Lecture 24

Laws of sines and cosines

MATH 0200

Dr. Boris Tsvelikhovskiy

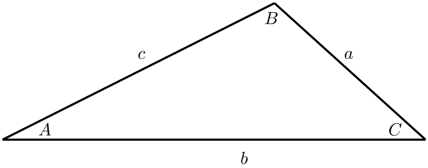
Outline

Law and Mathematics

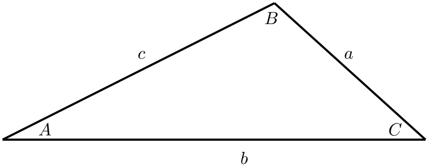
Pierre de Fermat
(1601-1665)

Pierre de Fermat was a French mathematician who is given credit for early developments that led to infinitesimal calculus. He made notable contributions to analytic geometry, probability, and optics. He is best known for his Fermat's principle for light propagation and his Fermat's Last Theorem in number theory. He was also a **lawyer** at the Parliament of Toulouse, France.

Law of Sines

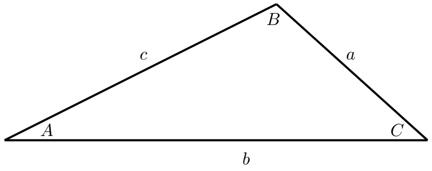


Law of Sines



$$\text{Area} = \frac{bc\sin(A)}{2} = \frac{ac\sin(B)}{2} = \frac{ab\sin(C)}{2}.$$

Law of Sines



$$\text{Area} = \frac{bc\sin(A)}{2} = \frac{ac\sin(B)}{2} = \frac{ab\sin(C)}{2}.$$

Law of Sines

In a triangle with sides of lengths a, b and c, and corresponding opposite angles A, B and C:

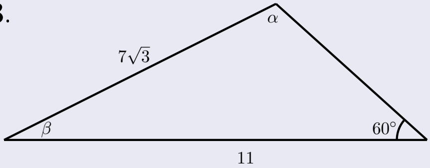
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}.$$

Example

Find all the angles in a triangle that has one side of length 11, one side of length $7\sqrt{3}$, and a 60° angle opposite the side of length $7\sqrt{3}$.

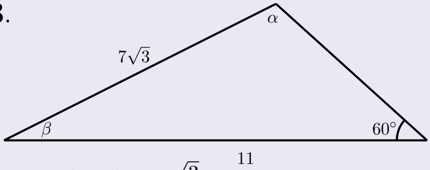
Example

Find all the angles in a triangle that has one side of length 11, one side of length $7\sqrt{3}$, and a 60° angle opposite the side of length $7\sqrt{3}$.



Example

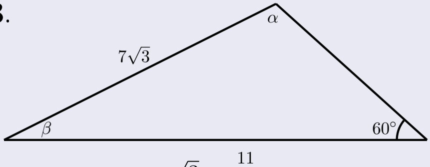
Find all the angles in a triangle that has one side of length 11, one side of length $7\sqrt{3}$, and a $60°$ angle opposite the side of length $7\sqrt{3}$.



We notice that $\sin(60°) = \frac{\sqrt{3}}{2}$ and apply the law of sines:

Example

Find all the angles in a triangle that has one side of length 11, one side of length $7\sqrt{3}$, and a 60° angle opposite the side of length $7\sqrt{3}$.

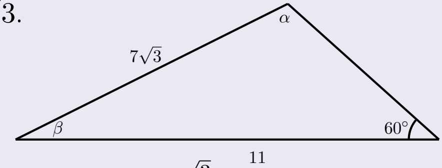


We notice that $\sin(60^\circ) = \frac{\sqrt{3}}{2}$ and apply the law of sines:

$$\frac{11}{\sin(\alpha)} = \frac{7\sqrt{3}}{\sin(60^\circ)} = \frac{7\sqrt{3}}{\frac{\sqrt{3}}{2}} = \frac{7 \cdot 2\sqrt{3}}{\sqrt{3}} = 14 \Leftrightarrow \sin(\alpha) = \frac{11}{14}.$$

Example

Find all the angles in a triangle that has one side of length 11, one side of length $7\sqrt{3}$, and a 60° angle opposite the side of length $7\sqrt{3}$.



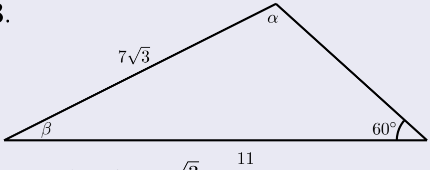
We notice that $\sin(60^\circ) = \frac{\sqrt{3}}{2}$ and apply the law of sines:

$$\frac{11}{\sin(\alpha)} = \frac{7\sqrt{3}}{\sin(60^\circ)} = \frac{7\sqrt{3}}{\frac{\sqrt{3}}{2}} = \frac{7\cdot 2\sqrt{3}}{\sqrt{3}} = 14 \Leftrightarrow \sin(\alpha) = \frac{11}{14}.$$

We conclude that $\alpha = \arcsin\left(\frac{11}{14}\right) \approx 51.79^\circ$ (notice that $180 - 51.79 \approx 128.21^\circ$ would give $128.21 + 60 = 188.21^\circ > 180^\circ$).

Example

Find all the angles in a triangle that has one side of length 11, one side of length $7\sqrt{3}$, and a $60°$ angle opposite the side of length $7\sqrt{3}$.



