Lecture 14

Exponential growth

MATH 0200

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Outline

1. A story of an imaginary trader
2. Exponential growth
3. Simple and compound interest

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The first equality gives $b = 1/c$,

which implies $cb^4 = \frac{c}{c^4} = \frac{1}{c^3} = 27 \Leftrightarrow c = 1/3$ and $b = 1/c = 3$ with $f(x) = \frac{1}{3}3^x = 3^{x-1}$.

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(b) Approximately how many cells will be in the colony after one hour?

$$f(1) = 100(\sqrt{3})^1 = 100\sqrt{3} \approx 173.$$

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Suppose you deposit \$3000 in a bank account that pays 5% annual interest. How much money will be on your account after 4 years? Here $P = 3000$, $r = 0.05$ and $t = 4$, giving $3000(1 + 0.05 \cdot 3) = 3000 \cdot 1.15 = 3450$ dollars.

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After t years the amount of money on the account is

$$P\left(1 + \frac{r}{4}\right)^{4t}, \text{ where}$$

t is the number of years and $4t$ the total number of compounds ('automatic withdrawal and deposit').

Generalizing the formula on previous slide, we get that the total accumulated value (principal sum plus compounded interest) after t years, is given by the formula

$$P\left(1+\frac{r}{n}\right)^{nt}, \text{ where}$$

- P is the original principal sum;
- r is the nominal annual interest rate;
- n is the compounding frequency (per year);
- t is the overall length of time the interest is applied (in years).

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$$500\left(1+\frac{0.03}{4}\right)^{4t} > 900.$$

Next we find the smallest value of t satisfying the inequality: $500\left(1+\frac{0.03}{4}\right)^{4t} > 900 \Leftrightarrow 5(1.0075)^{4t} > 9 \Leftrightarrow (1.0075)^{4t} \geq 1.8 \Leftrightarrow 4t > \ell og_{1.0075} 1.8 \approx 78.665 \Leftrightarrow t > \frac{78.665}{4} \approx 19.67$. The minimal number of years is 20.

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Remark

Notice that the simple interest (with same interest rate) would give $1000(1+0.06\cdot 2) = \$1120.$