

Lecture 25

MATH 0200

Double-
angle
formulas

Half-angle
formulas

Addition
and
subtraction
formulas

Lecture 25

More trigonometric formulas

MATH 0200

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Outline

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Double-
angle
formulas

Half-angle
formulas

Addition
and
subtraction
formulas

- 1 Double-angle formulas
- 2 Half-angle formulas
- 3 Addition and subtraction formulas

Double-angle formulas

- $\sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha)$;

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Double-angle formulas

- $\sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha)$;
- $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 2 \cos^2(\alpha) - 1 = 1 - 2 \sin^2(\alpha)$;

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Double-angle formulas

- $\sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha)$;
- $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 2 \cos^2(\alpha) - 1 = 1 - 2 \sin^2(\alpha)$;
- $\tan(2\alpha) = \frac{2 \tan(\alpha)}{1 - \tan^2(\alpha)}$.

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Double-angle formulas

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- $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 2 \cos^2(\alpha) - 1 = 1 - 2 \sin^2(\alpha)$;
- $\tan(2\alpha) = \frac{2 \tan(\alpha)}{1 - \tan^2(\alpha)}$.

Example

- ① Let's pretend that we don't know the value of $\sin(60^\circ)$, but know that $\sin(30^\circ) = \frac{1}{2}$ and $\cos(30^\circ) = \frac{\sqrt{3}}{2}$.

Double-angle formulas

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- $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 2 \cos^2(\alpha) - 1 = 1 - 2 \sin^2(\alpha)$;
- $\tan(2\alpha) = \frac{2 \tan(\alpha)}{1 - \tan^2(\alpha)}$.

Example

- ① Let's pretend that we don't know the value of $\sin(60^\circ)$, but know that $\sin(30^\circ) = \frac{1}{2}$ and $\cos(30^\circ) = \frac{\sqrt{3}}{2}$.
We get $\sin(60^\circ) = 2 \sin(30^\circ) \cos(30^\circ) = 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$.

Double-angle formulas

- $\sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha)$;
- $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 2 \cos^2(\alpha) - 1 = 1 - 2 \sin^2(\alpha)$;
- $\tan(2\alpha) = \frac{2 \tan(\alpha)}{1 - \tan^2(\alpha)}$.

Example

- 1 Let's pretend that we don't know the value of $\sin(60^\circ)$, but know that $\sin(30^\circ) = \frac{1}{2}$ and $\cos(30^\circ) = \frac{\sqrt{3}}{2}$.
We get $\sin(60^\circ) = 2 \sin(30^\circ) \cos(30^\circ) = 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$.
- 2 Let's pretend that we don't know the value of $\tan(60^\circ)$, but know that $\tan(30^\circ) = \frac{1}{\sqrt{3}}$.

Double-angle formulas

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Double-angle formulas

Half-angle formulas

Addition and subtraction formulas

- $\sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha)$;
- $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 2 \cos^2(\alpha) - 1 = 1 - 2 \sin^2(\alpha)$;
- $\tan(2\alpha) = \frac{2 \tan(\alpha)}{1 - \tan^2(\alpha)}$.

Example

- 1 Let's pretend that we don't know the value of $\sin(60^\circ)$, but know that $\sin(30^\circ) = \frac{1}{2}$ and $\cos(30^\circ) = \frac{\sqrt{3}}{2}$.
We get $\sin(60^\circ) = 2 \sin(30^\circ) \cos(30^\circ) = 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$.
- 2 Let's pretend that we don't know the value of $\tan(60^\circ)$, but know that $\tan(30^\circ) = \frac{1}{\sqrt{3}}$.

$$\text{We get } \tan(60^\circ) = \frac{2 \tan(30^\circ)}{1 - \tan^2(30^\circ)} = \frac{\frac{2}{\sqrt{3}}}{1 - \left(\frac{1}{\sqrt{3}}\right)^2} = \frac{2}{\sqrt{3} \frac{1}{3}} =$$

$$\frac{2}{\sqrt{3} \left(1 - \frac{1}{3}\right)} = \frac{2}{\frac{2\sqrt{3}}{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

Half-angle formulas

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Double-
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**Half-angle
formulas**

Addition
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- $\sin(\alpha/2) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}};$

Half-angle formulas

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Addition
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- $\sin(\alpha/2) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}};$
- $\cos(\alpha/2) = \pm\sqrt{\frac{1+\cos(\alpha)}{2}};$

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- $\sin(\alpha/2) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}};$
- $\cos(\alpha/2) = \pm\sqrt{\frac{1+\cos(\alpha)}{2}};$
- $\tan(\alpha/2) = \frac{1-\cos(\alpha)}{\sin(\alpha)}.$

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Double-angle formulas

Half-angle formulas

Addition and subtraction formulas

- $\sin(\alpha/2) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}};$
- $\cos(\alpha/2) = \pm\sqrt{\frac{1+\cos(\alpha)}{2}};$
- $\tan(\alpha/2) = \frac{1-\cos(\alpha)}{\sin(\alpha)}.$

Example

- 1 Find $\sin(22.5^\circ)$.

