Stirring coffee.

A real-life application of Topology in Analysis

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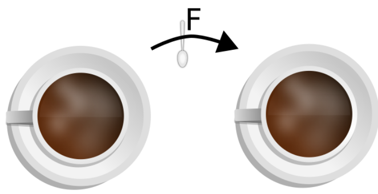
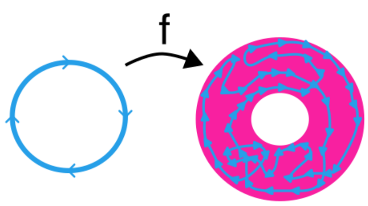


Have you ever pondered, after gently swirling a spoon through your steaming cup of coffee, whether, despite your meticulous stirring efforts, a solitary coffee molecule might have ended up exactly at the position where it was before stirring? Legend has it that Dutch mathematician Luitzen Egbertus Jan Brouwer (1881–1966) engaged in precisely such an experiment, ultimately establishing what is now celebrated as the Brouwer Fixed Point theorem – a beautiful result combining Topology with Analysis.

Let us describe the actual statement of the theorem we are going to discuss below. Take any continuous map F from the unit disk $\mathbb{D}$ to the unit disk $\mathbb{D}$. For example, the position of each coffee-molecule $\vec{p} \in \mathbb{D}$, before steering, gets mapped into the position $F(\vec{p}) \in \mathbb{D}$ after steering – where $\mathbb{D}$ denotes the coffee-mug. The theorem states that there must be at least one point $\vec{q}$ that has not changed at all: $F(\vec{q}) = \vec{q}$ – i.e. one molecule of coffee has not moved.

Topology: Winding Number

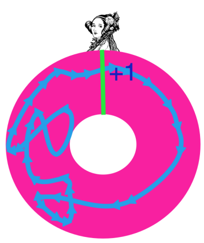
The winding number is a special case of a more general concept called the mapping degree



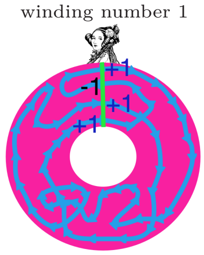
A continuous map f from $\mathbb{S}_1$ (left) to $\mathbb{S}_2$ (right) can be drawn as a curve on $\mathbb{S}_2$. To better visualize the behavior of curve f we visually "fatten" $\mathbb{S}_2$. Observe the orientation: when we walk around $\vec{v} \in \mathbb{S}_1$ clockwise, we have a direction of the curve in $\mathbb{S}_2$. This is indicated by the arrow signs. Observe that we don't really need to draw the left circle $\mathbb{S}_1$ to understand what f does, so in later pictures we won't draw $\mathbb{S}_1$.

We start by introducing our main tool, which is the winding number. For this, take a continuous map from a circle into a circle. Essentially, the circle represents the boundary of our (for simplicity: 2-dimensional) coffee mug. In order to distinguish between the domain (circle before stirring) and the target (after stirring), we will denote them by $\mathbb{S}_1$ and $\mathbb{S}_2$ respectively, where $\mathbb{S}$ stands for *sphere*. A map f from $\mathbb{S}_1$ to $\mathbb{S}_2$ (denoted as $f\colon \mathbb{S}_1 \to \mathbb{S}_2$) is simply any rule that to each point $\vec{v}$ in $\mathbb{S}_1$ assigns some point $f(\vec{v})$ in $\mathbb{S}_2$.

Continuous means that if we change a little bit the points in the domain, say I wiggle $\vec{v}$ into a $\vec{u}$ which is very close to $\vec{v}$, then $f(\vec{u})$ is not too far away from $f(\vec{v})$. Equivalent, but more geometrically intuitive: I can draw the curve defined by f (i.e. draw the corresponding values of $f(\vec{v})$ as $\vec{v}$ traverses the circle) without ever having to lift the pen. Since $\mathbb{S}_1$ is a circle, it has no start point and no end point. If I draw all the values $f(\vec{v})$ for all $\vec{v}$ in $\mathbb{S}_1$ then the picture I obtain is that of a curve inside of $\mathbb{S}_2$. Maybe like a rubber band somehow fiddled onto $\mathbb{S}_2$.



