. Regularity and Cauchy-estimates.

**Theorem 3.21 (Smoothness).** Let \( u \in C^2(U_T) \) satisfy
\[ \partial_t u = \Delta u \quad \text{in } U_T. \]
Then \( u \in C^\infty(\text{int}(U_T)). \)

**Proof.** This is a standard technique to transfer local questions to global situations, using a cut-off function. Let
\[ C(x, t; r) = \{(y, s) : |x - y| \leq r, t - r^2 \leq s \leq t\} \]
and
\[ C_1 = C(x_0, t_0; r), \quad C_2 = C \left( x_0, t_0; \frac{3}{4}r \right), \quad C_3 = C \left( x_0, t_0; \frac{r}{2} \right) \]
for some \( r \) such that \( C_1 \subset U_T \). Choose a cut-off function
\[ \eta \in C^\infty(\mathbb{R}^n \times [0, t_0]) \]
with \( 0 \leq \eta \leq 1, \eta|_{C_2} \equiv 1, \eta \equiv 0 \) around \( \mathbb{R}^n \times [0, t_0] \setminus C_1 \). Suppose first that \( u \) is smooth. Set
\[ v(x, t) = \eta(x, t)u(x, t) \quad \forall (x, t) \in \mathbb{R}^n \times (0, t_0], \]
extended by 0. Then
\[ \partial_t v - \Delta v = \partial_t u \eta + \eta \Delta u - \eta \Delta u - u \Delta \eta - 2 \langle \nabla u, \nabla \eta \rangle \]
\[ = \eta u - u \Delta \eta - 2 \langle \nabla u, \nabla \eta \rangle \]
\[ =: f(x, t) \]
with bounded $v$ and $f \in C^2$ by smoothness of $u$. Let $(x, t) \in C_3$. Then by Theorem 3.4

$$v(x, t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s) f(y, s) \, dy \, ds$$

(3.39)

$$= \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s) \left( u(y, s) \eta_t(y, s) - u(y, s) \Delta \eta(y, s) - 2 \langle \nabla u(y, s), \nabla \eta(y, s) \rangle \right) \, dy \, ds$$

We note: The singularity of $\Phi(x - y, t - s)$ at $y = x$ and $s = t$ is cut off due to $(x, t) \in C_3$.

Hence ($\eta \equiv 1$ around $C_1$)

$$v(x, t) = \int_{C_1(x_0, t_0; r)} \Phi(x - y, t - s) \left( \partial_t - \Delta \right) \eta(y, s) u(y, s) \, dy \, ds$$

(3.40)

By convolution: If $u \in C^2(U_T)$, we have a representation

$$v(x, t) = \int_C K(x, y, s, t) u(y, s) \, dy \, ds$$

with no singularities in the kernel. Thus $v$ is smooth and so is $u$ around $(x_0, t_0)$. □

**Theorem 3.22** (Cauchy estimates). For all $k, l \in \mathbb{N}$ there exists $C > 0$ such that for all $u \in C^{2,1}(U_T)$ ($u \in L^1_{loc}$ will be sufficient), solving

$$(\partial_t - \Delta) u = 0,$$

there holds

$$\max_{C(x_0, t_0; r)} |D_x^k \partial_t^l u| \leq \frac{C}{r^{k+2l+n+2}} \|u\|_{L^1(C(x_0, t_0; r))}$$

(3.43)

for all $C(x_0, t_0; r) \subset U_T$.

**Proof.** Suppose first $(x_0, t_0) = (0, 0)$ and $r = 1$. Set

$$C(1) = C(0, 0; 1).$$

(3.44)

Then as in the proof of Theorem 3.21 we have

$$u(x, t) = \int_{C(1)} K(x, t, y, s) u(y, s) \, dy \, ds \quad \forall (x, t) \in C \left( \frac{1}{2} \right).$$

(3.45)

Then

$$D_x^k \partial_t^l u(x, t) = \int_{C(1)} \left( D_x^k \partial_t^l K(x, t, y, s) \right) u(y, s) \, dy \, ds$$

and hence

$$|D_x^k \partial_t^l u(x, t)| \leq C_{k,l} \|u\|_{L^1(C(1))} \quad \forall (x, t) \in C \left( \frac{1}{2} \right).$$

(3.46)
Thus the claim is proven for $r = 1$. For $r > 0$ and $(x_0, t_0) \in \mathbb{R}^{n+1}$ set

$$v(x, t) = u(x_0 + rx, t_0 + r^2t).$$

Then

$$\max_{C(\frac{1}{2})} |D_x^k \partial_t^l v| \leq C_{k,l}\|v\|_{L^1(C(1))}.$$ 

(3.48)

Hence

$$\max_{C(x_0,r_0;\frac{r}{2})} |D_x^k \partial_t^l u| r^{k+2l} \leq C_{k,l}r^{-(n+2)}\|u\|_{L^1(C(1))}.$$ 

(3.49)

$$\max_{C(x_0,r_0;\frac{r}{2})} |D_x^k \partial_t^l u| r^{k+2l} \leq C_{k,l}r^{-(n+2)}\|u\|_{L^1(C(1))}.$$ 

(3.50)

\[ \square \]

3.8. Variational Methods. Consider

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega \in (0, T) \\ u = g & \text{on } \Omega \times \{0\}, \partial \Omega \times [0, T), \end{cases}$$

and we want to discuss uniuqeness – but for some reason we dont want to use maximum principles.

