Classical circuits

Problem 1 (15 pts) Let $\mathbb{B}^n$ be the set of n-bit expressions and $u \in \mathbb{B}^n$ a fixed expression. Write a characteristic Boolean function

$$\chi_u : \mathbb{B}^n \to \mathbb{B}$$
$$\chi_u(x) = \begin{cases} 1, x = u \\ 0, x \neq u. \end{cases}$$

as a composition of AND, OR and NOT operators (over the basis $\{AND, OR, NOT\}$).

(a) $u = 10$.

(b) $u = 101$.

(c) $u = 1001$.

(d) $u = u_1 u_2 \dots u_n \in \mathbb{B}^n$.

Problem 2 (15 pts) Let $x_1 \oplus x_2 \oplus \ldots \oplus x_n$ stand for $x_1 + x_2 + \ldots + x_n \pmod 2$. Implement the function $(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_n, x_1 \oplus x_2 \oplus \ldots \oplus x_n)$ using a few CNOTs.

(a) $n = 2$.

(b) $n = 3$.

(c) Any $n \in \mathbb{Z}_{>1}$.

Problem 3 (10 pts) Let $a = a_1 \ldots a_n$ and $b = b_1 \ldots b_n \in \mathbb{B}^n$ and construct a function that returns 1 if $a = b$ and 0 if $a \neq b$ over the basis $\{\mathsf{AND}, \mathsf{OR}, \mathsf{NOT}\}$.

Problem 4 (20 pts) Let $f : \mathbb{B}^n \to \mathbb{B}^m$ be any map. Define the map $f_\oplus : \mathbb{B}^{n+m} \to \mathbb{B}^{n+m}$ via

$$f_\oplus(x, y) = (x, f(x) \oplus y),$$

here $x \in \mathbb{B}^n$ and $y \in \mathbb{B}^m$.

(a) Show that $f_\oplus$ is injective, i.e. if $b_1 = (x_1, y_1) \neq b_2 = (x_2, y_2) \in \mathbb{B}^{n+m}$, then $f_\oplus(b_1) \neq f_\oplus(b_2)$.

(b) Show that $f_\oplus$ is surjective, i.e. for any $b \in \mathbb{B}^{n+m}$, there exists $b' \in \mathbb{B}^{n+m}$ with $f_\oplus(b') = b$.

Problem 5 (10 pts) A SWAP map interchanges two bits: it maps a state ab to ba. Build up a Boolean circuit computing SWAP using a few CNOTs.

Problem 6 (10 pts)

(a) Draw a Boolean circuit computing the characteristic Boolean function χ_{0000} using NOT and $\mathsf{CCCCNOT}$ operators.

(b) Draw a Boolean circuit computing the characteristic Boolean function χ_{1100} using NOT and $\mathsf{CCCCNOT}$ operators.

Problem 7 (20 pts) Consider the four Teenage Mutant Ninja Turtles: Donatello, Leonardo, Michelangelo, Raphael and their sensei Splinter. Michelangelo wants to throw a party, however, a recent ' incident' resulted in the following restrictions:

(1) Donatello and Leonardo will come to the party only together or not show up.

(2) Raphael will join only together with Michelangelo and in case Leonardo doesn't show up.

(3) In turn, Michelangelo will take part only if all his three friends join.

(a) Help sensei Splinter by listing all arrangements of participants, satisfying $(1)-(3)$.

(b) Let $\mathbb{B}^4$ be the set responsible for the possible arrangements. In other words, if a turtle comes to the party, the corresponding value is 1 and 0, otherwise. The 'bit-turtle participance' correspondence is as follows:

Donatello $\leftrightarrow$ first bit
Leonardo $\leftrightarrow$ second bit
Michelangelo $\leftrightarrow$ third bit
Raphael $\leftrightarrow$ fourth bit

For instance, if Donatello and Raphael participate, while Michelangelo and Leonardo don't the corresponding vector is $(1,0,0,1) \in \mathbb{B}^4$. Write $\mathcal{P} : \mathbb{B}^4 \to \mathbb{B}$ for the function, which gives 1 ('True') if the list of participants satisfies requirements $(1)-(3)$ and 0 ('False') otherwise. For instance $\mathcal{P}(1001) = 0$, since all conditions are violated. Write a circuit over $\mathcal{A} = \{\mathsf{NOT}, \underbrace{\mathsf{C}\dots\mathsf{C}}_{k}\mathsf{NOT}, k \leq 4\}$ that computes $\mathcal{P}$.