winding number 1



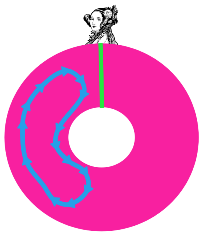
winding number 2

Let us assume we draw this rubber band clockwise (this fixes the *orientation* of $\mathbb{S}_1$). Then we find a picture of an oriented rubber band on $\mathbb{S}_2$. It is important to identify each of the strands of the rubber band, so on the margin we draw the rubber band on a "fattened" image of $\mathbb{S}_2$ – purely for visibility reasons.

We now define the *winding number* of the map f, we call it $w(f)$. Assume you stand on the top of the circle $\mathbb{S}_2$. The rubber band is possibly passing through the point you are standing on several times. You count how many times the rubber band passes through your position, but if it passes clockwise, you count $+1$. And every time the rubber bands passes through your position counterclockwise you count it as -1. Summing these numbers up you get the *winding number* $w(f)$. See the examples on the margin. Indeed, try and draw yourself some more pictures.

Exercise 1. *Draw curves with winding number* $0, +1, -1, +2, -2, \ldots$. *Try to find many different looking curves that have winding number* 2. *What do all the curves that have winding number* 2 *have in common?*

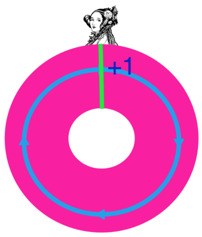
There are two pathological situations that you might have encountered when trying the above exercise: if the rubber band never passes through you, the winding number is simply zero. If a strand of the rubber band does not pass through you, but it just touches you, we count that as a $+0$ (or rather a $+1$ *and* a -1, which is effectively a $+0$.) Another subtlety I don't really want to discuss in detail here: who says the rubber band passes through us only *finitely* may times? Indeed it could pass through us infinitely many times. But if that happens then all but finitely many times the rubber band comes back again in the opposite direction – so these infinitely many times cancel out. Indeed, for any continuous map f the winding number is a finite number.



Winding number 0

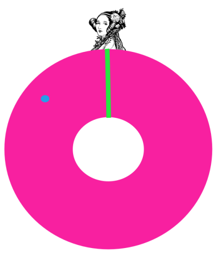
With the notion of winding number at hand let us look at two important examples:

Example 2. Assume the map f is simply $f(\vec{v}) = \vec{v}$. That is, each point on $\mathbb{S}_1$ gets mapped to its exact corresponding point on $\mathbb{S}_2$. Then the corresponding rubber band goes around the circle $\mathbb{S}_2$, in clockwise direction, exactly once. So the winding number of this particular configuration is $w(f) = 1$.



Example 2: The map $f(\vec{v}) = \vec{v}$ has winding number 1.

Example 3. Fix any vector $\vec{u} \in \mathbb{S}$. Denote $h(\vec{v}) = \vec{u}$. That is the rubber band is not going around $\mathbb{S}_1$ at all, it is collapsed onto a single point. Then the winding number is $w(h) = 0$.



Example 3: Any constant map has winding number 0.

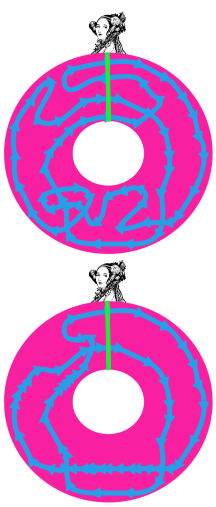
It is also good to know the following, although it will not be used here:

Exercise 4. *Justify that in our definition of winding number, it does not matter where exactly the person is standing. That is, changing the position of the person does not change the winding number.*

Exercise 4: The winding number is the same wherever Ada is.

Now the most important property: the winding number does not change under continuous changes of the curve. Continuous changes are called *homotopies*. If we can continuously transform the rubber band that f forms into a rubber band that g forms (without rupture and without leaving $\mathbb{S}_2$), we say that f and g are *homotopic* to each other.

