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# Partial Differential Equations 2 – Fall 2019
# Exercise Sheet 7 — Due Date: Dec 02

Work in groups, write in LaTeX!

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**Problem 14** Prove the following statement:

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. Let $f \in L^2(\Omega)$ and denote by

$$(f)_\Omega := |\Omega|^{-1} \int_\Omega f.$$

Show that

$$\|f - (f)_\Omega\|_{L^2(\Omega)} \leq \|f - c\|_{L^2(\Omega)} \quad \forall c \in \mathbb{R}.$$

*Hint:* Use the direct method of Calculus of Variations to find a minimizer of

$$G(c) := \|f - c\|^2_{L^2(\Omega)}.$$

Compute the Euler-Lagrange equations. Show that $c = (f)_\Omega$.

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**Problem 15** Prove the following statement:

(i) There is no solution $u \in C^1((-1,1)) \cap C^0([-1,1])$ that satisfies

$$\begin{cases} |u'(t)| = 1 & \text{in } (-1,1) \\ u(-1) = u(1) = 0 & \text{in } (-1,1) \end{cases}$$

(ii) There are infinitely many Lipschitz-continuous solutions $u \in C^{0,1}((-1,1)) \cap C^0([-1,1])$ that satisfy

$$\begin{cases} |u'(t)| = 1 & \text{for } \textit{almost every } t \in (-1,1) \\ u(-1) = u(1) = 0 & \text{in } (-1,1) \end{cases}$$

(iii) There are even solutions $u_k \in C^{0,1}((-1,1)) \cap C^0([-1,1])$ that satisfy

$$\begin{cases} |u_k'(t)| = 1 & \text{for almost every } t \in (-1,1) \\ u_k(-1) = u_k(1) = 0 & \text{in } (-1,1) \end{cases}$$

and such that $u_k \xrightarrow{k\to\infty} 0$ uniformly in $[-1,1]$.

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