Assume there is another solution of the same problem, lets call it $v$. Then set $w := u - v$, then that would solve

$$\begin{cases} \partial_t w - \Delta w = 0 & \text{in } \Omega \in (0, T) \\ w = 0 & \text{on } \Omega \times \{0\}, \partial \Omega \times [0, T), \end{cases}$$

As in the Laplace equation case, we multiply this equation by $w$, and we find

$$\partial_t \int_\Omega |w|^2 = -2 \int_\Omega |Dw|^2$$

Observe the right-hand side is negative (unless $w$ is constant, then it is zero – this is called the energy decay). Anyways, integrating this equation we obtain

$$\int_\Omega |w(t)|^2 - \int_\Omega |w(0)|^2 = -2 \int \int |Dw|^2.$$ 

That implies that if $w(0) = 0$ (which it is by assumption), then $w(T) = 0$. That is $w(t) \equiv 0$, i.e. $u = v$.

4. Wave Equation

The wave equation is written as

$$\partial_{tt} u - \Delta u = 0.$$ 

Alternatively we can think of it as

$$\partial_{tt} u = \Delta u.$$
In this form, we can consider it as Newton’s law: Force equals mass times acceleration. The mass is set to 1. If we think about $u(x, t)$ as the dilation of a surface from an equilibrium state (if $x$ is one dimensional, then height of string) then $\Delta u(x, t)$ is proportional to the stress that this dilation exacts on the surface, i.e. the force. By Newton’s law, this force $\Delta u$ is equal to the acceleration $\partial_t u$ – and this is the wave equation.

In one space dimension

$$\partial_{tt} u - \partial_{xx} u = (\partial_t - \partial_x)(\partial_t + \partial_x) u.$$ 

So we could hope by solving the one-dimension wave equation by considering solutions of

$$\partial_t u \pm u_x = 0.$$ 

This is a transport equation which could be solved via the method of characteristics.

In more than one space dimension this is more complicated, because $Du$ is a vector, so

$$\partial_{tt} u - \Delta u = (\partial_t - D)(\partial_t + D)$$

does not really make sense. What would make sense it so

$$\partial_{tt} u - \Delta u = \left(\partial_t - i\sqrt{-\Delta}\right)\left(\partial_t + i\sqrt{-\Delta}\right) u$$

if only we understood $\sqrt{-\Delta}$ (we can e.g. via Fourier transform). This is called the halfwave decomposition.

4.1. **Global Solution via Fourier transform.** We want to consider the wave equation

$$\left\{ (\partial_{tt} - \Delta_x)u = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R} \right.$$ 

If we again take the point of view that this is an ODE in time then this is a second order ODE, so the initial value problem should depend on $u(0)$ and $\partial_t u(0)$.

$$\left\{ \begin{array}{l} (\partial_{tt} - \Delta_x)u = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R} \\
 u(0, x) = u_0(x) \quad \text{in} \quad \mathbb{R}^n \\
 \partial_t u(0, x) = v_0(x) \quad \text{in} \quad \mathbb{R}^n \end{array} \right.$$ 

Let us take the Fourier transform in space, then the above becomes

$$\left\{ \begin{array}{l} \partial_{tt} u(\hat{\xi}, t) + c|\xi|^2 u(\hat{\xi}, t) = 0, \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R} \\
 u(0, \hat{\xi}) = u_0(\hat{\xi}) \quad \text{in} \quad \mathbb{R}^n \\
 \partial_t u(0, \hat{\xi}) = v_0(\hat{\xi}) \quad \text{in} \quad \mathbb{R}^n \end{array} \right.$$ 

This is an equation of the type

$$g''(t) = -cg(t)$$

Fundamental solutions to this equation are $\sin(\sqrt{c}t)$ and $\cos(\sqrt{c}t)$ – which gets messy. It is more convenient to use complex notation: For some $A \in \mathbb{C}$,

$$g(t) = A e^{i\sqrt{c}|\xi|t} + B e^{-i\sqrt{c}|\xi|t}$$
and we must choose $A, B \in \mathbb{C}$ so that

$$\hat{u}_0(\xi) = g(0) = A + B$$

and

$$\hat{v}_0(\xi) = g'(0) = i\sqrt{c} |\xi| (A - B).$$

or equivalently (unless $|\xi| = 0$)

$$\frac{\hat{v}_0(\xi)}{i\sqrt{c} |\xi|} = (A - B).$$

We add the equation for $A + B$ to the equation for $A - B$ and find

$$A = \frac{1}{2} \hat{u}_0(\xi) + \frac{1}{2i\sqrt{c} |\xi|} \hat{v}_0(\xi)$$

and subtracting the equation for $A - B$ from the equation for $A + B$ we have

$$B = \frac{1}{2} \hat{u}_0(\xi) - \frac{1}{2i\sqrt{c} |\xi|} \hat{v}_0(\xi)$$

Together we have found that

$$g(t) = \hat{u}_0(\xi) \left( \frac{1}{2} e^{i\sqrt{c} |\xi| t} + \frac{1}{2} e^{-i\sqrt{c} |\xi| t} \right) + \frac{1}{2i\sqrt{c} |\xi|} \hat{v}_0(\xi) \left( e^{i\sqrt{c} |\xi| t} - e^{-i\sqrt{c} |\xi| t} \right)$$

