Lecture 11

Rational functions

MATH 0200

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Outline

Rational functions

Definition

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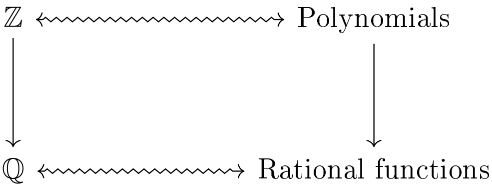

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We conclude that $623 = 13 \cdot 47 + 12$.

Long division of polynomials

Let $p(x) = 3x^3 - 12x + 2$ and divide it by $x - 3$.

$$
\begin{array}{r|rrrr}
 & & 3x^2 & +9x & +15 \\
\hline
x-3 & 3x^3 & & -12x & +2 \\
 & 3x^3 & -9x^2 & & \\
\hline
 & & 9x^2 & -12x & +2 \\
 & & 9x^2 & -27x & \\
\hline
 & & & 15x & +2 \\
 & & & 15x & -45 \\
\hline
 & & & & 47
\end{array}
$$

We get $p(x) = (3x^2 + 9x + 15)(x - 3) + 47$. Notice that $p(3) = 3 \cdot 3^3 - 12 \cdot 3 + 2 = 81 - 36 + 2 = 47$.

- Let p and q be two polynomials with $\deg(p) \geq \deg(q)$, then $p(x) = q(x)h(x) + r(x)$ for some polynomials $h(x)$ and $r(x)$ with $\deg(r) < \deg(q)$.

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- A real number a is a zero of $p(x)$ if and only if $p(x) = (x - a)h(x)$ for some polynomial $h(x)$ with $\deg(h) = \deg(p) - 1$. In other words, $(x - a)$ is a factor of $p(x)$.

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- More generally, the value of $p(a)$ is equal to the residue of division of $p(x)$ by $x - a$:

$$p(x) = h(x)(x-a) + r \Rightarrow p(a) = h(a)(a-a) + r = r.$$

Behavior of a rational function near $\pm\infty$

Similar to polynomials, the behavior of a rational function $f(x) = \dfrac{p(x)}{q(x)}$ is determined by the (ratio of) leading terms of p and q. We will denote the ratio of leading coefficients of p and q by α and the difference $\deg(p) - \deg(q)$ by s.

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For instance, the lines $y = \alpha$ and $y = 0$ appearing in the table above are (horizontal) asymptotes.

Example

- $f(x) = \dfrac{27 - x^3}{(10+x)(2x+5)}$. The ratio of leading terms is $\dfrac{-x^3}{2x^2} = -0.5x$ and $\deg(p) = 3 > 2 = \deg(q)$.

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- $f(x) = \dfrac{x^2 - 7}{(10-x)(3x+5)}$. The ratio of leading terms is $\dfrac{x^2}{-3x^2} = -\dfrac{1}{3}$ and $\deg(p) = \deg(q) = 2$.

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- $f(x) = \dfrac{x^2 - 7}{(10-x)(3x+5)}$. The ratio of leading terms is $\dfrac{x^2}{-3x^2} = -\dfrac{1}{3}$ and $\deg(p) = \deg(q) = 2$.As $x \to \infty$ or $x \to -\infty$, we get $f(x) \to -\dfrac{1}{3}$ and the line $y = -\dfrac{1}{3}$ is a horizontal asymptote of $f(x)$.

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Answer: the degrees of the polynomials in numerator and denominator are both equal to three. The leading terms are $-2x^3$ and $-4x^3$, hence, the ratio of leading coefficients is $\alpha = \dfrac{-2}{-4} = 0.5$. The horizontal asymptote is the line $y = 0.5$.

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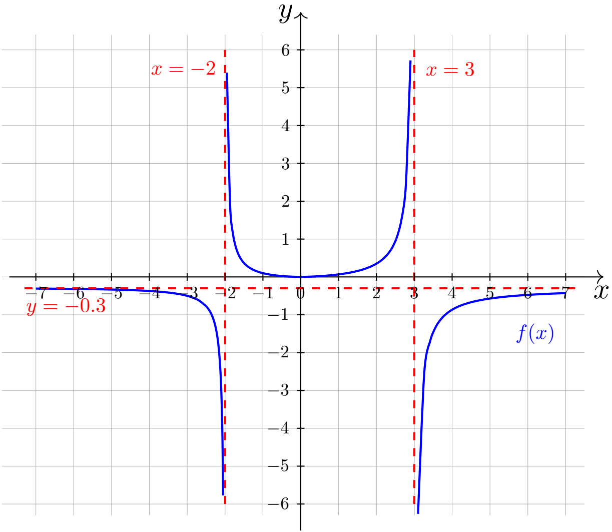
y
x
6
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
−7
1
2
3
4
5
6
7
x = −2
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f(x)