# Lecture 21

## Inverse trigonometric functions

MATH 0200

Dr. Boris Tsvelikhovskiy

# Outline

# Arccosine, the inverse of $\cos(x)$



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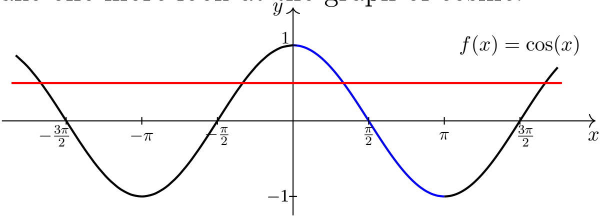

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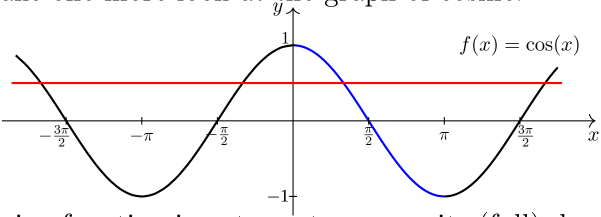


The cosine function is not one-to-one on its (full) domain $(-\infty, \infty)$. However, it is one-to-one if we restrict the domain, for instance, to $[0, \pi]$ and, therefore is invertible on the interval $[0, \pi]$:

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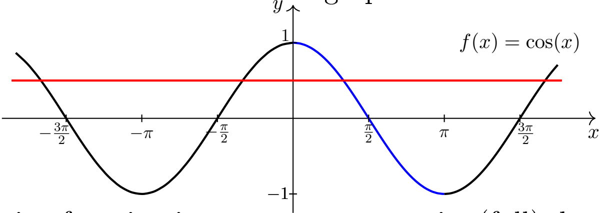


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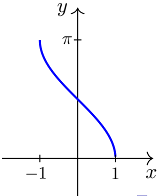
$f(x) = \arccos(x)$
Domain: $[-1, 1]$
Range: $[0, \pi]$



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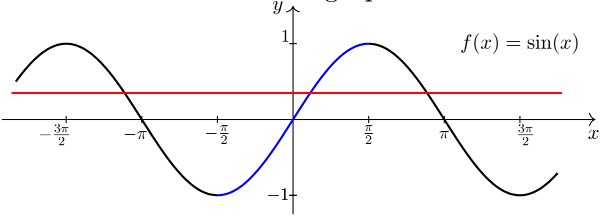

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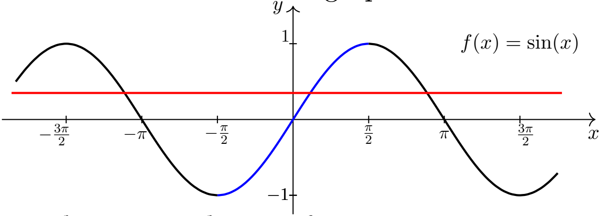


Similar to the cosine, the sine function is not one-to-one on its (full) domain $(-\infty, \infty)$. However, it is one-to-one if we restrict the domain, for instance, to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and, therefore is invertible on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$:

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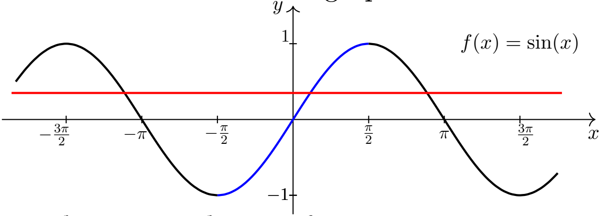


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$f(x) = \arcsin(x)$

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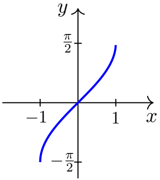

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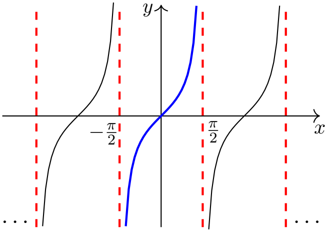