If we call suggestively

$$e^{it\sqrt{-\Delta}} f := \mathcal{F}^{-1}(e^{it\sqrt{c} |\xi|} \mathcal{F} f)$$

we have the semigroup representation

$$u(x, t) = \frac{e^{it\sqrt{-\Delta}} + e^{-it\sqrt{-\Delta}}}{2} u_0(x) + \frac{e^{it\sqrt{-\Delta}} - e^{-it\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} v_0(x)$$

Duhamel principle If we want to consider

$$\left\{ \begin{array}{l}
(\partial_t - \Delta_x) u = f \\
u(0, x) = u_0(x) \\
\partial_t u(0, x) = v_0(x)
\end{array} \right. \quad \text{in } \mathbb{R}^n \times \mathbb{R}$$

$$u(x, t) = \frac{e^{it\sqrt{-\Delta}} + e^{-it\sqrt{-\Delta}}}{2} u_0(x) + \frac{e^{it\sqrt{-\Delta}} - e^{-it\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} v_0(x)$$

$$+ \int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}} - e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x, s) ds.$$ 

Indeed,

$$\left. \int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}} - e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x, s) ds \right|_{t=0} = 0$$

$$\left. \partial_t \right|_{t=0} \int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}} - e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x, s) ds = 0$$
and

\[
\partial_t \int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}} - e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,s) \, ds
= \partial_t \left( \frac{e^{i0\sqrt{-\Delta}} - e^{-i0\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,t) + \int_0^t \partial_t \frac{e^{i(t-s)\sqrt{-\Delta}} - e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,s) \, ds \right)
= \partial_t \left( \int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}} - e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,s) \, ds \right)
= \partial_t \left( \int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}} - e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,s) \, ds \right)
= e^{i0\sqrt{-\Delta}} \frac{i\sqrt{-\Delta} + i\sqrt{-\Delta} e^{-i0\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,t)
+ \int_0^t \partial_t \frac{e^{i(t-s)\sqrt{-\Delta}} - e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,s) \, ds
= \sqrt{-\Delta} \sqrt{-\Delta}^{-1} f(x,t) + \int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}} - e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,s) \, ds
= f(x,t) - \sqrt{-\Delta} \int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}} - e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,s) \, ds
= f(x,t) \Delta \int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}} - e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,s) \, ds
\]

or, in other words,

\[
(\partial_t - \Delta) \int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}} - e^{-i(t-s)\sqrt{-\Delta}}}{2i} \sqrt{-\Delta}^{-1} f(x,s) \, ds
= f(x,t)
\]

4.2. Energy methods. Cf. [Evans, 2010, 2.4.3].

Consider solutions to the inhomogeneous wave equation.

\[
\begin{cases}
(\partial_t - \Delta)u = f & \text{in } \Omega \times (0,T) \\
u = g & \text{on } \Omega \times \{0\} \cup \partial \Omega \times (0,T) \\
\partial_t u = h & \text{on } \Omega \times \{0\}
\end{cases}
\]

\[
\textbf{Theorem 4.1 (Uniqueness).} \text{ There exist at most one function \( u \in C^2(\overline{\Omega} \times [0,T]) \) which solves (4.1).}
\]
Proof. Assume there are two solutions \( u, v \in C^2(\Omega \times (0, T)) \). Then we can consider \( w := u - v \) which solves
\[
\begin{cases}
(\partial_t - \Delta)w = 0 & \text{in } \Omega \times (0, T) \\
w = 0 & \text{on } \Omega \times \{0\} \cup \partial \Omega \times (0, T) \\
\partial_t w = 0 & \text{on } \Omega \times \{0\}
\end{cases}
\]

For \( t \in [0, T) \) define
\[
E(t) := \frac{1}{2} \int_{\Omega} |\partial_t w(x, t)|^2 \, dx + \int_{\Omega} |Dw(x, t)|^2 \, dx.
\]
We compute the derivative of \( E \) (which we can do since \( w \in C^2 \),
\[
\dot{E}(t) = \int_{\Omega} \partial_t w(x, t) \, \partial_t w(x, t) \, dx + \int_{\Omega} Dw(x, t) D\partial_t w(x, t) \, dx
= \int_{\Omega} \partial_t w(x, t) \, \partial_t w(x, t) \, dx - \int_{\Omega} \text{div} (Dw(x, t)) \partial_t w(x, t) \, dx
= \int_{\Omega} \partial_t w(x, t) \, (\partial_t - \Delta)w(x, t) \, dx
\equiv (4.2) \int_{\Omega} \partial_t w(x, t) \, 0 \, dx = 0.
\]
That is we have \( \dot{E}(t) = 0 \) for all \( t \in (0, T) \)
\[
E(t) = E(0) \quad (4.2) \equiv 0
\]
In particular \( Dw \equiv 0 \), so \( w \) is constant, and because of the boundary conditions in (4.2) we conclude \( w \equiv 0 \). Thus \( u \equiv v \).

\[\square\]

5. **Black Box – Sobolev Spaces**


**Definition 5.1.**

1. Let \( 1 \leq p \leq \infty, k \in \mathbb{N} \) and \( \Omega \subset \mathbb{R}^n \) open, nonempty. The **Sobolev space** \( W^{k,p}(\Omega) \) is the set of functions
\[
u \in L^p(\Omega)
\]
such that for any multiindex \( \gamma, |\gamma| \leq k \) we find a function (the **distributional \( \gamma \)-derivative** or **weak \( \gamma \)-derivative**) \( \partial^\gamma u \in L^p(\Omega) \) such that
\[
\int_{\Omega} u \, \partial^\gamma \varphi = (-1)^{|\gamma|} \int_{\Omega} \partial^\gamma u \, \varphi \quad \forall \varphi \in C_0^\infty(\Omega).
\]
Such \( u \) are also sometimes called Sobolev-functions.