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- $\sin(\alpha/2) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}};$
- $\cos(\alpha/2) = \pm\sqrt{\frac{1+\cos(\alpha)}{2}};$
- $\tan(\alpha/2) = \frac{1-\cos(\alpha)}{\sin(\alpha)}.$

Example

- 1 Find $\sin(22.5^\circ)$.

$$\text{We get } \sin(22.5^\circ) = \sin((45/2)^\circ) = \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}.$$

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- $\sin(\alpha/2) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}};$
- $\cos(\alpha/2) = \pm\sqrt{\frac{1+\cos(\alpha)}{2}};$
- $\tan(\alpha/2) = \frac{1-\cos(\alpha)}{\sin(\alpha)}.$

Example

- 1 Find $\sin(22.5^\circ)$.

$$\text{We get } \sin(22.5^\circ) = \sin((45/2)^\circ) = \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}.$$

- 2 Find $\cos(75^\circ)$.

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Addition and subtraction formulas

- $\sin(\alpha/2) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}}$;
- $\cos(\alpha/2) = \pm\sqrt{\frac{1+\cos(\alpha)}{2}}$;
- $\tan(\alpha/2) = \frac{1-\cos(\alpha)}{\sin(\alpha)}$.

Example

- 1 Find $\sin(22.5^\circ)$.

We get $\sin(22.5^\circ) = \sin((45/2)^\circ) = \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$.

- 2 Find $\cos(75^\circ)$.

Notice that $75 = 150/2$ and

$\cos(150^\circ) = -\cos(30^\circ) = -\frac{\sqrt{3}}{2}$. We get

$$\cos(75^\circ) = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

Addition and subtraction formulas

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Addition

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Addition

- $\sin(a + b) = \sin(a) \cos(b) + \sin(b) \cos(a);$

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Addition

- $\sin(a + b) = \sin(a) \cos(b) + \sin(b) \cos(a)$;
- $\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)$.

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Addition

- $\sin(a + b) = \sin(a) \cos(b) + \sin(b) \cos(a)$;
- $\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)$.

Subtraction

Addition and subtraction formulas

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Addition

- $\sin(a + b) = \sin(a) \cos(b) + \sin(b) \cos(a)$;
- $\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)$.

Subtraction

- $\sin(a - b) = \sin(a) \cos(b) - \sin(b) \cos(a)$;

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Addition

- $\sin(a + b) = \sin(a) \cos(b) + \sin(b) \cos(a)$;
- $\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)$.

Subtraction

- $\sin(a - b) = \sin(a) \cos(b) - \sin(b) \cos(a)$;
- $\cos(a - b) = \cos(a) \cos(b) + \sin(a) \sin(b)$.

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- $\sin(a + b) = \sin(a) \cos(b) + \sin(b) \cos(a)$;
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Subtraction

- $\sin(a - b) = \sin(a) \cos(b) - \sin(b) \cos(a)$;
- $\cos(a - b) = \cos(a) \cos(b) + \sin(a) \sin(b)$.

Example

- 1 Find $\sin(105^\circ)$.

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Addition

- $\sin(a + b) = \sin(a) \cos(b) + \sin(b) \cos(a)$;
- $\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)$.

Subtraction

- $\sin(a - b) = \sin(a) \cos(b) - \sin(b) \cos(a)$;
- $\cos(a - b) = \cos(a) \cos(b) + \sin(a) \sin(b)$.

Example

- 1 Find $\sin(105^\circ)$.

We notice that $105 = 60 + 45$, therefore

$$\begin{aligned}\sin(105^\circ) &= \sin(60^\circ) \cos(45^\circ) + \sin(45^\circ) \cos(60^\circ) = \\ &= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}.\end{aligned}$$

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Addition

- $\sin(a + b) = \sin(a) \cos(b) + \sin(b) \cos(a)$;
- $\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)$.

Subtraction

- $\sin(a - b) = \sin(a) \cos(b) - \sin(b) \cos(a)$;
- $\cos(a - b) = \cos(a) \cos(b) + \sin(a) \sin(b)$.

Example

- 1 Find $\sin(105^\circ)$.

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- 2 Express $\cos\left(a + \frac{\pi}{3}\right)$ in terms of $\sin(a)$ and $\cos(a)$.

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Addition

- $\sin(a + b) = \sin(a) \cos(b) + \sin(b) \cos(a)$;
- $\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)$.

Subtraction

- $\sin(a - b) = \sin(a) \cos(b) - \sin(b) \cos(a)$;
- $\cos(a - b) = \cos(a) \cos(b) + \sin(a) \sin(b)$.

Example

- 1 Find $\sin(105^\circ)$.

We notice that $105 = 60 + 45$, therefore

$$\begin{aligned}\sin(105^\circ) &= \sin(60^\circ) \cos(45^\circ) + \sin(45^\circ) \cos(60^\circ) = \\ &= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}.\end{aligned}$$

- 2 Express $\cos\left(a + \frac{\pi}{3}\right)$ in terms of $\sin(a)$ and $\cos(a)$.
$$\begin{aligned}\cos\left(a + \frac{\pi}{3}\right) &= \cos(a) \cos\left(\frac{\pi}{3}\right) - \sin(a) \sin\left(\frac{\pi}{3}\right) = \\ &= \frac{1}{2} \cos(a) - \frac{\sqrt{3}}{2} \sin(a).\end{aligned}$$