Problem 1  
In Theorem VI.3.7, we learned of the strong maximum principle in parabolic Cylinders. Use this to obtain the strong maximum principle in general open sets $X$:

Let $X \subset \mathbb{R}^{n+1}$ be a bounded, open set. Assume that $u \in C^\infty(X)$ and

$$\partial_t u - \Delta u = 0 \quad \text{in } X.$$

Assume moreover that for some $(x_0, t_0) \in X$ we have

$$M := u(x_0, t_0) = \sup_{(x,t) \in X} u(x,t).$$

(i) Describe (in words) in which set $C$ the function is necessarily constant

$$C := \{(x,t) \in X : u(x,t) = M\}.$$

(ii) Assume the set $X$ (grey) and the point $(x_0, t_0)$ are given in the picture. Draw (in orange) the set $C$ from the question above.
Problem 2  Show Theorem VI.3.7: Let $g \in C^0(\mathbb{R}^n)$, $f \in C^0(\mathbb{R}^n \times [0, T])$ for some $T > 0$. Assume there are two solutions $u^1$ and $u^2 \in C^2_1(\mathbb{R}^n \times (0, T)) \cap C^0(\mathbb{R}^n \times [0, T])$ of the problem

$$
\begin{cases}
(\partial_t - \Delta)u = 0 & \text{in } \mathbb{R}^n \times (0, T), \\
u(x, 0) = g(x) & \text{for } x \in \mathbb{R}^n.
\end{cases}
$$

If moreover we know that there are $a_1, a_2$ and $A_1, A_2 > 0$ with

$$
|u^1(x, t)| \leq A_1 e^{a_1|x|^2}, \quad |u^2(x, t)| \leq A_2 e^{a_2|x|^2} \quad \forall (x, t) \in \mathbb{R}^n \times [0, T],
$$

show that then

$$
u^1 \equiv u^2 \quad \text{in } \mathbb{R}^n \times [0, T].
$$

*Hint:* Theorem VI.3.6.

Problem 3  (cf. [John; PDE]) Define the following Tychonoff-function,

$$
u(x, t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.
$$

Here $g^{(k)}$ denotes the $k$-th derivative of $g$, given as

$$
g(t) := \begin{cases}
e^{-t-x} & t > 0 \\
0 & t \leq 0.
\end{cases}
$$

(i) Show that $\nu \in C^2_1(\mathbb{R}_{+}^n) \cap C^0(\mathbb{R} \times [0, \infty))$.

(ii) Show moreover that

$$
\begin{cases}
(\partial_t - \Delta)u = 0 & \text{in } \mathbb{R}^n \times (0, T), \\
u(x, 0) = 0 & \text{für } x \in \mathbb{R}^n.
\end{cases}
$$

(iii) Find a different solution $\nu \neq u$ of (1).

(iv) Why (without proof) does this not contradict Question 2?