2. For simplicity we write \( W^{0,p} = L^p \).
(3) The norm of the Sobolev space $W^{k,p}(\Omega)$ is given as
$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\gamma| \leq k} \|\partial^\gamma u\|_{L^p(\mathbb{R}^n)}$$
or equivalently (exercise!)
$$\|u\|_{W^{k,p}(\Omega)} = \left( \sum_{|\gamma| \leq k} \|\partial^\gamma u\|_{L^p(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}.$$

(4) We define another Sobolev space $H^{k,p}(\Omega)$ as follows
$$H^{k,p}(\Omega) = C^\infty(\Omega) \cap W^{k,p}(\Omega),$$
that is the (metric) closure or completion of the space $(C^\infty(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$. In yet other words, $H^{k,p}(\Omega)$ consists of such functions $u \in L^p(\Omega)$ such that there exist approximations $u_k \in C^\infty(\Omega)$ with
$$\|u_k - u\|_{W^{k,p}(\Omega)} \xrightarrow{k \to \infty} 0.$$We will later see that $H^{k,p}$ is the same as $W^{k,p}$ locally, or for nice enough domains; and use the notation $H$ or $W$ interchangeably.

(5) Now we introduce the Sobolev space $H^{k,p}_0(\Omega)$
$$H^{k,p}_0(\Omega) = C^\infty(\Omega)^\perp \|\cdot\|_{W^{k,p}(\Omega)}.$$We will later see that this space consists of all maps $u \in H^{k,p}(\Omega)$ that satisfy $u, \nabla u, \ldots, \nabla^{k-1} u \equiv 0$ on $\partial \Omega$ in a suitable sense (the trace sense, for a precise formulation see Theorem 5.21). – Again, later we see that $H = W$ and thus, $W^{k,p}_0(\Omega) = H^{k,p}_0(\Omega)$ for nice sets $\Omega$.

Observe that $L^p(\Omega) = W^{0,p}_0(\Omega) = W^{0,p}_0(\Omega)$.

(6) The local space $W^{k,p}_{loc}(\Omega)$ is similarly defined as $L^p_{loc}(\Omega)$. A map belongs to $u \in W^{k,p}_{loc}(\Omega)$ if for any $\Omega' \subset \subset \Omega$ we have $u \in W^{k,p}(\Omega')$.

Remark 5.2. Some people write $H^{k,p}(\Omega)$ instead of $W^{k,p}(\Omega)$. Other people use $H^k(\Omega)$ for $H^{k,2}$ – notation is inconsistent...

Some people claim that $W$ stand for Weyl, and $H$ for Hardy or Hilbert.

Example 5.3. For $s > 0$ let
$$f(x) := |x|^{-s}.$$Observe that $f$ is only defined for $x \neq 0$, but since measurable functions need only be defined outside of a null-set this is still a reasonable function.

We have already seen, when working with fundamental solutions, that $f \in L^p_{loc}(\mathbb{R}^n)$ for any $1 \leq p < \frac{n}{s}$.

We can compute for $x \neq 0$ that
$$\partial_i f(x) = -s |x|^{-s-2}x^i$$for $i = 1, \ldots, n$. 

(5.1)
and by the same argument as above we could conjecture that \( \partial_i f \in L^q_{\text{loc}}(\mathbb{R}^n) \) for any \( 1 \leq q < \frac{n}{s+1} \).

**Exercise 5.4.** It is an exercise to show that

(1) (5.1) holds in the distributional sense, i.e. that if \( n \geq 2 \) and \( 0 < s < n - 1 \) then for any \( \varphi \in C^\infty_c(\mathbb{R}^n) \),

\[
\int_{\mathbb{R}^n} f(x) \partial_i \varphi(x) \, dx = \int_{\mathbb{R}^n} s |x|^{-s-2} x^i \varphi(x) \, dx.
\]

(2) to conclude that \( f \in W^{1,q}_{\text{loc}}(\mathbb{R}^n) \) for any \( 1 \leq q < \frac{n}{s+1} \).

**Exercise 5.5.** Let

\[
f(x) := \log |x|.
\]

One can show that \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \) for any \( 1 \leq p < \infty \), and \( f \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \) for all \( p \in [1, n) \), if \( n \geq 2 \).

**Exercise 5.6.** Let

\[
f(x) := \log \log \frac{2}{|x|} \quad \text{in } B(0, 1)
\]

One can show that for \( n \geq 2 \), \( f \in W^{1,n}(B(0, 1)) \).

Moreover, for \( n = 2 \), in distributional sense

\[
\Delta f = |Df|^2
\]

Observe that this serves as an example for solutions to nice differential equations that are not continuous!