Exercise 5. *Convince yourself that if we continuously can transform a rubber band f into another rubber band g, so that during that transformation the rubber band always stays inside $\mathbb{S}_2$, then the winding number does not change.*



Exercise 5: When we can change two curves continuously into each other (without rupture and without leaving $\mathbb{S}_2$, the winding number must be the same. Fun fact: The converse is also true!

Let us reformulate this last observation in more analytic terms, which will be useful later.

Corollary 6. *Assume $f : \mathbb{S}_1 \to \mathbb{S}_2$ and $g : \mathbb{S}_1 \to \mathbb{S}_2$ are continuous maps and there exists another continuous map, called a homotopy,*

$$H(t, \vec{v}) \in \mathbb{S}_2 \quad \text{for } 0 \le t \le 1 \text{ and the circular variable } \vec{v} \in \mathbb{S}_1$$

such that:

- $H(0, \vec{v}) = f(\vec{v})$ *for all* $\vec{v} \in \mathbb{S}_1$, *and*
- $H(1, \vec{v}) = g(\vec{v})$ *for all* $\vec{v} \in \mathbb{S}_1$.

Then, the winding number of f and the winding number of g are the same.

This is indeed a restatement of Exercise 5. For fixed “time” $t \in [0, 1]$ one can treat the function $H(t, \cdot)\colon \mathbb{S}_1 \to \mathbb{S}_2$ as describing a curve. When changing t, this curve continuously changes, so we see that the map H describes how we continuously transform the curve of f (at time $t = 0$) to the curve of g (at time $t = 1$).

Analysis: Brouwer Fixed Point Theorem

First, some definitions: for a vector $\vec{p} = (x, y) \in \mathbb{R}^2$ we denote its length by $|\vec{p}| := \sqrt{x^2 + y^2}$. The closed unit disk is denoted by

$$\mathbb{D} := \{\text{all vectors } \vec{p} = (x, y) \in \mathbb{R}^2 : \text{ such that } |\vec{p}| \le 1\}.$$

The boundary of the unit disk $\mathbb{D}$ is the circle, denoted by

$$\mathbb{S} = \{\text{all vectors } \vec{v} = (x, y) \in \mathbb{R}^2 : \text{ such that } |\vec{v}| = 1\}.$$

We will also denote the origin as $\vec{0} = (0, 0)$. Now we are ready to formulate the precise statement of the Brouwer fixed point theorem.

Theorem 7. *Take any continuous map F from $\mathbb{D}$ into $\mathbb{D}$. There exists at least one point $\vec{p} \in \mathbb{D}$ such that $F(\vec{p}) = \vec{p}$. Such a point $\vec{p}$ is called a* fixed point *of F.*

If we think about our cup of coffee example from the beginning, then we may assume for simplicity that the mug is two-dimensional and moreover in the shape of a disk $\mathbb{D}$ – e.g. by looking on it from above like in the picture at the beginning. Then we could think of $F\colon \mathbb{D} \to \mathbb{D}$ as the map that describes the position of a drop of coffee that originally is at the position $\vec{p} \in \mathbb{D}$ and, after stirring, ends up at the position $F(\vec{p})$. Of course that position is still inside the mug $\mathbb{D}$ (no spills!). The Brouwer Fixed Point theorem says that one drop $\vec{p}$ has not changed position.

Proof of the Theorem. The logical strategy is to show that if we assume the claimed statement was false, then we find a logical contradiction – so the statement must have been true! The main observation to obtain this contradiction is: If the statement of the theorem was false, then we can divide by quantities like $|F(\vec{p}) - \vec{p}| \neq 0$ for all $\vec{p} \in \mathbb{D}$; And we can use this to continuously transform the rubber band from Example 2 to the rubber band from Example 3 without rupture – contradicting Corollary 6.

