Lecture 13

Rules for logarithms

MATH 0200

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Outline

1. Special bases
2. Properties of logarithm
3. Change of base

Special bases

- $\ell n(a) = \ell og_e(a)$ is called the **natural logarithm** (here $e = 2.7182\ldots$ is the Euler's number);

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Remark

In computer science, the most frequently appearing logarithm base is two.

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5. $\ell og_a(b) = \frac{1}{\ell og_b(a)}$;

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Question

Given that $\ell n(u) = -1$ and $\ell n(v) = 3$, compute $\ell n((v^2u)^{10})$.

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The change of base formula is very useful, since it allows to express logarithm with any base in terms of a ratio of two logarithms of some convenient base.

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Notice that we do get the same answer!

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   $t^2 - 5t - 14 = (t-7)(t+2) = 0$, giving $t = 7$ (as $-2 \leq 0$ we disregard this answer) with $e^x = 7$ and $x = \ell n(7)$.
   Check: $e^{\ell n(7)} - 5 = 7 - 5 = 2 = 14 \cdot \frac{1}{7} = 14e^{-\ell n(7)}$ ✓