**Proposition 5.7** (Basic properties of weak derivatives). Let \( u, v \in W^{k,p}(\Omega) \) and \( |\gamma| \leq k \). Then

(1) \( \partial^\gamma u \in W^{k-|\gamma|,p}(\Omega) \).
(2) Moreover \( \partial^\alpha \partial^\beta u = \partial^\beta \partial^\alpha u = \partial^{\alpha + \beta} u \) if \( |\alpha| + |\beta| \leq k \).
(3) For each \( \lambda, \mu \in \mathbb{R} \) we have \( \lambda u + \mu v \in W^{k,p}(\Omega) \) and

\[
\partial^\alpha (\lambda u + \mu v) = \lambda \partial^\alpha u + \mu \partial^\alpha v
\]

(4) If \( \Omega' \subset \Omega \) is open then \( u \in W^{k,p}(\Omega') \)
(5) For any \( \eta \in C^\infty_c(\Omega) \), \( \eta u \in W^{k,p} \) and (if \( k \geq 1 \)), and we have the Leibniz formula (aka product rule)

\[
\partial_i (\eta u) = \partial_i \eta \ u + \eta \partial_i u.
\]

(6) if \( f : \mathbb{R} \to \mathbb{R} \) is Lipschitz and bounded, and \( u \in W^{1,p}(\Omega) \) then \( f(u) \in W^{1,p}(\Omega) \), and

\[
Df(u) = Df(u) \cdot Du
\]
Proposition 5.8. \((W^{k,p}(\Omega), \| \cdot \|_{W^{k,p}(\Omega)}), (H^{k,p}(\Omega), \| \cdot \|_{W^{k,p}(\Omega)}), (H^{k,p}_0(\Omega), \| \cdot \|_{W^{k,p}(\Omega)})\) are all Banach spaces.

For \(p = 2\) they are Hilbert spaces, with inner product
\[
\langle u, v \rangle = \sum_{|\gamma| \leq k} \int \partial^\gamma u \partial^\gamma v.
\]

Theorem 5.9 (Weak compactness). Let \(1 < p < \infty, k \in \mathbb{N}, \Omega \subset \mathbb{R}^n\) open. Assume that \((f_i)_{i \in \mathbb{N}}\) is a bounded sequence in \(W^{k,p}(\Omega)\), that is
\[
\sup_{i \in \mathbb{N}} \| f_i \|_{W^{k,p}(\Omega)} < \infty.
\]
Then there exists a function \(f \in W^{k,p}(\Omega)\) and a subsequence \(f_{i_j}\) such that \(f_{i_j}\) weakly \(W^{k,p}\)-converges to \(f\), that is for any \(|\gamma| \leq k\) and any \(g \in L^p(\Omega)\), where \(p' = \frac{p}{p-1}\) is the Hölder dual of \(p\), we have
\[
\int \partial^\gamma f_{i_j} g \xrightarrow{i \to \infty} \int \partial^\gamma f g.
\]
In particular we have
\[
\| f \|_{W^{k,p}(\Omega)} \leq \sup_i \| f_i \|_{W^{k,p}(\Omega)}.
\]

5.1. Approximation by smooth functions. It is often ok to think of Sobolev maps as (essentially) smooth functions with bounded \(W^{k,p}\)-norm. The reason is approximation:

Proposition 5.10 (Local approximation by smooth functions). Let \(\Omega\) be open, \(u \in W^{k,p}(\Omega), 1 \leq p < \infty\). Set
\[
u_{\varepsilon}(x) := \eta_{\varepsilon} * u(x) = \int_{\mathbb{R}^n} \eta_{\varepsilon}(y-x) u(y) \, dy \quad x \in \Omega_{-\varepsilon}.
\]
Here \(\eta_{\varepsilon}(z) = \varepsilon^{-n} \eta(z/\varepsilon)\) for the usual bump function \(\eta \in C^\infty_c(B(0,1),[0,1]), \int_{B(0,1)} \eta = 1\).

Then
\(1\) \(\nu_{\varepsilon} \in C^\infty(\Omega_{-\varepsilon})\), where as before
\[
\Omega_{-\varepsilon} := \{ x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon \}
\]
for each \(\varepsilon > 0\) such that \(\Omega_{-\varepsilon} \neq \emptyset\).
\(2\) Moreover for any \(\Omega' \subset \subset \Omega\),
\[
\| \nu_{\varepsilon} - u \|_{W^{k,p}(\Omega')} \xrightarrow{\varepsilon \to 0} 0.
\]
We call this \(W^{k,p}_{\text{loc}}\)-approximation.

If we want to approximate \(W^{k,p}(\Omega)\) with functions \(u \in C^\infty(\bar{\Omega})\) we need regularity of \(\Omega\).

Theorem 5.11 (Smooth approximation for Sobolev functions). Let \(\Omega \subset \mathbb{R}^n\) be open and bounded, and \(\partial \Omega \in C^1\). For any \(u \in W^{k,p}(\Omega)\) there exist a smooth approximating sequence \(u_i \in C^\infty(\bar{\Omega})\) such that
\[
\| u_i - u \|_{W^{k,p}(\Omega)} \xrightarrow{i \to \infty} 0.
\]
On $\mathbb{R}^n$ approximation is much easier, indeed we can approximate with respect to the $W^{k,p}$-norm any $u \in W^{k,p}(\mathbb{R}^n)$ by functions $u_k \in C_c^\infty(\mathbb{R}^n)$. That is, $W^{k,p}(\mathbb{R}^n) = W^{k,p}_0(\mathbb{R}^n)$. We could describe this as “$u \in W^{k,p}(\mathbb{R}^n)$ implies that $u$ and $k-1$-derivatives of $u$ all vanish at infinity”.