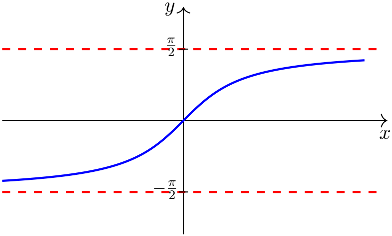

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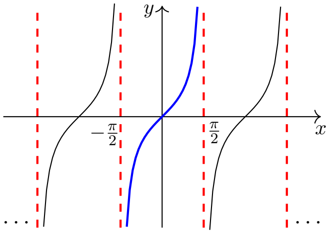


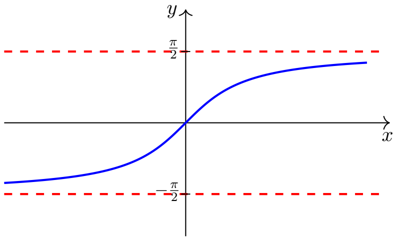


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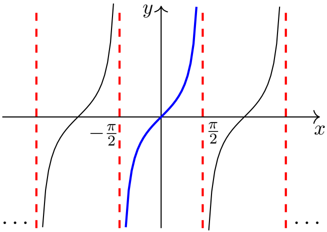


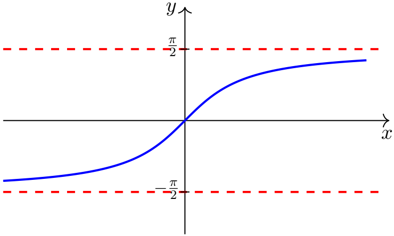


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# Definitions of arccosine, arcsine and arctangent

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- The arccosine of $-1 \leq c \leq 1$, denoted $\arccos(c)$, is the angle $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ with $\cos(\alpha) = c$.

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- $\arctan(\sqrt{3}) = \frac{\pi}{3}$ as $-\frac{\pi}{2} < \frac{\pi}{3} < \frac{\pi}{2}$ and $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$;

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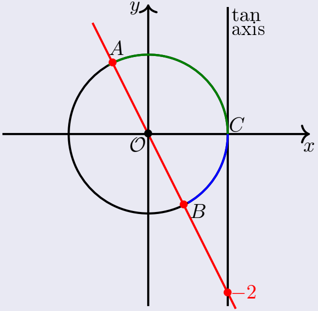

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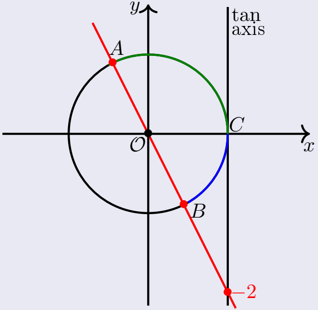


Notice that $\arctan(-2)$ is a negative number, minus the length of the arc $CB$, shaded in blue. The smallest positive value of $x$ with $\tan(x) = -2$ is the length of the arc $CA$, shaded in green. It is equal to $\pi + \arctan(-2)$.

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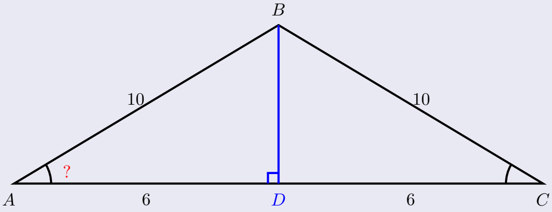

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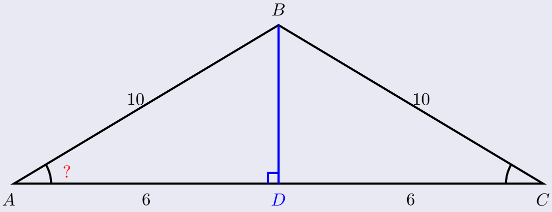


We drop a perpendicular from vertex $B$ to the base $AC$ and denote the point of intersection by $D$. Recall that since triangle $\triangle ABC$ is isosceles, $D$ is the midpoint of $AC$, so $|AD| = |DC| = 12/2 = 6$. We get $\cos(\angle BAC) = \frac{6}{10} = 0.6$, hence, $\angle BAC = \arccos(0.6) \approx 53.13°$.