Lecture 16

The unit circle

MATH 0200

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Outline

Unit circle

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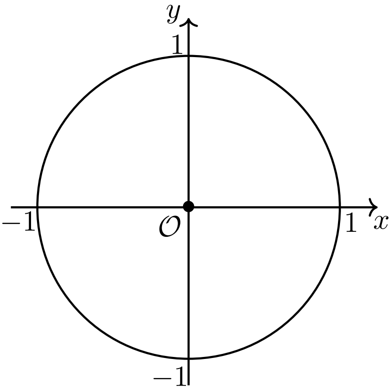

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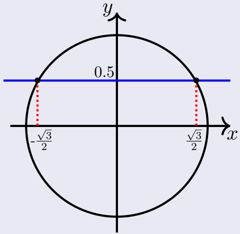

Positive angles

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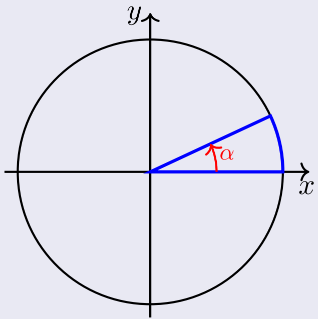

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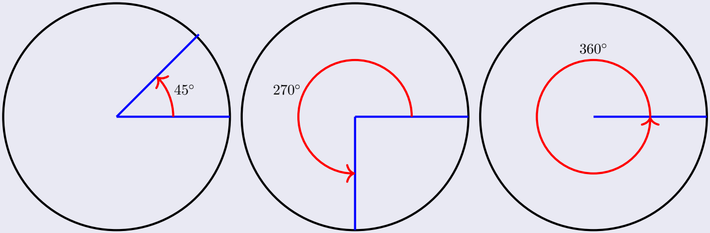

Negative angles

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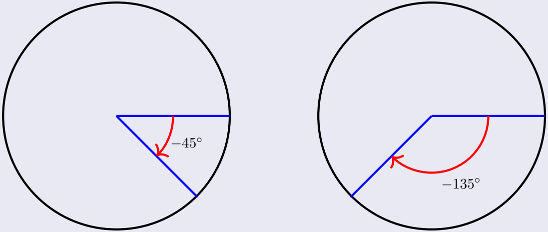

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- $1 \text{ rad} = \left(\frac{180}{\pi}\right)^{\circ}$ and
- $1° = \frac{\pi}{180}$ rad.

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   $\frac{2\pi}{15}$ rad $= \frac{180}{\pi} \cdot \frac{2\pi}{15} = \frac{180 \cdot 2\pi}{15\pi} = 12 \cdot 2 = 24^\circ$.

Question

What is the radian measure of the angle $\alpha = \frac{18}{\pi}$ degrees?

Special points on the unit circle

angle	endpoint of radius
$0 = 0°$	$(1, 0)$
$\frac{\pi}{6} = 30°$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{\pi}{4} = 45°$	$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
$\frac{\pi}{3} = 60°$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2} = 90°$	$(0, 1)$
$\pi = 180°$	$(-1, 0)$

Length of a circular arc and area of a sector

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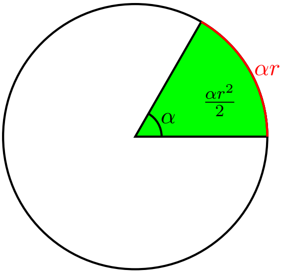

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$$\mathcal{S}\left(\text{slice} \right) = \frac{1}{2} \cdot \frac{4}{7} \cdot 7^2 = 14 \text{ in}^2.$$

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