**Proposition 5.12.** (1) Let $u \in W^{k,p}(\Omega)$, $p \in [1, \infty)$. If $\text{supp} \ u \subset \subset \mathbb{R}^n$ then there exists $u_k \in C_c^\infty(\Omega)$ such that

$$\| u - u_k \|_{W^{k,p}(\Omega)} \xrightarrow{k \to \infty} 0.$$ 

(2) Let $u \in W^{k,p}(\mathbb{R}^n)$, $p \in [1, \infty)$. Then there exists $u_k \in C_c^\infty(\mathbb{R}^n)$ such that

$$\| u - u_k \|_{W^{k,p}(\mathbb{R}^n)} \xrightarrow{k \to \infty} 0.$$ 

(3) Let $u \in W^{k,p}(\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, \infty))$. Then there exists $u \in C_c^\infty(\mathbb{R}^{n-1} \times [0, \infty))$ (i.e. $u$ may not be zero on $(x', 0)$ for small $x'$) such that

$$\| u - u_k \|_{W^{k,p}(\mathbb{R}^n_+)} \xrightarrow{k \to \infty} 0.$$ 

5.2. **Embedding Theorems.**

**Theorem 5.13** (**Rellich-Kondrachov**). Let $\Omega \subset \subset \mathbb{R}^n$, $\partial \Omega \subset C^{0,1}$, $1 \leq p \leq \infty$. Assume that $(u_k)_{k \in \mathbb{N}} \in W^{1,p}(\Omega)$ is bounded, i.e.

$$\sup_{k \in \mathbb{N}} \| u_k \|_{W^{1,p}(\Omega)} < \infty.$$ 

Then there exists a subsequence $k_i \to \infty$ and $u \in L^p(\Omega)$ such that $u_{k_i}$ is (strongly) convergent in $L^p(\Omega)$, moreover the convergence is pointwise a.e.

**Theorem 5.14** (**Poincaré**). Let $\Omega \subset \subset \mathbb{R}^n$ be open and connected, $\partial \Omega \subset C^{0,1}$, $1 \leq p \leq \infty$.

Let $K \subset W^{1,p}(\Omega)$ be a closed (with respect to the $W^{1,p}$-norm) cone with only one constant function $u \equiv 0$. That is, let $K \subset W^{1,p}(\Omega)$ be a closed set such that

(1) $u \in K$ implies $\lambda u \in K$ for any $\lambda \geq 0$.

(2) if $u \in K$ and $u \equiv \text{const}$ then $u \equiv 0$.

Then there exists a constant $C = C(K, \Omega)$ such that

$$\| u \|_{L^p(\Omega)} \leq C \| \nabla u \|_{L^p(\Omega)} \quad \forall u \in K.$$
Corollary 5.15 (Poincaré type lemma). Let \( \Omega \subset \subset \mathbb{R}^n \) be open, connected, and \( \partial \Omega \in C^{0,1} \).

1. There exists \( C = C'(\Omega) \) such that for all \( u \in W^{1,p}(\Omega) \) we have
   \[
   \|u - (u)\|_{L^p(\Omega)} \leq C'(\Omega)\|\nabla u\|_{L^p(\Omega)}
   \]

2. For any \( \Omega' \subset \subset \Omega \) open and nonempty there exists \( C = C(\Omega, \Omega') \) such that for all \( u \in W^{1,p}(\Omega) \) we have
   \[
   \|u - (u)_{\Omega'}\|_{L^p(\Omega)} \leq C(\Omega, \Omega')\|\nabla u\|_{L^p(\Omega)}
   \]

3. There exists \( C = C(\Omega) \) such that for all \( u \in W^{1,p}_0(\Omega) \)
   \[
   \|u\|_{L^p(\Omega)} \leq C(\Omega)\|\nabla u\|_{L^p(\Omega)}
   \]

If \( \Omega = B(x, r) \) (and in the second claim \( \Omega' B(x, \lambda r) \)) then \( C(\Omega) = C(B(0, 1)) r \) (and for the second claim: \( C(\Omega, \Omega') = C(B(0, 1), B(0, \lambda)) r \)).

Theorem 5.16 (Sobolev inequality). Let \( p \in [1, \infty) \) such that \( p^* := \frac{np}{n-p} \in (1, \infty) \) (equivalently: \( p \in (1, n) \)). Then \( W^{1,p}(\mathbb{R}^n) \) embeds into \( L^{p^*}(\mathbb{R}^n) \). That is, for \( u \in W^{1,p}(\mathbb{R}^n) \) then \( u \in L^{p^*}(\mathbb{R}^n) \) and we have\(^8\)

\[
\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C(p, n)\|Du\|_{L^p(\mathbb{R}^n)}.
\]

