University of Pittsburgh
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# Partial Differential Equations 2 – Fall 2019
# Exercise Sheet 1 — Due Date: Sep 16

Work in groups, write in LaTeX!

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**Problem 1** In Theorem VI.3.7. we learned of the strong maximum principle in parabolic Cylinders. Use this to obtain the strong maximum principle in general open sets $X$:

let $X \subset \mathbb{R}^{n+1}$ be a bounded, open set. Assume that $u \in C^\infty(\overline{X})$ and

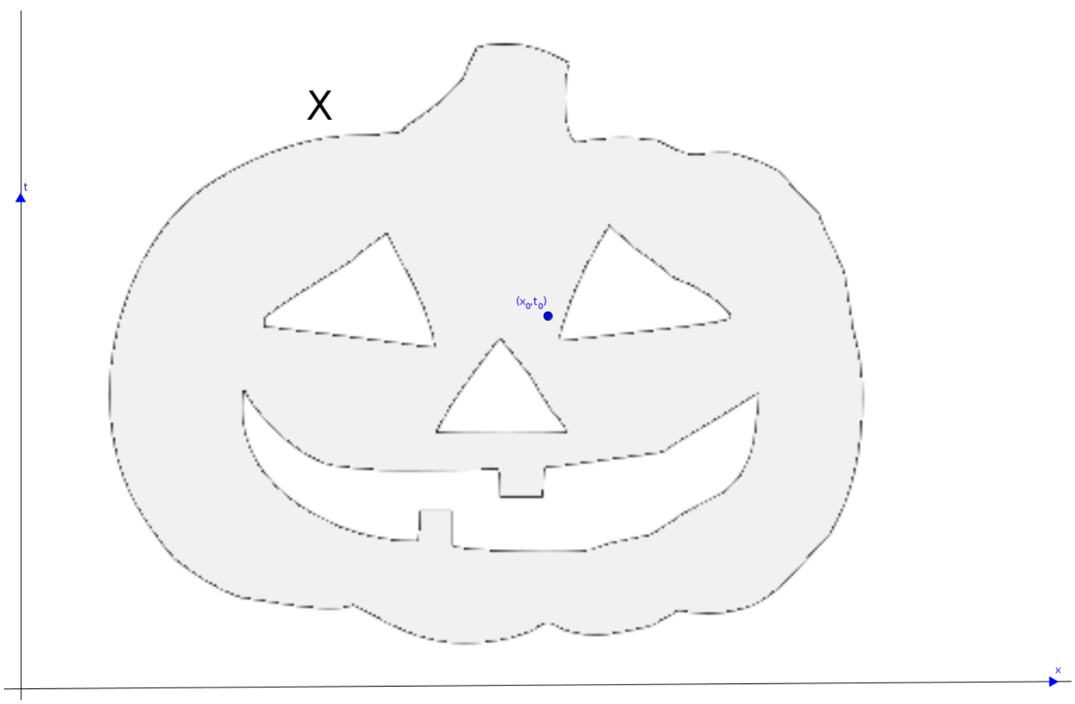
$$\partial_t u - \Delta u = 0 \quad \text{in X.}$$

Assume moreover that for some $(x_0, t_0) \in X$ we have

$$M := u(x_0, t_0) = \sup_{(x,t)\in X} u(x,t).$$

(i) Describe (in words) in which set $C$ the function is necessarily constant

$$C := \{(x,t) \in X : \quad u(x,t) = M\}.$$




(ii) Assume the set $X$ (grey) and the point $(x_0, t_0)$ are given in the picture. Draw (in orange) the set $C$ from the question above.

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## Problem 2

Show Theorem VI.3.7: Let $g \in C^0(\mathbb{R}^n)$, $f \in C^0(\mathbb{R}^n \times [0,T])$ for some $T > 0$. Assume there are two solutions $u^1$ and $u^2 \in C_1^2(\mathbb{R}^n \times (0,T)) \cap C^0(\mathbb{R}^n \times [0,T])$ of the problem

$$\begin{cases} (\partial_t - \Delta)u = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}^n. \end{cases}$$

If moreover we know that there are $a_1, a_2$ and $A_1, A_2 > 0$ with

$$|u^1(x,t)| \leq A_1\, e^{a_1\,|x|^2}, \quad |u^2(x,t)| \leq A_2\, e^{a_2\,|x|^2} \quad \forall (x,t) \in \mathbb{R}^n \times [0,T],$$

show that then

$$u^1 \equiv u^2 \quad \text{in } \mathbb{R}^n \times [0,T].$$

*Hint:* Theorem VI.3.6.

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## Problem 3

(cf. [John; PDE]) Define the following *Tychonoff*-function,

$$u(x,t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!}\, x^{2k}.$$

Here $g^{(k)}$ denotes the $k$-th derivative of $g$, given as

$$g(t) := \begin{cases} e^{(-t^{-\alpha})} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

(i) Show that $u \in C_1^2(\mathbb{R}^2_+) \cap C^0(\mathbb{R} \times [0,\infty))$.

(ii) Show moreover that

$$\begin{cases} (\partial_t - \Delta)u = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = 0 & \text{für } x \in \mathbb{R}^n. \end{cases} \tag{1}$$

(iii) Find a different solution $v \not\equiv u$ of (1).

(iv) Why (without proof) does this not contradict Question 2?

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