Lecture 15

Exponential growth (part II)

MATH 0200

Dr. Boris Tsvelikhovskiy

Outline

Continuous growth rate

Definition

If a quantity has a **continuous growth rate** (exponential growth rate) of r per unit time, then after t time units an initial amount P grows to Pe^{rt} units.

Continuous growth rate

Definition

If a quantity has a **continuous growth rate** (exponential growth rate) of r per unit time, then after t time units an initial amount P grows to Pe^{rt} units.

Example

Suppose a colony of bacteria has a continuous growth rate of 20% per hour.

Continuous growth rate

Definition

If a quantity has a **continuous growth rate** (exponential growth rate) of r per unit time, then after t time units an initial amount P grows to Pe^{rt} units.

Example

Suppose a colony of bacteria has a continuous growth rate of 20% per hour.

1. By what percent will the colony have grown after seven hours?

Continuous growth rate

Definition

If a quantity has a **continuous growth rate** (exponential growth rate) of r per unit time, then after t time units an initial amount P grows to Pe^{rt} units.

Example

Suppose a colony of bacteria has a continuous growth rate of 20% per hour.

1. By what percent will the colony have grown after seven hours?
   After seven hours the size of the colony is $Pe^{0.2\cdot 7}$, the answer is $(\frac{Pe^{0.2\cdot 7}}{P} - 1)\cdot 100\% \approx 306\%$

Continuous growth rate

Definition

If a quantity has a **continuous growth rate** (exponential growth rate) of r per unit time, then after t time units an initial amount P grows to Pe^{rt} units.

Example

Suppose a colony of bacteria has a continuous growth rate of 20% per hour.

1. By what percent will the colony have grown after seven hours?
   After seven hours the size of the colony is $Pe^{0.2 \cdot 7}$, the answer is $(\frac{Pe^{0.2\cdot 7}}{P} - 1) \cdot 100\% \approx 306\%$
2. If the colony contains 3000 cells now, how many did it contain three hours ago?

Continuous growth rate

Definition

If a quantity has a **continuous growth rate** (exponential growth rate) of r per unit time, then after t time units an initial amount P grows to Pe^{rt} units.

Example

Suppose a colony of bacteria has a continuous growth rate of 20% per hour.

1. By what percent will the colony have grown after seven hours?
   After seven hours the size of the colony is $Pe^{0.2\cdot 7}$, the answer is $(\frac{Pe^{0.2\cdot 7}}{P} - 1)\cdot 100\% \approx 306\%$
2. If the colony contains 3000 cells now, how many did it contain three hours ago?
   $Pe^{0.2\cdot 3} = 3000 \Leftrightarrow P = \frac{3000}{e^{0.6}} \approx 1646.$

Continuously compounded interest

As we have already observed in the previous lecture, an increasing frequency of interest compounds leads to a greater accumulation of interest earned.

Continuously compounded interest

As we have already observed in the previous lecture, an increasing frequency of interest compounds leads to a greater accumulation of interest earned.

Question

How many times a year can the interest be compounded?

Continuously compounded interest

As we have already observed in the previous lecture, an increasing frequency of interest compounds leads to a greater accumulation of interest earned.

Question

How many times a year can the interest be compounded?

There seems to be no upper bound. The formula for the interest of rate r compounded continuously is

Continuously compounded interest

As we have already observed in the previous lecture, an increasing frequency of interest compounds leads to a greater accumulation of interest earned.

Question

How many times a year can the interest be compounded?

There seems to be no upper bound. The formula for the interest of rate r compounded continuously is

$$P\left(1+\frac{r}{n}\right)^{nt} \xrightarrow{n\to\infty} Pe^{rt}.$$

Example

Suppose $\$10,000$ is placed in a bank account that pays 5% annual interest.

1. How much money will be in the bank account after 10 years?

Example

Suppose \$10,000 is placed in a bank account that pays 5% annual interest.

1. How much money will be in the bank account after 10 years?

$$10000(1 + 0.05 \cdot 10) = \$15,000.$$

Example

Suppose $\$10,000$ is placed in a bank account that pays 5% annual interest.