Corollary 5.17 (Sobolev-Poincaré embedding). Let \( u \in W^{1,p}(\mathbb{R}^n), 1 \leq p < n \). For any \( q \in [p, p^*] \) we have \( u \in L^q(\mathbb{R}^n) \) with the estimate

\[
\|f\|_{L^q(\mathbb{R}^n)} \leq C(q, n) \left( \|f\|_{L^p(\mathbb{R}^n)}^p + \|Df\|_{L^p(\mathbb{R}^n)}^{p^*} \right).
\]

Corollary 5.18 (Sobolev-Poincaré embedding on domains). Let \( \Omega \subset \mathbb{R}^n \) and \( \partial \Omega \) be \( C^1 \). For \( 1 \leq p < n \) we have for any \( u \in W^{1,p}(\Omega) \),

\[
\|u\|_{L^{p^*}(\Omega)} \leq C(\Omega) \left( \|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)} \right)
\]

\(^8\)The optimal constant \( C(p, n) \) has actually a geometric meaning, and is related to the isoperimetric inequality, cf. [Talenti, 1976]
Also, for any \( q \in [p, p^*] \) \(^9\)
\[
\|u\|_{L^q(\Omega)} \leq C(\Omega, q, \|u\|_{W^{1,p}(\Omega)}).
\]

If moreover \( \Omega \subset\subset \mathbb{R}^n \) and \( u \in W^{1,p}_0(\Omega) \) then
\[
\|u\|_{L^{p^*}(\Omega)} \leq C(\Omega) \|Du\|_{L^p(\Omega)}.
\]

Lastly, if \( 1 \leq p < \infty \) and \( \Omega \subset\subset \mathbb{R}^n \), \( u \in W^{1,p}(\Omega) \) then for any \( q \in [1, p^*] \) (if \( p < n \)) or for any \( q \in [1, \infty) \) (if \( p \geq n \))
\[
\|u\|_{L^q(\Omega)} \leq C(\Omega, q, p, n) \|u\|_{W^{1,p}(\Omega)}.
\]

**Theorem 5.19** (Sobolev Embedding). Let \( \Omega \subset\subset \mathbb{R}^n \) be open, \( \partial \Omega \in C^{0,1} \), \( k \geq \ell \) for \( k, \ell \in \mathbb{N} \cup \{0\} \), and \( 1 \leq p, q < \infty \) such that
\[
k - \frac{n}{p} \geq \ell - \frac{n}{q}.
\]

Then the identity is a continuous embedding \( W^{k,p}(\Omega) \hookrightarrow W^{\ell,q}(\Omega) \). That is,
\[
\|u\|_{W^{\ell,q}(\Omega)} \leq C(\|u\|_{W^{k,p}(\Omega)}).
\]

If \( k > \ell \) and we have the strict inequality
\[
k - \frac{n}{p} > \ell - \frac{n}{q},
\]
then the embedding above is compact. That is, whenever \( (u_i)_{i \in \mathbb{N}} \subset W^{k,p}(\Omega) \) such that
\[
\sup_i \|u_i\|_{W^{k,p}(\Omega)} < \infty
\]
then there exists a subsequence \( (u_{i_j})_{j \in \mathbb{N}} \) such that \( (u_{i_j})_{j \in \mathbb{N}} \) is convergent in \( W^{\ell,q}(\Omega) \).

**Theorem 5.20** (Morrey Embedding). Let \( \Omega \subset\subset \mathbb{R}^n \) with \( \partial \Omega \in C^k \), \( k \in \mathbb{N} \). Assume that for \( p \in (1, \infty), \alpha \in (0, 1) \) and \( \ell < k \) we have
\[
k - \frac{n}{p} \geq \ell + \alpha.
\]

Then the embedding \( W^{k,p}(\Omega) \hookrightarrow C^{\ell, \alpha}(\overline{\Omega}) \) is continuous.

If \( k - \frac{n}{p} > \ell + \alpha \) then the embedding is compact.

---

\(^9\)This means the following: For any \( \Lambda > 0 \) there exists a constant \( C(\Omega, q, \Lambda) \) such that
\[
\|u\|_{L^q(\Omega)} \leq C(\Omega, q, \Lambda) \forall u : \|u\|_{W^{1,p}(\Omega)} \leq \Lambda.
5.3. **Trace Theorems.** Let \( \Omega \) be a smoothly bounded domain, i.e. \( \partial \Omega \subset \mathbb{R}^n \) is a smooth (compact) manifold.

For \( s \in (0, 1) \), \( p \in [1, \infty) \) and for \( u \in C^\infty(\partial \Omega) \) we set (one of) the fractional Sobolev space norm, often called Gagliardo-seminorm or Sobolev-Slobodeckij-norm as

\[
[u]_{W^{s,p}(\partial \Omega)} := \left( \int_{\partial \Omega} \int_{\partial \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^s} \frac{dxdy}{|x - y|^{n-1}} \right)^{\frac{1}{p}}.
\]

We say \( u \in W^{s,p}(\partial \Omega) \) if \( u \in L^p(\partial \Omega) \) and \( [u]_{W^{s,p}(\partial \Omega)} < \infty \).

These spaces are sometimes called *trace space*, because of the following property: they describe the trace of \( W^{1,p} \)-functions.