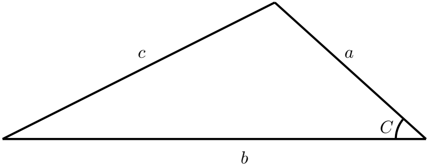
We notice that $\sin(60°) = \frac{\sqrt{3}}{2}$ and apply the law of sines:

$$\frac{11}{\sin(\alpha)} = \frac{7\sqrt{3}}{\sin(60°)} = \frac{7\sqrt{3}}{\frac{\sqrt{3}}{2}} = \frac{7 \cdot 2\sqrt{3}}{\sqrt{3}} = 14 \Leftrightarrow \sin(\alpha) = \frac{11}{14}.$$

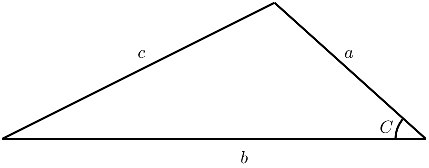
We conclude that $\alpha = \arcsin\left(\frac{11}{14}\right) \approx 51.79°$ (notice that $180 - 51.79 \approx 128.21°$ would give $128.21 + 60 = 188.21° > 180°$).

Finally, we compute $\beta \approx 180 - 60 - 51.79 \approx 68.21°$.

Law of Cosines



Law of Cosines



Law of Cosines

In a triangle with sides of lengths a, b and c, with an angle C opposite the side with length c:

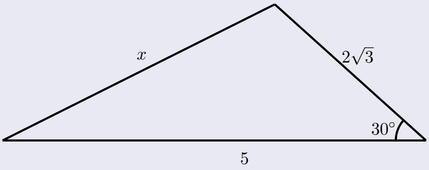
$$c^2 = a^2 + b^2 - 2ab\cos(C).$$

Example

A triangle has sides of length 5 and $2\sqrt{3}$, and a 30° angle between those two sides. Find the length of the third side of the triangle.

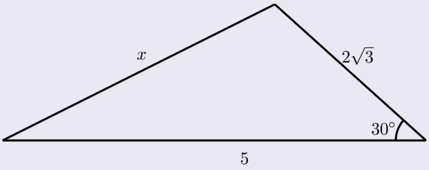
Example

A triangle has sides of length 5 and $2\sqrt{3}$, and a $30°$ angle between those two sides. Find the length of the third side of the triangle.



Example

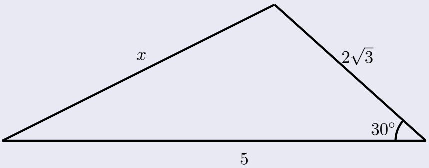
A triangle has sides of length 5 and $2\sqrt{3}$, and a $30°$ angle between those two sides. Find the length of the third side of the triangle.



We notice that $\cos(30°) = \frac{\sqrt{3}}{2}$ and apply the law of cosines:

Example

A triangle has sides of length 5 and $2\sqrt{3}$, and a 30° angle between those two sides. Find the length of the third side of the triangle.



We notice that $\cos(30^\circ) = \frac{\sqrt{3}}{2}$ and apply the law of cosines:

$$x^2 = 5^2 + (2\sqrt{3})^2 - 2 \cdot 5 \cdot 2\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 25 + 12 - 30 = 7 \Leftrightarrow x = \sqrt{7}.$$

When to use which law?

Law of Sines Vs Law of Cosines

Use the law of sines if you know

- two angles of a triangle and the length of one side;
- the length of two sides of a triangle and an angle other than the angle between those two sides.

When to use which law?

Law of Sines Vs Law of Cosines

Use the law of sines if you know

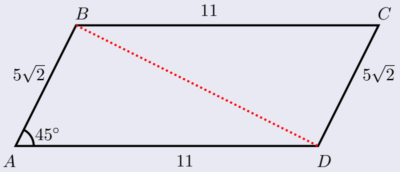
- two angles of a triangle and the length of one side;
- the length of two sides of a triangle and an angle other than the angle between those two sides.

Use the law of cosines if you know

- the lengths of all three sides of a triangle;
- the lengths of two sides of a triangle and the angle between them.

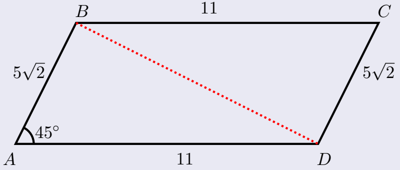
Question

Consider a parallelogram that has two sides of length $5\sqrt{2}$, two sides of length 11, and two angles of $45°$. Find the **square** of the length of the diagonal opposite the angles of $45°$.



Question

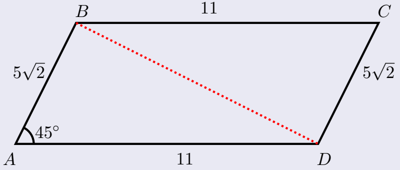
Consider a parallelogram that has two sides of length $5\sqrt{2}$, two sides of length 11, and two angles of $45°$. Find the **square** of the length of the diagonal opposite the angles of $45°$.



Answer: we recall that $\cos(45°) = \frac{1}{\sqrt{2}}$ and apply the law of cosines:

Question

Consider a parallelogram that has two sides of length $5\sqrt{2}$, two sides of length 11, and two angles of 45°. Find the **square** of the length of the diagonal opposite the angles of 45°.



Answer: we recall that $\cos(45°) = \frac{1}{\sqrt{2}}$ and apply the law of cosines:

$$x^2 = (5\sqrt{2})^2 + 11^2 - 2 \cdot 5\sqrt{2} \cdot 11 \cdot \frac{1}{\sqrt{2}} = 50 + 121 - 110 = 61.$$