1. How much money will be in the bank account after 10 years?
$$10000(1 + 0.05 \cdot 10) = \$15,000.$$
2. If interest is compounded quarterly, how much will be in the bank account after 10 years?

Example

Suppose $\$10,000$ is placed in a bank account that pays 5% annual interest.

1. How much money will be in the bank account after 10 years?

$$10000(1 + 0.05 \cdot 10) = \$15,000.$$

2. If interest is compounded quarterly, how much will be in the bank account after 10 years?

$$10000\left(1 + \frac{0.05}{4}\right)^{4 \cdot 10} \approx \$16,436.19.$$

Example

Suppose $\$10,000$ is placed in a bank account that pays 5% annual interest.

1. How much money will be in the bank account after 10 years?

$$10000(1 + 0.05 \cdot 10) = \$15,000.$$

2. If interest is compounded quarterly, how much will be in the bank account after 10 years?

$$10000 \left(1 + \frac{0.05}{4}\right)^{4 \cdot 10} \approx \$16,436.19.$$

3. If interest is compounded continuously, how much will be in the bank account after 10 years?

Example

Suppose \$10,000 is placed in a bank account that pays 5% annual interest.

1. How much money will be in the bank account after 10 years?

$$10000(1 + 0.05 \cdot 10) = \$15,000.$$

2. If interest is compounded quarterly, how much will be in the bank account after 10 years?

$$10000\left(1 + \frac{0.05}{4}\right)^{4 \cdot 10} \approx \$16,436.19.$$

3. If interest is compounded continuously, how much will be in the bank account after 10 years?

$$10000e^{0.05 \cdot 10} = 10000e^{0.5} \approx \$16,487.21.$$

Example

1. How many years does it take for money to double at 3% annual interest compounded continuously?

Example

1. How many years does it take for money to double at 3% annual interest compounded continuously?
$Pe^{0.03t} = 2P \Leftrightarrow e^{0.03t} = 2 \Leftrightarrow 0.03t = \ell n(2) \Leftrightarrow t = \frac{\ell n(2)}{0.03} \approx 23$ years.

Example

1. How many years does it take for money to double at 3% annual interest compounded continuously?
   $Pe^{0.03t} = 2P \Leftrightarrow e^{0.03t} = 2 \Leftrightarrow 0.03t = \ell n(2) \Leftrightarrow t = \frac{\ell n(2)}{0.03} \approx 23$ years.
2. How many years does it take for money to quadruple at 3% annual interest compounded continuously?

Example

1. How many years does it take for money to double at 3% annual interest compounded continuously?
   $Pe^{0.03t} = 2P \Leftrightarrow e^{0.03t} = 2 \Leftrightarrow 0.03t = \ell n(2) \Leftrightarrow t = \frac{\ell n(2)}{0.03} \approx 23$ years.
2. How many years does it take for money to quadruple at 3% annual interest compounded continuously?
   $Pe^{0.03t} = 4P \Leftrightarrow e^{0.03t} = 4 \Leftrightarrow 0.03t = \ell n(4) \Leftrightarrow t = \frac{\ell n(4)}{0.03} = \frac{2\ell n(2)}{0.03} \approx 46$ years.

Example

1. How many years does it take for money to double at 3% annual interest compounded continuously?
$Pe^{0.03t} = 2P \Leftrightarrow e^{0.03t} = 2 \Leftrightarrow 0.03t = \ell n(2) \Leftrightarrow t = \frac{\ell n(2)}{0.03} \approx 23$ years.
2. How many years does it take for money to quadruple at 3% annual interest compounded continuously?
$Pe^{0.03t} = 4P \Leftrightarrow e^{0.03t} = 4 \Leftrightarrow 0.03t = \ell n(4) \Leftrightarrow t = \frac{\ell n(4)}{0.03} = \frac{2\ell n(2)}{0.03} \approx 46$ years.
Notice that we have already shown that any principal amount doubles after 23 years, so the calculation was unnecessary.