**Theorem 5.21** (Trace theorem). Let \( \Omega \) be open, bounded domain with smooth boundary \( \partial \Omega \) and \( p \in (1, \infty) \). Then

- \( W^{1,p}(\Omega) \hookrightarrow W^{1-\frac{1}{p},p}(\partial \Omega) \) in the following sense. For every \( u \in W^{1,p}(\Omega) \), if we restrict \( u \rvert_{\partial \Omega} \) (in the right way), then

\[
[u]_{W^{1-\frac{1}{p},p}(\partial \Omega)} \lesssim \| \nabla u \|_{L^p(\Omega)}
\]

and

\[
\| u \rvert_{\partial \Omega} \|_{L^p(\partial \Omega)} + [u]_{W^{1-\frac{1}{p},p}(\partial \Omega)} \lesssim \| u \|_{L^p(\Omega)} + \| \nabla u \|_{L^p(\Omega)}.
\]

“In the right way” means that the restriction operator \( T : u \in C^\infty(\overline{\Omega}) \to C^\infty(\partial \Omega) \), \( u \mapsto u \rvert_{\partial \Omega} \) has the above estimates. By density this operator than is defined for any \( u \in W^{1,p}(\Omega) \).

- For any \( u \in W^{1-\frac{1}{p},p}(\partial \Omega) \) there exists \( U \in W^{1,p}(\Omega) \) and

\[
\| \nabla U \|_{L^p(\Omega)} \lesssim_{p,\Omega} [u]_{W^{1-\frac{1}{p},p}(\partial \Omega)}
\]

and

\[
\| U \|_{L^p(\Omega)} + \| \nabla u \|_{L^p(\Omega)} \lesssim \| u \|_{L^p(\Omega)} + [u]_{W^{1-\frac{1}{p},p}(\partial \Omega)}
\].

The statement above holds also for \( p = \infty \) (if we recall that \( W^{1,\infty} \) are simply Lipschitz maps. For \( p = 1 \) there are versions in the spirit of the above trace (observe \( 1 - 1/1 = 0 \))

5.4. **Difference Quotients.** We used above, e.g. for the Cauchy estimates, Proof of Lemma 2.41 the method of differentiating the equation (e.g. that if \( \Delta u = 0 \) then also for \( v := \partial_i u \) we have \( \Delta v = 0 \) – so we can easier estimates for \( \partial_i u \)). In the Sobolev space category this is also a useful technique. Sometimes, the “first assume that everything is smooth, then use mollification”-type argument as for Lemma 2.41 is difficult to put into
practice. In this case, a technique developed by Nirenberg, is discretely differentiating the equation (which does not require the function to be a priori differentiable):

$$\Delta u = 0 \Rightarrow v(x) := (\Delta_h^e u)(x) := \frac{u(x + he_i) - u(x)}{h} : \Delta v = 0$$

For this to work, we need some good estimates. Recall that (by the fundamental theorem of calculus), for $C^1$-functions $u$,

$$\|\Delta_h^e u\|_{L^\infty} \leq \|\partial\ell u\|_{L^\infty}.$$  

This also holds in $L^p$ for $W^{1,p}$-functions $u$, which is a result attributed to Nirenberg, see Proposition 5.23.

One important ingredient is that the fundamental theorem of calculus holds for Sobolev functions:

**Lemma 5.22.** Let $u \in W^{1,1}_\text{loc}(\Omega)$. Fix $v \in \mathbb{R}^n$. Then for almost every $x \in \Omega$ such that the path $[x, x + v] \subset \Omega$ we have

$$u(x + v) - u(x) = \int_0^1 \partial_\alpha u(x + tv) v^\alpha \, dt.$$  

**Proposition 5.23.**

1. Let $k \in \mathbb{N}$, (i.e. $k \neq 0$), and $1 \leq p < \infty$. Assume that $\Omega' \subset \subset \Omega$ are two open (nonempty) sets, and let $0 < |h| < \text{dist}(\Omega', \partial\Omega)$. For $u \in W^{k,p}(\Omega)$ we have

$$\|\Delta_h^e u\|_{W^{k-1,p}(\Omega')} \leq \|\partial\ell u\|_{W^{k-1,p}(\Omega)}.$$  

Moreover we have

$$\|\Delta_h^e u - \partial\ell u\|_{W^{k-1,p}(\Omega')} \xrightarrow{h \to 0} 0.$$

2. Let $u \in W^{k-1,p}(\Omega)$, $1 < p \leq \infty$. Assume that for any $\Omega' \subset \subset \Omega$ and any $\ell = 1, \ldots, n$ there exists a constant $C(\Omega')$ such that

$$\sup_{|h| < \text{dist}(\Omega', \partial\Omega)} \|\Delta_h^e u\|_{W^{k-1,p}(\Omega')} \leq C(\Omega', \ell)$$

Then we $u \in W^{k,p}_\text{loc}(\Omega)$, and for any $\Omega' \subset \subset \Omega$ we have

(5.6) $$\|\partial\ell u\|_{W^{k-1,p}(\Omega')} \leq \sup_{|h| < \text{dist}(\Omega', \partial\Omega)} \|\Delta_h^e u\|_{W^{k-1,p}(\Omega')}.$$  

If $p = \infty$ we even have $u \in W^{k,\infty}(\Omega)$ with the estimate

(5.7) $$\|\partial\ell u\|_{W^{k-1,\infty}(\Omega)} \leq \sup_{\Omega' \subset \subset \Omega} \sup_{|h| < \text{dist}(\Omega', \partial\Omega)} \|\Delta_h^e u\|_{W^{k-1,\infty}(\Omega')}.$$