So assume the claim of our theorem is false. That is, we assume there is no fixed point. In other words we assume $|F(\vec{p}) - \vec{p}| \neq 0$ for all $\vec{p} \in \mathbb{D}$. We will reach a contradiction by inspecting the winding number of the map $g\colon \mathbb{S}_1 \to \mathbb{S}_2$ given by $g(\vec{v}) := \frac{F(\vec{v})-\vec{v}}{|F(\vec{v})-\vec{v}|}$. First, observe that each $\vec{v}$ in the circle is also a point in the disk, so it makes sense to apply F to it. Second, we are not dividing by zero, which means g is indeed well defined.

In fact, we will consider three maps:

$$f(\vec{v}) = \vec{v}, \qquad g(\vec{v}) = \frac{F(\vec{v}) - \vec{v}}{|F(\vec{v}) - \vec{v}|} \qquad h(\vec{v}) = \frac{F(\vec{0})}{|F(\vec{0})|}.$$

as in Example 2, $w(f)=1$	as above	constant as in Example 3, $w(h)=0$

We are going to show that f is homotopic to g, and g is homotopic to h. Thus, by Corollary 6 the winding numbers of f, g, and h are the same – but the winding number of f is 1, the winding number of h is 0. This just cannot be true, it is a logical contradiction. Thus, somewhere along our argument there is a logical mistake. Since every step was a logical deduction, we conclude that it is the assumption "there is no fixed point" which must be false. But if it is not true that there is no fixed point, then logically there must be at least one fixed point. And that is what we wanted to prove!

In conclusion, we are done once we show that f, g, and h are homotopic: see below. □

Proof that f and g are homotopic. Consider the homotopy

$$H(t, \vec{v}) := \frac{tF(\vec{v}) - \vec{v}}{|tF(\vec{v}) - \vec{v}|}, \quad \vec{v} \in \mathbb{S}_1, \quad 0 \le t \le 1.$$

For each t and each $\vec{v} \in \mathbb{S}_1$ this is a well-defined and continuous map with values in $\mathbb{S}_2$ because we never divide by zero: $|tF(\vec{v}) - \vec{v}| \neq 0$ for any $\vec{v} \in \mathbb{D}$ and $t \in [0, 1]$. Indeed, for $t = 1$ this is a consequence of our assumption $F(\vec{v}) \neq \vec{v}$, whereas for $t < 1$ the point $tF(\vec{v})$ lies *strictly inside* the circle (note that $|F(\vec{v})| \le 1$), which in particular means that $tF(\vec{v}) \neq \vec{v}$.

Moreover, $H(0, \vec{v}) = f(\vec{v})$ (recall that $|\vec{v}| = 1$) and $H(1, \vec{v}) = g(\vec{v})$. Thus, we see that $H(t, \vec{v})$ is a homotopy in the sense of Corollary 6. □

Proof that g and h are homotopic. This time we set

$$H(\vec{v}, t) := \frac{F(t\vec{v}) - t\vec{v}}{|F(t\vec{v}) - t\vec{v}|}, \quad \vec{v} \in \mathbb{S}_1, \quad 0 \le t \le 1.$$

Then we see

- H is well-defined because we never divide by zero
- $H(\vec{v}, 1) = g(\vec{v})$
- $H(\vec{v}, 0) = h(\vec{v})$

Thus H is a continuous transformation of the rubber bands of g and h. $\square$

Concluding remarks

We restricted our attention to the two-dimensional disk $\mathbb{D}$ because its boundary is the one-dimensional circle – and the winding number is relatively easy to define on the circle. The same argument works in dimensions 3 and higher – but one needs to replace the winding number by the *degree*. Actually, there are even versions of the Brouwer Fixed Point theorem in infinite dimensions. The theory of mapping degree, and more generally homotopy groups, are the fundamental topics of a mathematical discipline called Algebraic Topology.

Let us also remark that one major property of the Brouwer fixed point theorem: We have absolutely no idea *where* the fixed point is, the proof is not constructive at all and gives us no hint on how to find this fixed point. We just know it exists. Also, the fixed point might not be unique, there might be many fixed points. Lastly, as a final exercise, I invite the reader to think about how to prove this result in one dimension.