Lecture 10

Polynomials

MATH 0200

Dr. Boris Tsvelikhovskiy

Outline

In this lecture we will go over examples and properties of polynomials as well as their graphs.

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A **polynomial** $P(x)$ in a single variable (indeterminate) x is an expression of *(possibly after substitution of variable)* the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$

where $a_0, a_1, \ldots, a_n$ are numbers, $n \geq 0$ and $a_n \neq 0$.

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Interesting fact: the word polynomial joins two diverse roots: the Greek poly, meaning "many", and the Latin nomen, or "name".

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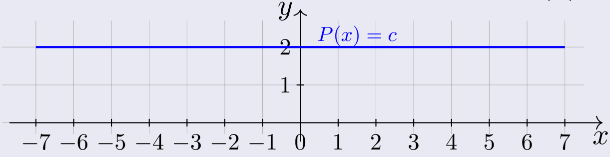

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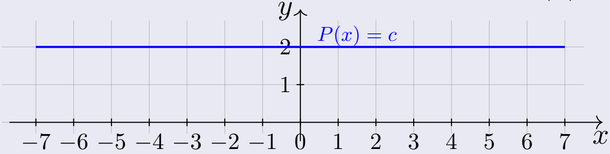

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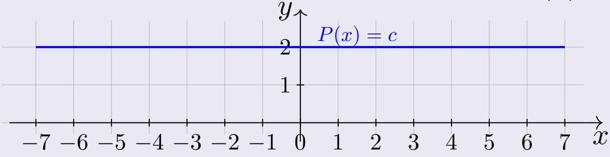

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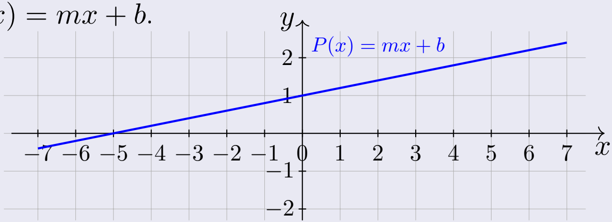

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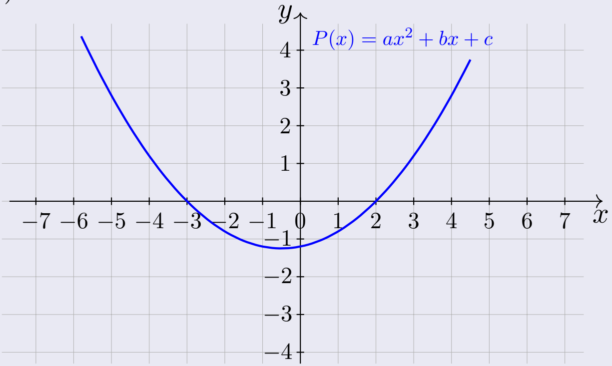

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$P(x) = 2 - x^2$ and $Q(x) = 4x^3 - 7x$, then $P(x)Q(x) = (2 - x^2)(4x^3 - 7x) = 8x^3 - 14x - 4x^5 - 7x^3 = x^3 - 14x - 4x^5$ and $\deg(PQ) = \deg(P) + \deg(Q) = 2 + 3 = 5$.

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Theorem

Let $p(x)$ be a polynomial, then a is a zero of p if and only if $x - a$ is a ***factor*** *of p, i.e. $p(x) = (x - a)q(x)$ for some polynomial $q(x)$ (notice that $deg(q) = deg(p) - 1$).*

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	n is even, $a_n > 0$	n is even, $a_n < 0$	n is odd, $a_n > 0$	n is odd, $a_n < 0$
$x \to \infty$	$P(x) \to \infty$	$P(x) \to -\infty$	$P(x) \to \infty$	$P(x) \to -\infty$
$x \to -\infty$	$P(x) \to \infty$	$P(x) \to -\infty$	$P(x) \to -\infty$	$P(x) \to \infty$

Example

Consider the polynomial $p(x) = 3 - 8x^2 - 10x^5$. The leading term of p is $-10x^5$ with the leading coefficient $-10 < 0$. We conclude that $p(x) \to -\infty$ as $x \to \infty$ and $p(x) \to \infty$ as $x \to -\infty$.

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2. What is the leading coefficient of $q(x)$?
3. What happens to $q(x)$ as x approaches ∞?

Answer: the leading term of $q(x)$ is the monomial $9x^6$. It follows that $\deg(q) = 6$, the leading coefficient is 9 and $q(x) \to \infty$ when $x \to \infty$.